Proceedings of the Edinburgh Mathematical Society (2006) **49**, 115–129 ©
DOI:10.1017/S0013091504001415 Printed in the United Kingdom

A REMARK ON THE EXISTENCE OF BREATHER SOLUTIONS FOR THE DISCRETE NONLINEAR SCHRÖDINGER EQUATION IN INFINITE LATTICES: THE CASE OF SITE-DEPENDENT ANHARMONIC PARAMETERS

NIKOS I. KARACHALIOS

Department of Mathematics, University of the Aegean, Karlovassi GR 83200, Samos, Greece (karan@aegean.gr)

(Received 27 November 2004)

Abstract We discuss the existence of breather solutions for a discrete nonlinear Schrödinger equation in an infinite N-dimensional lattice, involving site-dependent anharmonic parameters. We give a simple proof of the existence of (non-trivial) breather solutions based on a variational approach, assuming that the sequence of anharmonic parameters is in an appropriate sequence space (decays with an appropriate rate). We also give a proof of the non-existence of (non-trivial) breather solutions, and discuss a possible physical interpretation of the restrictions, in both the existence and non-existence cases.

Keywords: discrete nonlinear Schrödinger equation; lattice differential equations; breather solutions; variational methods

2000 *Mathematics subject classification:* Primary 37L60; 35Q55; 47J30

1. Introduction

The one-dimensional discrete nonlinear Schrödinger (DNLS) equation,

$$\mathrm{i}\dot{\psi}_n + \varepsilon(\psi_{n-1} - 2\psi_n + \psi_{n+1}) + \gamma|\psi_n|^2\psi_n = 0, \tag{1.1}$$

represents an infinite ($n \in \mathbb{Z}$) or a finite ($|n| \leqslant K$) one-dimensional array of coupled anharmonic oscillators, coupled to their nearest neighbours with a coupling strength ε. Here $\psi_n(t)$ stands for the complex mode amplitude of the oscillator at lattice site n, and γ denotes an anharmonic parameter. Setting $\varepsilon = 1/(\Delta x)^2$ reminds us that the model includes a finite spacing between molecules, and the formal continuum limit, the NLS partial differential equation, is obtained by taking $\Delta x \to 0$. The DNLS equation is one of the most important inherently discrete models, having a crucial role in the modelling of a great variety of phenomena, ranging from solid-state and condensed-matter physics to biology [**2**, **9**, **11**, **13**, **17**]. Depending on the size of the lattice, we have to deal with an infinite or finite system of ordinary differential equations, respectively.

The gauge invariance of the nonlinearity allows for the support of special solutions of (1.1) of the form $\psi_n(t) = \phi_n \exp(\mathrm{i}\omega t)$, $\omega > 0$. These solutions are called *breather* *solutions*, due to their periodic time behaviour. Inserting the ansatz of a breather solution into (1.1), it follows that ϕ_n satisfies the nonlinear system of algebraic equations

$$-\varepsilon(\phi_{n-1} - 2\phi_n + \phi_{n+1}) + \omega\phi_n = \gamma|\psi_n|^2\psi_n. \tag{1.2}$$

The problem of existence and stability properties of breather solutions of coupled oscillators has been developed as a fascinating subject of research. Starting from the derivation of the stationary DNLS equation [**14**], we refer among other contributions to the derivation of stationary solutions for the (coupled) DNLS equation, by numerical continuation from the so-called anticontinuum limit (the case $\varepsilon \to 0$) [**10**], and the ingenious construction of localized time-periodic or quasi-periodic solutions of general discrete systems (starting from periodic solutions of the corresponding anticontinuum limit equations) [**2**, **18**]. We refer to [**9**, **16**] for reviews of the existing results and the history of the problem and for a long list of references.

Motivated by [**9**, § 3] and [**3**], in this work we consider higher-dimensional generalizations of DNLS equations, involving an arbitrary power law nonlinearity and site dependence of the anharmonic parameter γ. For this particular case of nonlinearity, we also refer to [**19**–**21**]. For instance, for breather solutions of the DNLS equation in infinite higher-dimensional lattices ($n = (n_1, n_2, \dots, n_N) \in \mathbb{Z}^N$) we consider the equation

$$\mathrm{i}\dot{\psi}_n + (\boldsymbol{A}\psi)_n + \gamma_n|\psi_n|^{2\sigma}\psi_n = 0, \tag{1.3}$$

where

$$\begin{aligned}(\boldsymbol{A}\psi)_{n\in\mathbb{Z}^N} = \psi_{(n_1-1,n_2,\dots,n_N)} + \psi_{(n_1,n_2-1,\dots,n_N)} + \cdots \\ + \psi_{(n_1,n_2,\dots,n_N-1)} - 2N\psi_{(n_1,n_2,\dots,n_N)} \\ + \psi_{(n_1+1,n_2,\dots,n_N)} + \psi_{(n_1,n_2+1,\dots,n_N)} + \cdots + \psi_{(n_1,n_2,\dots,n_N+1)}.\end{aligned} \tag{1.4}$$

In this case, Equation (1.3) could be viewed as the discretization of the NLS partial differential equation

$$\mathrm{i}\psi_t + \Delta\psi + \gamma(x)|\psi|^{2\sigma}\psi = 0, \quad x \in \mathbb{R}^N. \tag{1.5}$$

As in the one-dimensional case, it can easily be seen that any breather solution $\psi_n(t) = \phi_n \exp(\mathrm{i}\omega t)$ of the DNLS equation (1.3) satisfies the infinite nonlinear system of algebraic equations

$$-(\boldsymbol{A}\phi)_n + \omega\phi_n = \gamma_n|\phi_n|^{2\sigma}\phi_n, \quad n \in \mathbb{Z}^N. \tag{1.6}$$

Based on a variational approach, which makes use of the famous mountain-pass theorem (MPT), we give a simple proof of the existence of (non-trivial) breather solutions for (1.3), by showing that the energy functional associated with (1.6) has a critical point of 'mountain-pass type'. Our main assumption is that γ_n decays at an appropriate rate, in the sense that γ_n is in an appropriate sequence space. This restriction enables us to use a compact inclusion between ordinary sequence spaces and *weighted* sequence spaces, in order to justify one of the crucial steps needed for the application of the MPT, namely the Palais–Smale condition. This is an important difference from the case of constant anharmonic parameters, as the analysis of our recent work [**15**] shows: the latter is associated with lack of compactness and restricted our study to a finite-dimensional problem (in a one-dimensional lattice, assuming Dirichlet boundary conditions). We note here that the case of lack of compactness can be treated by the concentration compactness arguments in the discrete context [**26**]. We also refer to [**22**] for an application of an envelope technique, characterizing the precompactness of minimizing sequences. The application of the MPT to (1.6) gives rise to some restrictions, which possibly have some physical interpretation. These restrictions, needed for the support of a non-trivial breather solution, are imposed on some energy quantities. The restrictions depend on the frequency ω, the nonlinearity σ and the sequence of anharmonic parameters γ_n.

