Bull. Aust. Math. Soc. **107** (2023), 204–214
doi:10.1017/S0004972722000600

ON LÜROTH EXPANSIONS IN WHICH THE LARGEST DIGIT GROWS WITH SLOWLY INCREASING SPEED

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(Received 23 April 2022; accepted 24 May 2022; first published online 23 June 2022)

Abstract

Let $0 \le \alpha \le \infty$, $0 \le a \le b \le \infty$ and ψ be a positive function defined on $(0, \infty)$. This paper is concerned with the growth of $L_n(x)$, the largest digit of the first n terms in the Lüroth expansion of $x \in (0, 1]$. Under some suitable assumptions on the function ψ, we completely determine the Hausdorff dimensions of the sets

$$E_\psi(\alpha) = \Big\{x \in (0, 1] : \lim_{n\to\infty} \frac{\log L_n(x)}{\log \psi(n)} = \alpha\Big\}$$

and

$$E_\psi(a, b) = \Big\{x \in (0, 1] : \liminf_{n\to\infty} \frac{\log L_n(x)}{\log \psi(n)} = a, \limsup_{n\to\infty} \frac{\log L_n(x)}{\log \psi(n)} = b\Big\}.$$

2020 *Mathematics subject classification*: primary 11K55; secondary 28A80.

Keywords and phrases: Lüroth expansion, largest digit, Hausdorff dimension.

1. Introduction

It is well known that every $x \in (0, 1]$ admits an infinite Lüroth expansion of the form

$$x = \frac{1}{d_1(x)} + \sum_{n\ge2} \frac{1}{d_1(x)(d_1(x) - 1)\cdots d_{n-1}(x)(d_{n-1}(x) - 1)d_n(x)}, \tag{1.1}$$

where $d_n(x) \in \mathbb{N}$ for all $n \ge 1$, which we write as $x = [d_1(x), d_2(x), \ldots]$. Lüroth [12] showed that the Lüroth expansion can be induced by the Lüroth map $T : [0, 1] \to [0, 1]$ defined by

$$T(x) = \begin{cases} 0 & \text{if } x = 0,\\ \Big\lfloor \frac{1}{x} \Big\rfloor \Big(\Big(\Big\lfloor \frac{1}{x} \Big\rfloor + 1\Big)x - 1\Big) & \text{if } x \in (0, 1]. \end{cases}$$

This research was supported by National Natural Science Foundation of China (No. 12101191), Natural Science Research Project of West Anhui University (No. WGKQ2021020) and Provincial Natural Science Research Project of Anhui Colleges (No. KJ2021A0950).

The digits $d_n := d_n(x)$ in (1.1) are defined by

$$d_1(x) = \left\lfloor \frac{1}{x} \right\rfloor + 1 \quad \text{and} \quad d_n(x) = d_1(T^{n-1}(x)) \quad \text{for all } n \in \mathbb{N},$$

where $\lfloor \cdot \rfloor$ denotes the integer part of some real number and T^n stands for the nth iterate of T ($T^0 = \mathrm{Id}_{(0,1]}$).

Clearly, the above algorithm gives $d_n \geq 2$ for each $n \geq 1$. Conversely, it is shown in [6] that any sequence of integers $\{d_n\}_{n\geq 1}$ with $d_n \geq 2$ for each $n \geq 1$ must be the Lüroth expansion of some $x \in (0, 1]$. The Lüroth expansion has been studied extensively in the representation theory of real numbers, probability theory and dynamical systems (see [1, 2, 5, 7] and the monograph of Dajani and Kraaikamp [3]).

Given $x \in (0, 1]$, let $L_n(x) = \max\{d_1(x), d_2(x), \ldots, d_n(x)\}$ be the largest digit among the first n terms of the Lüroth expansion of x. The first metrical result on $L_n(x)$ was given by Galambos [6] in 1976: for Lebesgue almost all $x \in (0, 1]$,

$$\lim_{n\to\infty} \frac{\log L_n(x)}{\log n} = 1. \tag{1.2}$$

That is, $\log L_n(x)$ tends to infinity steadily with the speed $\log n$.

From the point of view of multifractal analysis, Shen *et al.* [14] studied the level sets

$$\left\{x \in (0, 1] : \lim_{n\to\infty} \frac{\log L_n(x)}{\log n} = \gamma\right\}, \quad \gamma \geq 0, \tag{1.3}$$

and showed that they have full Hausdorff dimension. Recently, Lin and Li [11] generalised this result by considering the size of the sets for which the limit in (1.3) may not exist. More precisely, they proved that for $0 \leq \alpha \leq \beta \leq \infty$, the set

$$\left\{x \in (0, 1] : \liminf_{n\to\infty} \frac{\log L_n(x)}{\log n} = \alpha, \limsup_{n\to\infty} \frac{\log L_n(x)}{\log n} = \beta\right\} \tag{1.4}$$

has Hausdorff dimension one.

After (1.3) and (1.4), it is natural to wonder how large the sets are when $\log L_n(x)$ tends to infinity at a different rate. We will investigate the Hausdorff dimension of the sets when $\log L_n(x)$ grows with slowly increasing speed as defined below.

Definition 1.1 [8, 9]. Let $f(x)$ be a function defined on the interval $[c, \infty)$ such that $f(x) > 0$, $\lim_{x\to\infty} f(x) = \infty$ and with continuous derivative $f'(x) > 0$. We say the function $f(x)$ is slowly increasing if $\lim_{x\to\infty} x f'(x)/f(x) = 0$.

Slowly increasing functions were used recently by Jakimczuk [8, 9] as a tool to study the asymptotic properties of Bell numbers. Typical slowly increasing functions are $\log x$, $\log\log x$, $\log^2 x$, $\log x/\log\log x$. The elementary properties of slowly increasing functions will be presented in Section 2.

We complement the limit theorem (1.2) by studying the following two sets:

$$E_\psi(\alpha) := \Big\{x \in (0,1] : \lim_{n\to\infty} \frac{\log L_n(x)}{\log \psi(n)} = \alpha\Big\},$$

$$E_\psi(a,b) := \Big\{x \in (0,1] : \liminf_{n\to\infty} \frac{\log L_n(x)}{\log \psi(n)} = a, \limsup_{n\to\infty} \frac{\log L_n(x)}{\log \psi(n)} = b\Big\},$$

where $0 \le \alpha \le \infty$, $0 \le a \le b \le \infty$ and ψ is a positive function defined on $(0,\infty)$. We will establish the following two main theorems. We use $\dim_H$ to denote the Hausdorff dimension.

THEOREM 1.2. *If the function* $\log\psi$ *is slowly increasing, then* $\dim_H E_\psi(\alpha) = 1$ *for any real number* α *with* $0 \le \alpha \le \infty$.

THEOREM 1.3. *If the function* $\log\psi$ *is slowly increasing, then* $\dim_H E_\psi(a,b) = 1$ *for any real numbers* a, b *with* $0 \le a \le b \le \infty$.

In particular, we can take $\psi(x) = x^\gamma$ $(\gamma > 0)$, $\psi(x) = x^{\log x}$ and $\psi(x) = \log x$ in Theorem 1.3 to give the following result.

COROLLARY 1.4. *If* $0 \le a \le b \le \infty$ *and* $\gamma > 0$, *then*

$$\dim_H E_{\{n^\gamma\}}(a,b) = \dim_H E_{\{n^{\log n}\}}(a,b) = \dim_H E_{\{\log n\}}(a,b) = 1.$$

Notice that if we take $\psi(n) = n$ in Theorems 1.2 and 1.3, then we obtain the special results $\dim_H E_\psi(\alpha) = \dim_H E_\psi(a,b) = 1$ given in [11, 14]. Theorem 1.3 also implies the following result.

COROLLARY 1.5. *If the function* $\log\psi$ *is slowly increasing, the set*

$$\Big\{x \in (0,1] : \liminf_{n\to\infty} \frac{\log L_n(x)}{\log \psi(n)} < \limsup_{n\to\infty} \frac{\log L_n(x)}{\log \psi(n)}\Big\}$$

has full Hausdorff dimension.

For more results concerning the largest digits in Lüroth expansions and continued fraction expansions, see [10, 15–17]. For the definitions and elementary properties of Hausdorff dimension, Falconer's book [4] is recommended.

2. Preliminaries

In this section, we will list some elementary results related to Lüroth expansions and present some notation and basic facts that will be used later.