On the other hand, it is shown that non-trivial solutions of (1.6) do not exist in a ball of the space ℓ^2, of sufficiently small radius. The proof is based on a fixed-point argument that is also used in [**15**]. This result could imply that we should not expect the existence of breather solutions if the energy of the excitations of the lattice is sufficiently small.

The restrictions posed by the MPT can be combined with those needed for the proof of the non-existence result to derive a 'dispersion relation' of nonlinearity exponent σ versus the frequency ω of the breather solution, providing information on the behaviour of the associated energy quantities (see relations (2.47) and (2.48)). For a detailed discussion on the breather problem in higher-dimensional lattices and the dependence of the frequency ω on the conserved quantities of the DNLS equation, we refer to [**9**, § 6].

2. Preliminaries

In this introductory section, we describe the functional setting needed for the treatment of the infinite nonlinear system (1.6). We also introduce some weighted sequence spaces, and we prove a compact inclusion between the ordinary sequence spaces and their weighted counterparts.

For some positive integer N, we consider the complex sequence spaces

$$\ell^p = \left\{ \begin{matrix} \phi = \{\phi_n\}_{n\in\mathbb{Z}^N}, \quad n = (n_1, n_2, \dots, n_N) \in \mathbb{Z}^N, \quad \phi_n \in \mathbb{C} \\ \|\phi\|_{\ell^p} = \left(\sum_{n\in\mathbb{Z}^N} |\phi_n|^p \right)^{1/p} < \infty \end{matrix} \right\}. \tag{2.1}$$

Between ℓ^p spaces the following elementary embedding relation [**24**] holds:

$$\ell^q \subset \ell^p, \quad \|\phi\|_{\ell^p} \leqslant \|\phi\|_{\ell^q}, \quad 1 \leqslant q \leqslant p \leqslant \infty. \tag{2.2}$$

Note that the opposite holds for the $L^p(\Omega)$-spaces if $\Omega \subset \mathbb{R}^N$ has finite measure. For $p = 2$, we get the usual Hilbert space of square-summable sequences, which becomes a real Hilbert space if endowed with the real scalar product

$$(\phi, \psi)_{\ell^2} = \mathrm{Re} \sum_{n\in\mathbb{Z}^N} \phi_n \bar{\psi}_n, \quad \phi, \psi \in \ell^2. \tag{2.3}$$

For a sequence *of positive real numbers* $\delta = \{\delta_n\}_{n\in\mathbb{Z}^N}$, we define the weighted sequence spaces ℓ^2_δ by

$$\ell^p_\delta = \left\{ \begin{array}{c} \phi = \{\phi_n\}_{n\in\mathbb{Z}^N}, \quad n = (n_1, n_2, \dots, n_N) \in \mathbb{Z}^N, \quad \phi_n \in \mathbb{C} \\ \|\phi\|_{\ell^p_\delta} = \left(\sum_{n\in\mathbb{Z}^N} \delta_n |\phi_n|^p \right)^{1/p} < \infty \end{array} \right\}. \tag{2.4}$$

For the case $p = 2$, it is not hard to see that ℓ^2_δ is a Hilbert space, with scalar product

$$(\phi, \psi)_{\ell^2_\delta} = \mathrm{Re} \sum_{n\in\mathbb{Z}^N} \delta_n \phi_n \bar{\psi}_n, \quad \phi, \psi \in \ell^2_\delta. \tag{2.5}$$

For a certain class of weight δ, we have the following lemma, which will play a crucial role in our analysis.

Lemma 2.1. *We assume that the positive sequence of real numbers* $\delta \in \ell^\rho$, $\rho = (q-1)/(q-2)$ *for some* $q > 2$. *Then* $\ell^2 \hookrightarrow \ell^2_\delta$ *with compact inclusion.*

Proof. We use the ideas of [**5**, Lemma 2.3, p. 79] and (2.2). We consider a bounded sequence $\phi_k \in \ell^2$ and we denote by $(\phi_k)_n$ the nth coordinate of this sequence. It suffices to show that the sequence ϕ_k is a Cauchy sequence in ℓ^2_δ. For some $q > 2$ we consider its Hölder conjugate through the relation $p^{-1} + q^{-1} = 1$. Then, for all positive integers k, l, we have

$$\begin{aligned} \|\phi_k - \phi_l\|^2_{\ell^2_\delta} &= \sum_{n\in\mathbb{Z}^N} \delta_n |(\phi_k)_n - (\phi_l)_n|^2 \\ &\leqslant \left(\sum_{n\in\mathbb{Z}^N} \delta_n |(\phi_k)_n - (\phi_l)_n|^p \right)^{1/p} \left(\sum_{n\in\mathbb{Z}^N} \delta_n |(\phi_k)_n - (\phi_l)_n|^q \right)^{1/q}. \end{aligned} \tag{2.6}$$

Since ϕ_k is a bounded sequence in ℓ^2, it follows from (2.2) that ϕ_k is bounded in ℓ^q. Then from (2.6) we have that there exists a positive constant c, such that

$$\|\phi_k - \phi_l\|^2_{\ell^2_\delta} \leqslant c \left(\sum_{n\in\mathbb{Z}^N} \delta_n |(\phi_k)_n - (\phi_l)_n|^p \right)^{1/p}. \tag{2.7}$$

Since $\delta \in \ell^\rho$, it holds that for any $\varepsilon_1 > 0$ there exists $K_0(\varepsilon_1)$ such that for all $K > K_0(\varepsilon_1)$

$$\sum_{|n|>K} |\delta_n|^\rho < \varepsilon_1.$$

Thus, using the boundedness of ϕ_k in ℓ^q once again, we have

$$\begin{aligned} \sum_{|n|>K} \delta_n |(\phi_k)_n - (\phi_l)_n|^p &\leqslant \left(\sum_{|n|>K} |\delta_n|^\rho \right)^{1/\rho} \left(\sum_{|n|>K} |(\phi_k)_n - (\phi_l)_n|^q \right)^{p/q} \\ &< c\varepsilon_1^{1/\rho}. \end{aligned} \tag{2.8}$$

https://doi.org/10.1017/S0013091504001415 Published online by Cambridge University Press

On the other hand, since the sequence ϕ_k is a Cauchy sequence in the finite-dimensional space $\mathbb{C}^{(2K+1)^N}$, we get that, for k and l sufficiently large and for any $\varepsilon_2 > 0$, it holds that

$$\sum_{|n|\leqslant K} \delta_n |(\phi_k)_n - (\phi_l)_n|^p < \varepsilon_2. \tag{2.9}$$