Let $\{d_n\}_{n\ge1}$ be a sequence of integers not less than 2. We call

$$I_n(d_1,\ldots,d_n) = \{x \in (0,1] : d_k(x) = d_k \text{ for } 1 \le k \le n\}$$

a cylinder of level n, whose endpoints and length denoted by $|I_n(d_1,\ldots,d_n)|$ are determined by the following lemma.

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LEMMA 2.1 [6]. *Let $I_n(d_1, \ldots, d_n)$ be a cylinder of level n. Then the left and right endpoints are*

$$\frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \cdots + \prod_{k=1}^{n-1} \frac{1}{d_k(d_k-1)} \frac{1}{d_n}$$

and

$$\frac{1}{d_1} + \frac{1}{d_1(d_1-1)d_2} + \cdots + \prod_{k=1}^{n-1} \frac{1}{d_k(d_k-1)} \frac{1}{d_n} + \prod_{k=1}^{n} \frac{1}{d_k(d_k-1)}.$$

As a result,

$$|I_n(d_1, \ldots, d_n)| = \prod_{k=1}^{n} \frac{1}{d_k(d_k-1)}.$$

For $m \in \mathbb{N}$ with $m \geq 2$, write $\Sigma_m = \{2, 3, \ldots, m\}$. Let E_m be the set consisting of all points in $(0, 1]$ whose digits are less than m, that is,

$$E_m = \{x \in (0, 1] : d_n(x) \in \Sigma_m \text{ for all } n \geq 1\}.$$

It is known that the set E_m can be regarded as a self-similar set generated by contracting similarities $\{x/a(a-1) + 1/a\}_{a=2}^{m}$. The following lemma is a classic result which gives the dimension of E_m.

LEMMA 2.2 [7, 13]. *For any $m \geq 2$, $\dim_H E_m = s_m$, where s_m is the solution s of the equation*

$$\sum_{2 \leq a \leq m} \left(\frac{1}{a(a-1)}\right)^s = 1.$$

Moreover, $\lim_{m\to\infty} s_m = 1$.

Next, we present a key tool which indicates that the Hausdorff dimensions of some specific sets are stationary to the dimension of E_m under certain Hölder mappings defined below.

Let $\mathbb{J} = \{n_1 < n_2 < \cdots\} \subset \mathbb{N}$ and $f_{\mathbb{J}} : (0, 1] \to (0, 1]$ be a mapping satisfying

$$f_{\mathbb{J}} : x = [d_1(x), d_2(x), \ldots] \mapsto \overline{x} := \overline{[d_1(x), d_2(x), \ldots]} = [d_1(\overline{x}), d_2(\overline{x}), \ldots],$$

where the number $\overline{x}$ is obtained by deleting all $\{d_{n_k}(x)\}_{k\geq 1}$ in the Lüroth expansion of x. For $m \geq 2$ and $\{a_n\}_{n\geq 1}$ a sequence of integers, set

$$F_m(\mathbb{J}, \{a_k\}) := \{x \in (0, 1] : d_{n_k}(x) = a_k, d_n(x) \in \Sigma_m \text{ for } n \neq n_k \text{ for all } k \geq 1\}.$$

LEMMA 2.3. *Fix $m \geq 2$ and a set of positive integers $\mathbb{J} = \{n_1 < n_2 < \cdots\}$. Let $\{a_k\}_{k\geq 1}$ be an increasing positive integer sequence satisfying $a_k \to \infty$ as $k \to \infty$ and*

$$\lim_{k\to\infty} \frac{k \log a_k}{n_k} = 0. \tag{2.1}$$

Then $\dim_H F_m(\mathbb{J}, \{a_k\}) = \dim_H f_{\mathbb{J}}(F_m(\mathbb{J}, \{a_k\})) = \dim_H E_m = s_m$.

https://doi.org/10.1017/S0004972722000600 Published online by Cambridge University Press

PROOF. The main idea of the proof of Lemma 2.3 comes from [16]. Here we will modify the calculations in [14] and give a sketch of the proof of this argument.