Inequality (2.7) can be rewritten as

$$\|\phi_k - \phi_l\|_{\ell^2_\delta}^{2p} \leqslant c\biggl\{\sum_{|n|\leqslant K} \delta_n |(\phi_k)_n - (\phi_l)_n|^p + \sum_{|n|>K} \delta_n |(\phi_k)_n - (\phi_l)_n|^p\biggr\}. \tag{2.10}$$

Now, from (2.8)–(2.10), and appropriate choices of ε_1 and ε_2, we may derive that, for sufficiently large k and l,

$$\|\phi_k - \phi_l\|_{\ell^2_\delta} < \varepsilon, \quad \text{for any } \varepsilon > 0.$$

That is, ϕ_k is a Cauchy sequence in ℓ^2_δ. □

Let $\boldsymbol{A} : D(\boldsymbol{A}) \subseteq X \to X$ be a $\mathbb{C}$-linear, self-adjoint, dissipative operator with dense domain $D(\boldsymbol{A})$ on the Hilbert space X, equipped with the scalar product $(\cdot\,,\cdot)_X$. The space $X_{\boldsymbol{A}}$ is the completion of $D(\boldsymbol{A})$ in the norm $\|u\|^2_{\boldsymbol{A}} = \|u\|^2_X - (\boldsymbol{A}u, u)_X$, for $u \in X_{\boldsymbol{A}}$, and we denote by $X^*_{\boldsymbol{A}}$ its dual and by $\boldsymbol{A}^*$ the extension of $\boldsymbol{A}$ to the dual of $D(\boldsymbol{A})$, denoted by $D(\boldsymbol{A})^*$ (Friedrichs extension theory (see [**8**] and [**27**, vol. II/A])).

Considering the operator $\boldsymbol{A}$ defined by (1.4), we observe that for any $\phi \in \ell^2$

$$\|\boldsymbol{A}\phi\|^2_{\ell^2} \leqslant 4N\|\phi\|^2_{\ell^2}; \tag{2.11}$$

that is, $\boldsymbol{A} : \ell^2 \to \ell^2$ is a continuous operator. Now we consider the discrete operator $\boldsymbol{L}^+ : \ell^2 \to \ell^2$ defined by

$$\begin{aligned}(\boldsymbol{L}^+\psi)_{n\in\mathbb{Z}^N} &= \{\psi_{(n_1+1,n_2,\dots,n_N)} - \psi_{(n_1,n_2,\dots,n_N)}\}\\ &\quad + \{\psi_{(n_1,n_2+1,\dots,n_N)} - \psi_{(n_1,n_2,\dots,n_N)}\}\\ &\quad\ \vdots\\ &\quad + \{\psi_{(n_1,n_2,\dots,n_N+1)} - \psi_{(n_1,n_2,\dots,n_N)}\},\end{aligned} \tag{2.12}$$

and $\boldsymbol{L}^- : \ell^2 \to \ell^2$ defined by

$$\begin{aligned}(\boldsymbol{L}^-\psi)_{n\in\mathbb{Z}^N} &= \{\psi_{(n_1-1,n_2,\dots,n_N)} - \psi_{(n_1,n_2,\dots,n_N)}\}\\ &\quad + \{\psi_{(n_1,n_2-1,\dots,n_N)} - \psi_{(n_1,n_2,\dots,n_N)}\}\\ &\quad\ \vdots\\ &\quad + \{\psi_{(n_1,n_2,\dots,n_N-1)} - \psi_{(n_1,n_2,\dots,n_N)}\}.\end{aligned} \tag{2.13}$$

https://doi.org/10.1017/S0013091504001415 Published online by Cambridge University Press

Setting

$$(\boldsymbol{L}_\nu^+\psi)_{n\in\mathbb{Z}^N} = \psi_{(n_1,n_2,\dots,n_{\nu-1},n_\nu+1,n_{\nu+1},\dots,n_N)} - \psi_{(n_1,n_2,\dots,n_N)}, \tag{2.14}$$

$$(\boldsymbol{L}_\nu^-\psi)_{n\in\mathbb{Z}^N} = \psi_{(n_1,n_2,\dots,n_{\nu-1},n_\nu-1,n_{\nu+1},\dots,n_N)} - \psi_{(n_1,n_2,\dots,n_N)}, \tag{2.15}$$

we observe that the operator $\boldsymbol{A}$ satisfies the relations

$$(-\boldsymbol{A}\psi_1,\psi_2)_{\ell^2} = \sum_{\nu=1}^{N}(\boldsymbol{L}_\nu^+\psi_1,\boldsymbol{L}_\nu^+\psi_2)_{\ell^2}, \quad \text{for all } \psi_1,\psi_2\in\ell^2, \tag{2.16}$$

$$(\boldsymbol{L}_\nu^+\psi_1,\psi_2)_{\ell^2} = (\psi_1,\boldsymbol{L}_\nu^-\psi_2)_{\ell^2}, \qquad \text{for all } \psi_1,\psi_2\in\ell^2. \tag{2.17}$$

From (2.16), it is clear that $\boldsymbol{A}:\ell^2\to\ell^2$ defines a self-adjoint operator on ℓ^2, and $\boldsymbol{A}\leqslant 0$. The graph-norm

$$\|\phi\|^2_{D(\boldsymbol{A})} = \|\boldsymbol{A}\phi\|^2_{\ell^2} + \|\phi\|^2_{\ell^2}$$

is equivalent to that of ℓ^2, since

$$\|\phi\|^2_{\ell^2} \leqslant \|\phi\|^2_{D(\boldsymbol{A})} \leqslant (4N+1)\|\phi\|^2_{\ell^2}.$$

In our case, we have that $X_{\boldsymbol{A}} = \ell^2$ equipped with the norm

$$\|\phi\|^2_{\boldsymbol{A}} = \|\phi\|^2_X - (\boldsymbol{A}\phi,\phi)_X = \sum_{\nu=1}^{N}\|\boldsymbol{L}_\nu^+\phi\|^2_{\ell_2} + \|\phi\|^2_{\ell^2},$$

for $\phi\in\ell^2$, and is an equivalent norm with the usual one of ℓ^2. Moreover, $D(\boldsymbol{A}) = X = \ell^2 = D(\boldsymbol{A})^*$. Obviously, $\boldsymbol{A}^* = \boldsymbol{A}$ and the operator $\mathrm{i}\boldsymbol{A}:\ell^2\to\ell^2$, defined by $(\mathrm{i}\boldsymbol{A})\phi = \mathrm{i}\boldsymbol{A}\phi$ for $\phi\in\ell^2$, is $\mathbb{C}$-linear and skew-adjoint and $\mathrm{i}\boldsymbol{A}$ generates a group $(\mathcal{T}(t))_{t\in\mathbb{R}}$ of isometries on ℓ^2 (see [**6**]). The analysis of the operator $\boldsymbol{A}$ is useful if one would like to consider the DNLS equation (1.3) as an abstract evolution equation [**15**], and holds for other discrete operators that are not necessarily discretizations of the Laplacian (for examples of such operators, we refer the reader to [**28**]).

2.1. Existence of non-trivial breather solutions in the case of decaying anharmonic parameters

We shall seek non-trivial breather solutions as critical points of the functional

$$\boldsymbol{E}(\phi) = \frac{1}{2}\sum_{\nu=1}^{N}\|\boldsymbol{L}_\nu^+\phi\|^2_{\ell_2} + \frac{\omega^2}{2}\sum_{n\in\mathbb{Z}^N}|\phi_n|^2 - \frac{1}{2\sigma+2}\sum_{n\in\mathbb{Z}^N}\gamma_n|\phi_n|^{2\sigma+2}. \tag{2.18}$$

To establish differentiability of the functional $\boldsymbol{E}:\ell^2\to\mathbb{R}$, we will use the following discrete version of the dominated convergence theorem, provided by [**4**].

https://doi.org/10.1017/S0013091504001415 Published online by Cambridge University Press

Theorem 2.2. *Let $\{\psi_{i,k}\}$ be a double sequence of summable functions,*

$$\sum_{i\in\mathbb{Z}^N} |\psi_{i,k}| < \infty,$$

and let $\lim_{k\to\infty} \psi_{i,k} = \psi_i$ for all $i \in \mathbb{Z}^N$. If there exists a summable sequence $\{g_i\}$ such that $|\psi_{i,k}| \leqslant g_i$ for all values of i and k, we have that

$$\lim_{k\to\infty} \sum_{i\in\mathbb{Z}^N} \psi_{i,k} = \sum_{i\in\mathbb{Z}^N} \psi_i.$$

We then have the following lemma.

Lemma 2.3. *Let $(\phi_n)_{n\in\mathbb{Z}^N} = \phi \in \ell^{2\sigma+2}$ for some $0 < \sigma < \infty$. Moreover, we assume that $\gamma_n \in \ell^\rho$, $\rho = (q-1)/(q-2)$ for some $q > 2$. Then the functional*

$$\boldsymbol{F}(\phi) = \sum_{n\in\mathbb{Z}^N} \gamma_n |\phi_n|^{2\sigma+2}$$

is a $\mathrm{C}^1(\ell^{2\sigma+2}, \mathbb{R})$ functional and

$$\langle \boldsymbol{F}'(\phi), \psi\rangle = (2\sigma+2)\,\mathrm{Re} \sum_{n\in\mathbb{Z}^N} \gamma_n |\phi_n|^{2\sigma} \phi_n \bar{\psi}_n, \quad \psi = (\psi_n)_{n\in\mathbb{Z}^N} \in \ell^{2\sigma+2}. \tag{2.19}$$

Proof. We assume that $\phi, \psi \in \ell^{2\sigma+2}$. Then, for any $n \in \mathbb{Z}^N$, $0 < s < 1$, we get

$$\begin{aligned}\frac{\boldsymbol{F}(\phi + s\psi) - \boldsymbol{F}(\psi)}{s} &= \frac{1}{s}\,\mathrm{Re} \sum_{n\in\mathbb{Z}^N} \gamma_n \int_0^1 \frac{\mathrm{d}}{\mathrm{d}\theta} |\phi_n + \theta s\psi_n|^{2\sigma+2}\,\mathrm{d}\theta \\ &= (2\sigma+2)\,\mathrm{Re} \sum_{n\in\mathbb{Z}^N} \gamma_n \int_0^1 |\phi_n + \theta s\psi_n|^{2\sigma} (\phi_n + s\theta\psi_n)\bar{\psi}_n\,\mathrm{d}\theta. \end{aligned} \tag{2.20}$$

Since γ_n is in ℓ^ρ, it follows from (2.2) that

$$\sup_{n\in\mathbb{Z}^N} |\gamma_n| = M < \infty. \tag{2.21}$$

On the other hand, we have the inequality

$$\begin{aligned}&\sum_{n\in\mathbb{Z}^N} |\phi_n + \theta s\psi_n|^{2\sigma+1} |\psi_n| \\ &\qquad\leqslant \sum_{n\in\mathbb{Z}^N} (|\phi_n| + |\psi_n|)^{2\sigma+1} |\psi_n| \\ &\qquad\leqslant \biggl(\sum_{n\in\mathbb{Z}^N} (|\phi_n| + |\psi_n|)^{2\sigma+2}\biggr)^{(2\sigma+1)/(2\sigma+2)} \biggl(\sum_{n\in\mathbb{Z}^N} |\psi_n|^{2\sigma+2}\biggr)^{1/(2\sigma+2)}. \end{aligned} \tag{2.22}$$

Now, by using (2.21) and inserting (2.22) into (2.20), we see that Lemma 2.2 is applicable: letting $s \to 0$, we get the existence of the Gâteaux derivative (2.19) of the functional $\boldsymbol{F} : \ell^{2\sigma+2} \to \mathbb{R}$.

Next we show that the functional

$$\boldsymbol{F}' : \ell^{2\sigma+2} \to \ell^{(2\sigma+2)/(2\sigma+1)}$$

is continuous. For $\phi \in \ell^{2\sigma+2}$, we set $(F_1(\phi))_{n\in\mathbb{Z}^N} = |\phi_n|^{2\sigma}\phi_n$.

Let us note that for any $F \in \mathrm{C}(\mathbb{C},\mathbb{C})$ that takes the form $F(z) = g(|z|^2)z$, with g real and sufficiently smooth, the following equality holds:

$$F(\phi_1) - F(\phi_2) = \int_0^1 \{(\phi_1 - \phi_2)(g(r) + rg'(r)) + (\bar{\phi}_1 - \bar{\phi}_2)\Phi^2 g'(r)\}\,\mathrm{d}\theta, \tag{2.23}$$

for any $\phi_1, \phi_2 \in \mathbb{C}$, where $\Phi = \theta\phi_1 + (1-\theta)\phi_2$, $\theta \in (0,1)$ and $r = |\Phi|^2$ (see [**12**, p. 202]). Applying (2.23) for the case of F_1 ($g(r) = r^\sigma$), one obtains that

$$F_1(\phi_1) - F_1(\phi_2) = \int_0^1 [(\sigma+1)(\phi_1 - \phi_2)|\Phi|^{2\sigma} + \sigma(\bar{\phi}_1 - \bar{\phi}_2)\Phi^2|\Phi|^{2\sigma-2}]\,\mathrm{d}\theta,$$

which implies the inequality

$$|F_1(\phi_1) - F_1(\phi_2)| \leqslant (2\sigma+1)(|\phi_1| + |\phi_2|)^{2\sigma}|\phi_1 - \phi_2|. \tag{2.24}$$