To estimate the dimension of $F_m(\mathbb{J}, \{a_k\})$, we shall use the terminology of symbolic space. For each $n \geq 1$, let

$$D_n = \{(\sigma_1, \dots, \sigma_n) \in \mathbb{N}^n : \sigma_{n_k} = a_k \text{ and } \sigma_i \in \Sigma_m, 1 \leq i \neq n_k \leq n\}.$$

For any $n \geq 1$ and $(\sigma_1, \dots, \sigma_n) \in D_n$, we call

$$J_n(\sigma_1, \dots, \sigma_n) = \bigcup_{\sigma_{n+1}} I_{n+1}(\sigma_1, \dots, \sigma_n, \sigma_{n+1})$$

the fundamental interval of level n, where the union is taken over all σ_{n+1} such that $(\sigma_1, \dots, \sigma_n, \sigma_{n+1}) \in D_{n+1}$. Clearly,

$$F_m(\mathbb{J}, \{a_k\}) = \bigcap_{n\geq 1} \bigcup_{(\sigma_1,\dots,\sigma_n)\in D_n} I_n(\sigma_1, \dots, \sigma_n) = \bigcap_{n\geq 1} \bigcup_{(\sigma_1,\dots,\sigma_n)\in D_n} J_n(\sigma_1, \dots, \sigma_n).$$

By the definition of $f_{\mathbb{J}}$ with $\mathbb{J} = \{n_k\}_{k\geq 1}$, we can assume that $n_k \leq n < n_{k+1}$ for some $k \in \mathbb{N}$. Then $\overline{(\sigma_1, \dots, \sigma_n)} := f_{\mathbb{J}}((\sigma_1, \dots, \sigma_n))$ is obtained by deleting the k terms $\{\sigma_{n_i}\}_{i=1}^k$ in $(\sigma_1, \dots, \sigma_n)$. Write

$$\overline{I_n}(\sigma_1, \dots, \sigma_n) := I_{n-k}\overline{(\sigma_1, \dots, \sigma_n)}.$$

Then we have the following claim.

Claim 1. For any $\varepsilon > 0$, there exists $N_0 > 0$ such that for all $n \geq N_0$ and $(\sigma_1, \dots, \sigma_n) \in D_n$, we have

$$|I_n(\sigma_1, \dots, \sigma_n)| \geq |\overline{I_n}(\sigma_1, \dots, \sigma_n)|^{1+\varepsilon}.$$

In fact, (2.1) implies that for any $\varepsilon > 0$, there exists $N_0 > 0$ such that for all $k > N_0$, we have $k \log a_k < \frac{1}{2}\varepsilon \log 2n_k$. We can assume that $n_k \leq n < n_{k+1}$ and obtain

$$|I_n(\sigma_1, \dots, \sigma_n)|^{\varepsilon} \leq \frac{1}{2^{(n-k)\varepsilon}} \leq \frac{1}{2^{n_k\varepsilon}} \leq \frac{1}{a_k^{2k}}. \tag{2.2}$$

Since $\{a_k\}$ is increasing, (2.2) and Lemma 2.1 give

$$\begin{aligned} |I_n(\sigma_1, \cdots, \sigma_n)| &= |\overline{I_n}(\sigma_1, \dots, \sigma_n)| / \sigma_{n_1}(\sigma_{n_1} - 1) \cdots \sigma_{n_k}(\sigma_{n_k} - 1) \\ &\geq |\overline{I_n}(\sigma_1, \dots, \sigma_n)| / a_k^{2k} \\ &\geq |\overline{I_n}(\sigma_1, \dots, \sigma_n)|^{1+\varepsilon}. \end{aligned}$$

Let x and y belong to the set $F_m(\mathbb{J}, \{a_k\})$ with $x \neq y$. It follows that there exists a largest integer n such that x and y are both contained in the same cylinder of level n. The next claim is devoted to estimating the distance between x and y, which is very similar to [14, Lemma 3.3], so we omit the details.