We consider a sequence $\phi_m \in \ell^{2\sigma+2}$ such that $\phi_m \to \phi$ in $\ell^{2\sigma+2}$. Using (2.21), we get the inequality

$$|\langle \boldsymbol{F}'(\phi_m) - \boldsymbol{F}'(\phi), \psi\rangle| \leqslant c(M)\|F_1(\phi_m) - F_1(\phi)\|_{\ell^q}\|\psi\|_{\ell^p}, \quad q = \frac{2\sigma+2}{2\sigma+1}, \quad p = 2\sigma+2. \tag{2.25}$$

We denote by $(\phi_m)_n$ the nth coordinate of the sequence $\phi_m \in \ell^2$. By setting

$$\Phi_n = (|(\phi_m)_n| + |\phi_n|)^{2\sigma},$$

we get from (2.24) that, for some constant $c > 0$,

$$\begin{aligned}\sum_{n\in\mathbb{Z}^N} &|F_1((\phi_m)_n) - F_1(\phi_n)|^q \\ &\leqslant c\sum_{n\in\mathbb{Z}^N}(\Phi_n)^q|(\phi_m)_n - \phi_n|^q \\ &\leqslant c\biggl(\sum_{n\in\mathbb{Z}^N}|(\phi_m)_n - \phi_n|^{2\sigma+2}\biggr)^{1/(2\sigma+1)}\biggl(\sum_{n\in\mathbb{Z}^N}(\Phi_n)^{(\sigma+1)/\sigma}\biggr)^{2\sigma/(2\sigma+1)} \to 0,\end{aligned}$$

as $m \to \infty$. □

By using (2.16), we easily see that the rest of the terms of the functional $\boldsymbol{E}$ given by (2.18) define $\mathrm{C}^1(\ell^2,\mathbb{R})$ functionals. Since Lemma 2.3 holds for any $\phi \in \ell^2$ (by (2.2)), we finally obtain that the functional $\boldsymbol{E}$ is $\mathrm{C}^1(\ell^2,\mathbb{R})$. Moreover, by using the analysis of § 1 for the self-adjoint operator $\boldsymbol{A} : \ell^2 \to \ell^2$, it appears that any solution of (1.6) satisfies the formula

$$(-\boldsymbol{A}\phi, \psi)_{\ell^2} + \omega(\phi,\psi)_{\ell^2} = (\gamma_n F_1(\phi), \psi)_{\ell^2}, \quad \text{for all } \psi \in \ell^2,$$

https://doi.org/10.1017/S0013091504001415 Published online by Cambridge University Press

and vice versa. Equivalently, due to the differentiability of the functional $\boldsymbol{E}$, any solution of (1.6) is a critical point of $\boldsymbol{E}$. For convenience, we recall [**7**, Definition 4.1, p. 130] (hereafter referred to as condition (PS)) and [**7**, Theorem 6.1, p. 140] or [**25**, Theorem 6.1, p. 109] (the mountain-pass theorem of Ambrosseti and Rabinowitz [**1**]).

Definition 2.4. Let X be a Banach space and let $\boldsymbol{E} : X \to \mathbb{R}$ be C^1. We say that $\boldsymbol{E}$ satisfies condition (PS) if, for any sequence $\{\phi_n\} \in X$ such that $|\boldsymbol{E}(\phi_n)|$ is bounded and $\boldsymbol{E}'(\phi_n) \to 0$ as $n \to \infty$, there exists a convergent subsequence. If this condition is only satisfied in the region where $\boldsymbol{E} \geqslant \alpha > 0$ (respectively, $\boldsymbol{E} \leqslant -\alpha < 0$) for all $\alpha > 0$, we say that $\boldsymbol{E}$ satisfies condition (PS^+) (respectively, (PS^-)).

Theorem 2.5. *Let $\boldsymbol{E} : X \to \mathbb{R}$ be C^1 and satisfy*

(a) $\boldsymbol{E}(0) = 0$,

(b) $\exists \rho > 0,\ \alpha > 0 : \|\phi\|_X = \rho$ *implies* $\boldsymbol{E}(\phi) \geqslant \alpha$,

(c) $\exists \phi_1 \in X : \|\phi_1\|_X \geqslant \rho$ *and* $\boldsymbol{E}(\phi_1) < \alpha$.

Define

$$\Gamma = \{\gamma \in \mathrm{C}^0([0,1], X) : \gamma(0) = 0,\ \gamma(1) = \phi_1\}.$$

Let $F_\gamma = \{\gamma(t) \in X : 0 \leqslant t \leqslant 1\}$ *and let* $\mathcal{L} = \{F_\gamma : \gamma \in \Gamma\}$. *If $\boldsymbol{E}$ satisfies condition (PS), then*

$$\beta := \inf_{F_\gamma \in \mathcal{L}} \sup\{\boldsymbol{E}(v) : v \in F_\gamma\} \geqslant \alpha$$

is a critical point of the functional $\boldsymbol{E}$.

For fixed $\omega > 0$, we shall consider a norm in ℓ^2 defined by

$$\|\phi\|^2_{\ell^2_\omega} = \sum_{\nu=1}^{\nu=N} \|\boldsymbol{L}^+_\nu \phi\|^2_{\ell_2} + \omega \|\phi\|^2_{\ell^2}, \quad \phi \in \ell^2. \tag{2.26}$$

The norm (2.26) is an equivalent norm with the usual one of ℓ^2, since

$$\omega \|\phi\|^2_{\ell^2} \leqslant \|\phi\|^2_{\ell^2_\omega} \leqslant (2N + \omega)\|\phi\|^2_{\ell^2}. \tag{2.27}$$

We first check the behaviour of the functional $\boldsymbol{E}$. Using (2.27), we observe that

$$\begin{aligned} |\boldsymbol{F}(\phi)| &\leqslant M \sum_{n \in \mathbb{Z}^N} |\phi_n|^{2\sigma+2} \\ &\leqslant M \|\phi\|^{2\sigma+2}_{\ell^2} \\ &\leqslant \frac{M}{\omega^{\sigma+1}} \|\phi\|^{2\sigma+2}_{\ell^2_\omega}. \end{aligned} \tag{2.28}$$

Now setting $M_0 = M/\omega^{\sigma+1}$ we observe that

$$\begin{aligned} \boldsymbol{E}(\phi) &= \tfrac{1}{2}\|\phi\|^2_{\ell^2_\omega} - \frac{1}{2\sigma+2}\boldsymbol{F}(\phi) \\ &\geqslant \tfrac{1}{2}\|\phi\|^2_{\ell^2_\omega} - \frac{M_0}{2\sigma+2}\|\phi\|^{2\sigma+2}_{\ell^2_\omega}. \end{aligned} \tag{2.29}$$