Claim 2. Let n be the largest level of the cylinders which contain both x and y. Then

$$|y - x| \geq \min\left\{\frac{|I_n(\sigma_1, \dots, \sigma_n)|}{m^2 \cdot a_n}, \frac{|I_n(\sigma_1, \dots, \sigma_n)|}{m^3}\right\}.$$

Therefore, when $x, y \in F_m(\mathbb{J}, \{a_k\})$ with

$$|x-y| < \min_{(\sigma_1,\ldots,\sigma_{N_0})\in D_{N_0}} \Big\{\frac{I_{N_0}(\sigma_1,\ldots,\sigma_{N_0})}{m^2 a_{N_0+2}}, \frac{I_{N_0}(\sigma_1,\ldots,\sigma_{N_0})}{m^3}\Big\},$$

we have

$$|f(x)-f(y)| \le \max\{m^2 a_{N_0+2}, m^3\}^{1+\varepsilon} \cdot |x-y|^{1/\varepsilon}.$$

From these two claims and [4, Proposition 2.3], we obtain

$$\dim_H F_m(\mathbb{J}, \{a_k\}) \ge \frac{1}{1+\varepsilon} \dim_H f_{\mathbb{J}}(F_m(\mathbb{J}, \{a_k\})) = \frac{1}{1+\varepsilon} \dim_H E_m$$

and so $\dim_H F_m(\mathbb{J}, \{a_k\}) \ge \dim_H f_{\mathbb{J}}(F_m(\mathbb{J}, \{a_k\}))$ by letting $\varepsilon \to 0$.

To see that $\dim_H F_m(\mathbb{J}, \{a_k\}) \le \dim_H f_{\mathbb{J}}(F_m(\mathbb{J}, \{a_k\}))$, it suffices to show that the mapping

$$f_{\mathbb{J}}^{-1} : f_{\mathbb{J}}(F_m(\mathbb{J}, \{a_k\})) \to F_m(\mathbb{J}, \{a_k\})$$

is 1-Hölder. For any $y_1, y_2 \in f_{\mathbb{J}}(F_m(\mathbb{J}, \{a_k\}))$, let $y_1, y_2 \in I_n(\sigma_1, \ldots, \sigma_n)$ with $\sigma_{n+1}(y_1) \neq \sigma_{n+1}(y_2)$. Let $x_1 = f_{\mathbb{J}}^{-1}(y_1), x_2 = f_{\mathbb{J}}^{-1}(y_2)$. By the definition of $f_{\mathbb{J}}^{-1}$, we know that x_1 and x_2 are obtained by inserting the sequence $\{a_k\}_{k\ge1}$ in the Lüroth expansions of y_1 and y_2 at the positions $\{n_k\}_{k\ge1}$, respectively. Let $M \in \mathbb{N}$ be such that we can insert just M integers $\{a_i\}_{i=1}^{M}$ into the block $(\sigma_1, \ldots, \sigma_n)$. Then x_1 and x_2 have at least $n+M$ common digits in their Lüroth expansions. By Lemma 2.1,

$$\begin{aligned} |x_1 - x_2| &\le |I_{n+M}(\sigma_1, \ldots, \sigma_{n+M})| \\ &\le |I_n(\sigma_1, \ldots, \sigma_n)|/\sigma_{n_1}(\sigma_{n_1}-1)\cdots\sigma_{n_k}(\sigma_{n_k}-1) \\ &\le \tfrac{1}{2}|I_n(\sigma_1, \ldots, \sigma_n)|. \end{aligned}$$

However, similar to the argument in Claim 2, we also have

$$|y_1 - y_2| \ge \min\Big\{\frac{|I_n(\sigma_1, \ldots, \sigma_n)|}{m^2 \cdot a_n}, \frac{|I_n(\sigma_1, \ldots, \sigma_n)|}{m^3}\Big\}.$$

It follows that

$$|f_{\mathbb{J}}^{-1}(y_1) - f_{\mathbb{J}}^{-1}(y_2)| = |x_1 - x_2| \le \tfrac{1}{2}\max\{m^2 a_{N_0+2}, m^3\} \cdot |y_1 - y_2|,$$

showing that $f_{\mathbb{J}}^{-1}$ is 1-Hölder and $\dim_H F_m(\mathbb{J}, \{a_k\}) \le \dim_H f_{\mathbb{J}}(F_m(\mathbb{J}, \{a_k\}))$.

We end this section by presenting the following lemma which exhibits some basic properties of slowly increasing functions.

LEMMA 2.4 [8]. *Let the functions $f(x)$ and $g(x)$ be slowly increasing and γ be a positive constant. Then,*

(1) *the function $f(x^\gamma)$ is slowly increasing;*
(2) *the function $f(x^\gamma g(x))$ is slowly increasing;*
(3) $\lim_{n\to\infty} \log f(x)/\log x = 0$;
(4) $\lim_{n\to\infty} f(x+1)/f(x) = 1$.

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3. Proofs

This section is devoted to the proofs of our main results. To prove Theorem 1.2, we will construct a suitable subset $F_m(\mathbb{J}, \{a_k\})$ of $E_\psi(\alpha)$, so that the result can be established by using Lemma 2.3. As for the proof of Theorem 1.3, since the nonexistence of the limit in $E_\psi(a, b)$ describes the essence of the question compared with the known results, we need to carefully construct a nice Cantor subset in the lower bound estimations for the Hausdorff dimension. Our proof provides a convenient method to estimate the lower bound for the Hausdorff dimension, which is very different from the method used in [11].

PROOF OF THEOREM 1.2. The proof is divided into three cases according as $\alpha = 0$, $0 < \alpha < \infty$ and $\alpha = \infty$.

Case 1: $\alpha = 0$. In this case, it is clear that $E_m \subset E_\psi(0)$. Therefore the result follows directly by Lemma 2.2.

Case 2: $0 < \alpha < \infty$. Let $m \geq 2$ and $\{a_n\}_{n\geq 1}$ be a sequence of integers and recall the set

$$F_m(\mathbb{J}, \{a_k\}) := \{x \in (0, 1] : d_{n_k}(x) = a_k, d_n(x) \in \Sigma_m \text{ for } n \neq n_k \text{ for all } k \geq 1\}$$

defined in Lemma 2.3. Here we take $n_k = k^2$ and $a_k = \lfloor \psi(k^2)^\alpha \rfloor$ for each $k \geq 1$.

On the one hand, for any $x \in F_m(\mathbb{J}, \{a_k\})$, if $k^2 \leq n < (k+1)^2$ for some integer k, then

$$\frac{\log\lfloor \psi(k^2)^\alpha \rfloor}{\log \psi((k+1)^2)} \leq \frac{\log L_n(x)}{\log \psi(n)} \leq \frac{\log\lfloor \psi(k^2)^\alpha \rfloor}{\log \psi(k^2)}.$$

From Lemma 2.4(1) and (4),

$$\lim_{n\to\infty} \frac{\log \psi(k^2)}{\log \psi((k+1)^2)} = 1.$$

Consequently,

$$\lim_{n\to\infty} \frac{\log L_n(x)}{\log \psi(n)} = \alpha,$$

which yields $F_m(\mathbb{J}, \{a_k\}) \subset E_\psi(\alpha)$.

On the other hand, since $\log \psi$ is slowly increasing, Lemma 2.4(3) implies that

$$\lim_{k\to\infty} \frac{\log \log \psi(k)}{\log k} = 0,$$

which ensures that for any ε with $0 < \varepsilon < \frac{1}{2}$ and sufficiently large k,

$$\log \psi(k) < k^\varepsilon. \tag{3.1}$$

This gives

$$\lim_{k\to\infty} \frac{k \log\lfloor \psi(k^2)^\alpha \rfloor}{k^2} \leq \lim_{k\to\infty} \frac{\alpha \cdot k^{1+2\varepsilon}}{k^2} = 0,$$

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that is, (2.1) in Lemma 2.3 holds. From Lemma 2.3,

$$\dim_H E_\psi(\alpha) \geq \dim_H F_m(\mathbb{J}, \{a_k\}) = \dim_H E_m = s_m$$

and we obtain the result in Theorem 1.2 by letting $m \to \infty$.

Case 3: $\alpha = \infty$. In this case, for each $k \geq 1$, we take

$$n_k = \lfloor k^2 \log k \rfloor \quad \text{and} \quad a_k = \lfloor (\psi(k^2 \log k))^{(\log k)^{1/2}} \rfloor$$

in the definition of the set $F_m(\mathbb{J}, \{a_k\})$ in Lemma 2.3.