We now select some $\phi \in \ell^2$ such that $\|\phi\|_{\ell^2_\omega} = R > 0$. Then, if

$$0 < R < \left(\frac{\sigma+1}{M_0}\right)^{1/2\sigma} = \left(\frac{(\sigma+1)\omega^{\sigma+1}}{M}\right)^{1/2\sigma} := E^*_{\ell^2_\omega}(\sigma,\omega,M), \tag{2.30}$$

it follows from (2.29) that

$$\boldsymbol{E}(\phi) \geqslant \alpha > 0, \quad \alpha = R^2\left(\frac{1}{2} - \frac{M_0}{2\sigma+2}R^{2\sigma}\right).$$

We assume that $\gamma_n > 0$ for all $n \in \boldsymbol{S}_+ \subseteq \mathbb{Z}^N$. Next we will consider some $\psi \in \ell^2$ such that $\|\psi\|_{\ell^2_\omega} = 1$ and

$$\{\psi_n\}_{n\in\mathbb{Z}} = \{\psi_n\}_{n\in\boldsymbol{S}_+} + \{\psi_n\}_{n\in(\mathbb{Z}^N\setminus\boldsymbol{S}_+)}, \quad \text{where } \{\psi_n\}_{n\in\boldsymbol{S}_+} > 0,\ \{\psi_n\}_{n\in(\mathbb{Z}^N\setminus\boldsymbol{S}_+)} = 0.$$

For some $t > 0$ we consider the element $\chi = t\psi \in \ell^2$. We have that

$$\boldsymbol{E}(\chi) = \frac{t^2}{2} - \frac{1}{2\sigma+2}t^{2\sigma+2}\sum_{n\in\boldsymbol{S}_+}\gamma_n|\psi_n|^{2\sigma+2}. \tag{2.31}$$

Now letting $t \to +\infty$ we find that $\boldsymbol{E}(t\psi) \to -\infty$.

For fixed $\phi \neq 0$ and choosing t sufficiently large, we may set $\phi_1 = t\phi$ to satisfy the second condition of Theorem 2.5. To conclude with the existence of a non-trivial breather solution, it remains to show that the functional $\boldsymbol{E}$ satisfies Lemma 2.4.

To this end, we consider a sequence ϕ_m of ℓ^2 such that $|\boldsymbol{E}(\phi_m)| < M'$ for some $M' > 0$ and $\boldsymbol{E}'(\phi_m) \to 0$ as $m \to \infty$. By using (2.18) and Lemma 2.3, we observe that, for m sufficiently large,

$$M' \geqslant \boldsymbol{E}(\phi_m) - \frac{1}{2\sigma+2}\langle \boldsymbol{E}'(\phi_m), \phi_m\rangle = \left(\frac{1}{2} - \frac{1}{2\sigma+2}\right)\|\phi_m\|^2_{\ell^2_\omega}. \tag{2.32}$$

Therefore, the sequence ϕ_m is bounded. Thus, we may extract a subsequence, still denoted by ϕ_m, such that

$$\phi_m \rightharpoonup \phi \text{ in } \ell^2, \quad \text{as } m \to \infty. \tag{2.33}$$

For this subsequence it follows once again from (2.18) and Lemma 2.3 that

$$\begin{aligned}\|\phi_m - \phi\|^2_{\ell^2_\omega} &= \langle \boldsymbol{E}'(\phi_m) - \boldsymbol{E}'(\phi), \phi_m - \phi\rangle \\ &\quad + \sum_{n\in\mathbb{Z}^N}\gamma_n[|(\phi_m)_n|^{2\sigma}(\phi_m)_n - |\phi_n|^{2\sigma}\phi_n]((\phi_m)_n - \phi_n).\end{aligned} \tag{2.34}$$

Another assumption on the sequence γ_n is that the sequence $|\gamma_n| = (\delta_n)_{n\in\mathbb{Z}^N}$ satisfies the assumptions of Lemma 2.1. We consider the associated Hilbert space ℓ^2_δ. Now, by using the inequality (2.24), we get the following estimate for the second term of the right-hand

https://doi.org/10.1017/S0013091504001415 Published online by Cambridge University Press

side of (2.34):

$$\sum_{n\in\mathbb{Z}^N}\gamma_n[|(\phi_m)_n|^{2\sigma}(\phi_m)_n-|\phi_n|^{2\sigma}\phi_n]((\phi_m)_n-\phi_n)$$
$$\leqslant c\sum_{n\in\mathbb{Z}^N}\Phi_n|\gamma_n||(\phi_m)_n-\phi_n|^2$$
$$\leqslant c\sup_{n\in\mathbb{Z}^N}\Phi_n\sum_{n\in\mathbb{Z}^N}|\gamma_n||(\phi_m)_n-\phi_n|^2=c_2\|\phi_m-\phi\|^2_{\ell^2_\delta},\qquad(2.35)$$

where $c_2=c\sup_{n\in\mathbb{Z}^N}\Phi_n$. Obviously, ϕ_m is bounded in ℓ^2_δ and from Lemma 2.1 it follows that

$$\phi_m\to\phi\text{ in }\ell^2_\delta,\quad\text{as }m\to\infty.\qquad(2.36)$$

Combining (2.34), (2.35) and (2.36) we obtain that

$$\|\phi_m-\phi\|_{\ell^2_\omega}\to0,\quad\text{as }m\to\infty.$$

Hence ϕ_m has a (strongly) convergent subsequence. The assumptions of Theorem 2.5 are satisfied, and we may summarize in the following theorem.

Theorem 2.6. *Assume that the site-dependent anharmonic parameter $\gamma_n>0$ in some $\boldsymbol{S}_+\subseteq\mathbb{Z}^N$. Moreover, we assume that $|\gamma_n|=\delta_n\in\ell^\rho$, $\rho=(q-1)/(q-2)$ for some positive integer $q>2$. Then, for any $\omega>0$, there exists a non-trivial breather solution $\psi_n(t)=\phi_n\exp(\mathrm{i}\omega t)$ of the DNLS equation (1.3).*

We remark here that the assumptions on the sequence of anharmonic parameters γ_n are crucial for the derivation of the strong convergence of the subsequence ϕ_m. If γ_n is constant for all $n\in\mathbb{Z}^N$, then due to the fact that the space ℓ^2 lacks the Schur property (in contrast with the space ℓ^1 which has the Schur property (i.e. weak convergence coincides with strong convergence)), it is not possible to derive the strong convergence of the subsequence from its weak convergence. Of course the strong convergence is valid in the case of a finite lattice: in this case, the problem is formulated in finite-dimensional spaces where weak convergence is equivalent to strong convergence [**15**]. For the case of constant anharmonic parameters in infinite lattices, we refer the reader to [**26**] (concentration compactness arguments) and [**22**] (envelope technique). We also refer to the recent work [**23**], on the existence of gap solitons in periodic DNLS equations, via a variational approach.