We show first that $F_m(\mathbb{J}, \{a_k\}) \subset E_\psi(\infty)$. For every $x \in F_m(\mathbb{J}, \{a_k\})$, since the functions $\log \psi(x)$ and $\log x$ are slowly increasing, Lemma 2.4(2) and (4) give

$$\lim_{n\to\infty} \frac{\log(\psi(k^2 \log k))}{\log(\psi((k+1)^2 \log(k+1)))} = 1.$$

So if $\lfloor k^2 \log k \rfloor \leq n < \lfloor (k+1)^2 \log(k+1) \rfloor$ for some integer k, then

$$\lim_{n\to\infty} \frac{\log L_n(x)}{\log \psi(n)} \geq \lim_{k\to\infty} \frac{(\log k)^{1/2} \cdot \log(\lfloor \psi(k^2 \log k) \rfloor)}{\log(\psi(\lfloor (k+1)^2 \log(k+1) \rfloor))} = \lim_{k\to\infty} (\log k)^{1/2} = \infty,$$

which means $F_m(\mathbb{J}, \{a_k\}) \subset E_\psi(\infty)$.

Next, (3.1) holds for any ε with $0 < \varepsilon < \frac{1}{2}$ as in the last case, and we can check that (2.1) still holds here, namely

$$\lim_{k\to\infty} \frac{k \log \lfloor (\psi(k^2 \log k))^{(\log k)^{1/2}} \rfloor}{\lfloor k^2 \log k \rfloor} \leq \lim_{k\to\infty} \frac{\log \psi(k^2 \log k)}{k (\log k)^{1/2}} \leq \lim_{k\to\infty} k^{2\varepsilon - 1} (\log k)^{\varepsilon - 1/2} = 0.$$

Hence, by Lemma 2.3,

$$\dim_H E_\psi(\infty) \geq \dim_H F_m(\mathbb{J}, \{a_k\}) = \dim_H E_m = s_m.$$

Then we finish the proof of Theorem 1.2 by letting $m \to \infty$.

PROOF OF THEOREM 1.3. We give the proof of Theorem 1.3 for the case $0 < a < b < \infty$ in detail. The argument for other cases involves minor modifications. In the following, we will write $\phi := \log \psi$ for simplicity.

Case 1: $0 < a < b < \infty$. Let ϕ be a slowly increasing function. Our strategy is to find a nice Cantor subset of $E_\psi(a, b)$ with full Hausdorff dimension. To this end, we construct another slowly increasing function $\widetilde{\phi}$ satisfying some specific properties with respect to ϕ. Then the proof can be completed by using the result mentioned in Theorem 1.2.

For $0 < a < b < \infty$, define $\widetilde{\phi}(x)$ on $(0, \infty)$ such that $\widetilde{\phi}(x) > 0$ and, for any $n \in \mathbb{N}$,

$$\widetilde{\phi}(n) = \frac{a+b}{2} \phi(n) + \frac{b-a}{2} \phi(n) \sin\left(\frac{a}{b-a} \log \phi(n) \right).$$

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PROPOSITION 3.1. *Let $\phi(n)$ be slowly increasing and define the function $\widetilde{\phi}$ as above. Then $\widetilde{\phi}$ is also slowly increasing and*

$$\liminf_{n\to\infty} \frac{\widetilde{\phi}(n)}{\phi(n)} = a, \quad \limsup_{n\to\infty} \frac{\widetilde{\phi}(n)}{\phi(n)} = b. \tag{3.2}$$

PROOF. First, $0 < a \cdot \phi \le \widetilde{\phi} \le b \cdot \phi$ and $\widetilde{\phi} \to \infty$ as $x \to \infty$. Next, we check that the function $\widetilde{\phi}(x)$ has positive derivative. In fact,

$$\begin{aligned}
\widetilde{\phi}'(x) &= \Big(\frac{a+b}{2}\phi(x) + \frac{b-a}{2}\phi(x)\sin\Big(\frac{a}{b-a}\log\phi(x)\Big)\Big)' \\
&= \frac{a+b}{2}\phi'(x) + \frac{b-a}{2}\phi'(x)\sin\Big(\frac{a}{b-a}\log\phi(x)\Big) \\
&\quad + \frac{b-a}{2}\phi(x)\cos\Big(\frac{a}{b-a}\log\phi(x)\Big)\frac{a}{b-a}\cdot\phi^{-1}(x)\cdot\phi'(x) \\
&\ge \frac{a+b}{2}\phi'(x) - \frac{b-a}{2}\phi'(x) - \frac{a}{2}\phi'(x) = \frac{a}{2}\phi'(x) > 0,
\end{aligned}$$

where the last inequality follows from the fact that ϕ is slowly increasing. The calculation also implies that

$$\left|\frac{x\widetilde{\phi}'(x)}{\widetilde{\phi}(x)}\right| \le \left|\frac{x}{a\phi(x)}\Big(\frac{a+b}{2}\phi'(x) + \frac{b-a}{2}\phi'(x) + \frac{a}{2}\phi'(x)\Big)\right| \to 0 \quad \text{as } x \to \infty.$$

Therefore, $\widetilde{\phi}$ is also a slowly increasing function. By the construction of $\widetilde{\phi}$, (3.2) holds immediately. □

Let $\widetilde{\phi} = \log\widetilde{\psi}$ be the slowly increasing function defined above, where $\widetilde{\psi}$ is a positive function defined on $(0,\infty)$. We replace $\widetilde{\phi}$ with $\phi = \log\psi$ and take $\alpha = 1$ in the set $E_\psi(\alpha)$ in Theorem 1.2. The Hausdorff dimension of the set

$$E_{\widetilde{\psi}}(1) := \Big\{x \in (0,1] : \lim_{n\to\infty} \frac{\log L_n(x)}{\widetilde{\phi}(n)} = 1\Big\}$$

is full. The lower bound of $\dim_H E_\psi(a,b)$ follows directly by Proposition 3.1 and the fact that $E_{\widetilde{\psi}}(1) \subset E_\psi(a,b)$. To see this, note that for any $x \in E_{\widetilde{\psi}}(1)$,

$$\begin{aligned}
\liminf_{n\to\infty} \frac{\log L_n(x)}{\phi(n)} &= \lim_{n\to\infty} \frac{\log L_n(x)}{\widetilde{\phi}(n)} \cdot \liminf_{n\to\infty} \frac{\widetilde{\phi}(n)}{\phi(n)} = a, \\
\limsup_{n\to\infty} \frac{\log L_n(x)}{\phi(n)} &= \lim_{n\to\infty} \frac{\log L_n(x)}{\widetilde{\phi}(n)} \cdot \limsup_{n\to\infty} \frac{\widetilde{\phi}(n)}{\phi(n)} = b,
\end{aligned}$$

which means that $x \in E_\psi(a,b)$.

Case 2: $0 = a < b < \infty$. The proof is similar to the case when $0 < a < b < \infty$. We only need to modify the construction of the function $\widetilde{\phi}$ to make sure that Proposition 3.1

https://doi.org/10.1017/S0004972722000600 Published online by Cambridge University Press

still holds. We define $\widetilde{\phi}(x)$ on $(0, \infty)$ such that $\widetilde{\phi}(x) > 0$ by taking

$$\widetilde{\phi}(x) = \frac{b\phi(x)}{\log\log\phi(x)} + \frac{1}{2}b\phi(x)(\sin(\log\log\phi(x)) + 1).$$

Equation (3.2) holds directly and we can check that $\widetilde{\phi}(x)$ satisfies $\widetilde{\phi}'(x) > 0$ and $|x\widetilde{\phi}'(x)/\widetilde{\phi}(x)| \to 0$ as $x \to \infty$. Thus $\widetilde{\phi}(x)$ is slowly increasing.

For the remaining cases, the discussions run as before, so we only give the constructions of the slowly increasing functions $\widetilde{\phi}(x)$ as follows.

Case 3: $0 < a < b = \infty$. Take

$$\widetilde{\phi}(x) = a\phi(x) + \phi(x)\log\phi(x)\Big(\sin\Big(\frac{a}{2}\log\log\phi(x)\Big) + 1\Big).$$

Case 4: $0 = a < b = \infty$. Take

$$\widetilde{\phi}(x) = \frac{\phi(x)}{\log\log\phi(x)} + \frac{1}{2}\phi(x)\log\phi(x)(\sin(\log\log\log\phi(x)) + 1).$$

Case 5: $0 < a = b < \infty$. Take $\widetilde{\phi}(x) = a\phi(x)$.

Case 6: $a = b = \infty$. Take $\widetilde{\phi}(x) = \phi(x)\log x$.

Case 7: $a = b = 0$. Take $\widetilde{\phi}(x) = \log\phi(x)$.

Acknowledgment

The authors wish to warmly thank the anonymous referee for the helpful suggestions to improve the readability of this paper.

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https://doi.org/10.1017/S0004972722000600 Published online by Cambridge University Press