Inequality (2.30) could have some physical interpretation with respect to the existence of non-trivial breather solutions of frequency $\omega>0$ if one considers (2.30) as a possible (local) upper bound for the ℓ^2_ω norm. It is one of the conditions which determines the necessary behaviour of the functional $\boldsymbol{E}$, in order to have a non-trivial breather solution of frequency ω. This relation could be seen as some kind of dispersion relation of frequency versus energy for breather solutions. It contains information on the type of nonlinearity and the sequence of anharmonic parameters through its dependence on the nonlinearity exponent σ and M. Such types of relation seem to be reasonable, as the next result concerning non-existence of non-trivial breather solutions shows.

For the sake of completeness, we recall [**27**, Theorem 18.E, p. 68] (the theorem of Lax and Milgram). This theorem will be used to establish the existence of solutions for an auxiliary infinite linear system of algebraic equations related to (1.6).

Theorem 2.7. *Let X be a Hilbert space and let $\boldsymbol{A} : X \to X$ be a linear continuous operator. Suppose that there exists $c^* > 0$ such that*

$$\mathrm{Re}(\boldsymbol{A}u, u)_X \geqslant c^* \|u\|_X^2, \quad \text{for all } u \in X. \tag{2.37}$$

Then, for a given $f \in X$, the operator equation $\boldsymbol{A}u = f$, $u \in X$, has a unique solution.

The non-existence result can be stated as follows.

Theorem 2.8. *There exist no non-trivial breather solutions with an energy of less than*

$$E_{\min}(\omega, \sigma, M) := \frac{1}{2}\left(\frac{\omega}{M(2\sigma+1)}\right)^{1/2\sigma}. \tag{2.38}$$

Proof. For some $\omega > 0$, we consider the operator $\boldsymbol{A}_\omega : \ell^2 \to \ell^2$, defined by

$$(\boldsymbol{A}_\omega \phi)_{n\in\mathbb{Z}^N} = (\boldsymbol{A}\phi)_{n\in\mathbb{Z}^N} + \omega\phi_n. \tag{2.39}$$

It is linear and continuous and satisfies assumption (2.37) of Theorem 2.7: using (2.16), we find that

$$(\boldsymbol{A}_\omega \phi, \phi)_{\ell^2} = \sum_{\nu=1}^{N} \|\boldsymbol{L}_\nu^+ \phi\|_{\ell^2}^2 + \omega\|\phi\|^2 \geqslant \omega\|\phi\|_{\ell^2}^2, \quad \text{for all } \phi \in \ell^2. \tag{2.40}$$

For given $z \in \ell^2$, we consider the linear operator equation

$$(\boldsymbol{A}_\omega \phi)_{n\in\mathbb{Z}^N} = \gamma_n |z_n|^{2\sigma} z_n. \tag{2.41}$$

For the map

$$(\boldsymbol{T}(z))_{n\in\mathbb{Z}^N} = \gamma_n |z_n|^{2\sigma} z_n, \tag{2.42}$$

we observe that

$$\|\boldsymbol{T}(z)\|_{\ell^2}^2 \leqslant M^2 \sum_{n\in\mathbb{Z}^N} |z_n|^{4\sigma+2} \leqslant M^2 \|z\|_{\ell^2}^{4\sigma+2}.$$

Hence, the assumptions of Theorem 2.7 are satisfied, and (2.41) has a unique solution $\phi \in \ell^2$. For some $R > 0$, we consider the closed ball of ℓ^2, $B_R := \{z \in \ell^2 : \|z\|_{\ell^2} \leqslant R\}$, and we define the map $\mathcal{P} : \ell^2 \to \ell^2$ by $\mathcal{P}(z) := \phi$, where ϕ is the unique solution of the operator equation (2.41). Clearly, the map $\mathcal{P}$ is well defined.

Let $\zeta, \xi \in B_R$ such that $\phi = \mathcal{P}(\zeta)$, $\psi = \mathcal{P}(\xi)$. The difference $\chi := \phi - \psi$ satisfies the equation

$$(\boldsymbol{A}_\omega \chi)_{n\in\mathbb{Z}^N} = (\boldsymbol{T}(z))_{n\in\mathbb{Z}^N} - (\boldsymbol{T}(\xi))_{n\in\mathbb{Z}^N}. \tag{2.43}$$

The map $\boldsymbol{T} : \ell^2 \to \ell^2$ is locally Lipschitz, since we may use (2.24) once again, to get

$$\begin{aligned}
\|\boldsymbol{T}(\zeta) - \boldsymbol{T}(\xi)\|_{\ell^2}^2 &\leqslant (2\sigma+1)^2 M^2 \sum_{n\in\mathbb{Z}^N} (((|\zeta_n| + |\xi_n|)^{2\sigma})^2 |\zeta_n - \xi_n|^2 \\
&\leqslant (2\sigma+1)^2 M^2 \Big[\sup_{n\in\mathbb{Z}^N} (|\zeta_n| + |\xi_n|)^{2\sigma}\Big]^2 \sum_{n\in\mathbb{Z}^N} |\zeta_n - \xi_n|^2 \\
&\leqslant M_1^2 R^{4\sigma} \|\zeta - \xi\|_{\ell^2}^2,
\end{aligned} \tag{2.44}$$

with $M_1 = 2^{2\sigma} M(2\sigma+1)$. Now, taking the scalar product of (2.43) with χ in ℓ^2 and using (2.44), we have

$$\begin{aligned}
\sum_{\nu=1}^{N} \|\boldsymbol{L}_\nu^+ \chi\|_{\ell^2}^2 + \omega \|\chi\|_{\ell^2}^2 &\leqslant \|\boldsymbol{T}(\zeta) - \boldsymbol{T}(\xi)\|_{\ell^2} \|\chi\|_{\ell^2} \\
&\leqslant M_1 R^{2\sigma} \|\zeta - \xi\|_{\ell^2} \|\chi\|_{\ell^2} \\
&\leqslant \tfrac{1}{2}\omega \|\chi\|_{\ell^2}^2 + \frac{1}{2\omega} M_1^2 R^{4\sigma} \|z - \xi\|_{\ell^2}^2.
\end{aligned} \tag{2.45}$$

From (2.45), we obtain the inequality

$$\|\chi\|_{\ell^2}^2 = \|\mathcal{P}(z) - \mathcal{P}(\xi)\|_{\ell^2}^2 \leqslant \frac{1}{\omega^2} M_1^2 R^{4\sigma} \|z - \xi\|_{\ell^2}^2. \tag{2.46}$$

Since $\mathcal{P}(0) = 0$, from inequality (2.46), we derive that, for $R < E_{\min}$, the map $\mathcal{P} : B_R \to B_R$ is a contraction. Therefore, $\mathcal{P}$ satisfies the assumptions of the Banach fixed-point theorem and has a unique fixed point, the trivial one. Hence, for $R < E_{\min}$ the only breather solution is the trivial one. □

If the energy of the excitation is less than $E_{\min}$, the lattice may not support a standing wave of frequency ω. This time, relation (2.38) could be seen as some kind of dispersion relation of frequency versus energy for the non-existence of breather solutions of the DNLS equation (1.3). Taking into account the dependence $E^*_{\ell^2_\omega}$ and $E_{\min}$ on ω, σ, M, we observe that the inequality $E_{\min} < E^*_{\ell^2_\omega}$ is satisfied if

$$\left(\frac{1}{2^{2\sigma}(\sigma+1)(2\sigma+1)}\right)^{1/\sigma} < \omega. \tag{2.47}$$

For example, in the case $\sigma = 1$ (cubic nonlinearity) we get the lower bound $\omega > 24^{-1} \sim 0.041\,66$ for the frequency of the breather solution, satisfying

$$\|\phi\|_{\ell^2_\omega} > E_{\min}. \tag{2.48}$$

Let us also remark that a similar non-existence result to Theorem 2.8 can be proved in either of the following cases:

(a) that of an infinite lattice with $\gamma = \text{const.}$ [**15**], and

(b) that of a finite lattice (assuming Dirichlet boundary conditions).

Numerical simulations for testing (2.38) or (2.47)–(2.48) could be of interest. Further developments could consider DNLS equations involving a site-dependent coupling strength ϵ_n, or operators which are not necessarily discretizations of the Laplacian (for examples of such operators see [**28**]).

Acknowledgements. I thank Professor J. C. Eilbeck and Professor J. Cuevas for their valuable discussions (especially for resolving the significance of relation (2.47)) and their interest, improving considerably the presentation of the final version of the manuscript, and my colleagues A. N. Yannacopoulos and H. Nistazakis for their suggestions. I also thank the referee for useful comments. This work was partly supported by the research project proposal 'Pythagoras I: Dynamics of Discrete and Continuous Systems and Applications' from the National Technical University of Athens and University of the Aegean.

References

1. A. Ambrosetti and P. H. Rabinowitz, Dual variational methods in critical point theory and applications, *J. Funct. Analysis* **14** (1973), 349–381.
2. S. Aubry, Breathers in nonlinear lattices: existence, linear stability and quantization, *Physica* D **103** (1997), 201–250.
3. O. Bang, J. Rasmussen and P. Christiansen, Subcritical localization in the discrete nonlinear Schrödinger equation with arbitrary nonlinearity, *Nonlinearity* **7** (1994), 205–218.
4. P. W. Bates and A. Chmaj, A discrete convolution model for phase transitions, *Arch. Ration. Mech. Analysis* **150** (1999), 281–305.
5. K. J. Brown and N. M. Stavrakakis, Global bifurcation results for a semilinear elliptic equation on all of $\mathbb{R}^N$, *Duke Math. J.* **85**(1) (1996), 77–94.
6. T. Cazenave, *An introduction to nonlinear Schrödinger equations*, Textos des Métodos Matemáticos, vol. 26 (IMUFRJ, Rio de Janeiro, 1996).
7. S. N. Chow and J. K. Hale, *Methods of bifurcation theory*, Grundlehren der mathematischen Wissenschaften, A Series of Comprehensive Studies in Mathematics, vol. 251 (Springer, 1982).
8. E. B. Davies, *Heat kernels and spectral theory*, Cambridge Tracts in Mathematics, vol. 92 (Cambridge University Press, 1990).
9. J. C. Eilbeck and M. Johansson, The discrete nonlinear Schrödinger equation: 20 years on, in *Localization and energy transfer in nonlinear systems* (ed. L. Vasquez, R. S. MacKay and M. P. Zorzano), pp. 44–67 (World Scientific, Singapore, 2003).
10. J. C. Eilbeck, P. S. Lomdahl and A. C. Scott, Soliton structure in crystalline acetanilide, *Phys. Rev.* B **30** (1984), 4703–4712.
11. S. Flach and C. R. Willis, Discrete breathers, *Phys. Rep.* **295** (1998), 181–264.
12. J. Ginibre and G. Velo, The Cauchy problem in local spaces for the complex Ginzburg–Landau equation, I, Compactness methods, *Physica* D **95** (1996), 191–228.
13. D. Hennig and G. P. Tsironis, Wave transmission in nonlinear lattices, *Phys. Rep.* **307** (1999), 333–432
14. T. Holstein, Studies of polaron motion, *Ann. Phys.* **8** (1959), 325–389.

15. N. Karachalios and A. Yannacopoulos, Global existence and global attractors for the discrete nonlinear Schrödinger equation, *J. Diff. Eqns* **217**(1) (2005), 88–123.
16. P. G. Kevrekidis, K. O. Rasmussen and A. R. Bishop, The discrete nonlinear Schrödinger equation: a survey of recent results, *Int. J. Mod. Phys.* B **15** (2001), 2833–2900.
17. Y. S. Kivshar, M. Haelterman and A. Sheppard, Standing localized modes in nonlinear lattices, *Phys. Rev.* E **50**(4) (1994), 3161–3170.
18. R. S. MacKay and S. Aubry, Proof of existence of breathers for time-reversible or Hamiltonian networks of weakly coupled oscillators, *Nonlinearity* **7** (1994), 1623–1643.
19. M. I. Molina, Nonlinear impurity in a square lattice, *Phys. Rev.* B **60**(4) (1999), 2276–2280.
20. M. I. Molina, Nonlinear impurity in a lattice: dispersion effects, *Phys. Rev.* B **67**(5) (2003), 054202.
21. M. I. Molina and H. Bahlouli, Conductance through a single nonlinear impurity, *Phys. Lett.* A **294**(2) (2002), 87–94.
22. A. Pankov and N. Zakharenko, On some discrete variational problems, Special issue dedicated to Antonio Avantaggiati on the occasion of his 70th birthday, *Acta Appl. Math.* **65** (2001), 295–303.
23. A. Pankov, Gap solitons in periodic discrete nonlinear Schrödinger equation, Preprint arxiv:nlin.PS/0502043 (2005).
24. M. Reed and B. Simon, *Methods of mathematical physics*, vol. I, *Functional analysis* (Academic, 1979).
25. M. Struwe, *Variational methods—applications to nonlinear partial differential equations and Hamiltonian systems*, 2nd edn, A Series of Modern Surveys in Mathematics, vol. 34 (Springer, 1996).
26. M. Weinstein, Excitation thresholds for nonlinear localized modes on lattices, *Nonlinearity* **12** (1999), 673–691.
27. E. Zeidler, *Nonlinear functional analysis and its applications*, vols I, II: *Fixed point theorems, monotone operators* (Springer, 1990).
28. S. Zhou, Attractors for first order dissipative lattice dynamical systems, *Physica* D **178** (2003), 51–61.

https://doi.org/10.1017/S0013091504001415 Published online by Cambridge University Press