ARCS OF PARABOLIC ORDER FOUR

N. D. LANE

1. Introduction.

1.1. This paper is concerned with some of the properties of arcs in the real affine plane which are met by every parabola at not more than four points. Many of the properties of arcs of parabolic order four which we consider here are analogous to the corresponding properties of arcs of cyclic order three in the conformal plane which are described in **(1)**. The paper **(2)**, on parabolic differentiation, provides the background for the present discussion.

In Section 2, general tangent, osculating, and superosculating parabolas are introduced. The concept of strong differentiability is introduced in Section 3; cf. Theorem 1. Section 4 deals with arcs of finite parabolic order, and it is proved (Theorem 2) that an end point p of an arc A of finite parabolic order is twice parabolically differentiable. Moreover, we show that if the osculating parabolas of A at p are non-degenerate, then A satisfies the third condition for parabolic differentiability at p. Section 5 is concerned with arcs of parabolic order four. Multiplicities at a point of an arc are defined, and it is observed (Theorem 3) that the parabolic order of an open arc of parabolic order four is not increased when one end point is added and certain multiplicities are introduced. Theorem 4 indicates that the types of parabolically differentiable end points of arcs of parabolic order four are restricted, and Theorem 5 shows that there is a further restriction in the case of parabolically differentiable interior points. We prove that an end point of an arc of parabolic order four is three times strongly parabolically differentiable (Theorem 6) and that its interior points are twice strongly parabolically differentiable (Theorem 7). In Section 6, we use a monotony property of arcs of parabolic order four (Theorem 9) to prove that all but a countable number of points of A_4 are strongly parabolically differentiable (Theorem 10).

1.2. Prerequisites. The definitions and notations which will be used in this paper are the same as in **(2)**. For the convenience of the reader, some of the results in **(2)** which are needed for this paper are summarized below.

The letters $P, Q, \ldots$ denote points and $\mathfrak{T}, \mathfrak{L}, \ldots$ denote lines. π denotes a parabola. The interior [exterior] of a non-degenerate parabola π is denoted by $\pi_*[\pi^*]$.

The family of all the non-degenerate parabolas which touch a line $\mathfrak{T}$ at a

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point P is denoted by τ and the compactified family by $\bar{\tau}$; cf. **(2**, 3.2**)**. If P, Q, R are not collinear and Q and R lie on the same side of $\mathfrak{T}$, the two parabolas of τ through Q and R are denoted by $\pi_i(\tau; Q, \mathrm{R})$, $i = 1, 2$. If π_0 is a fixed parabola of τ, $\phi(\pi_0)$ denotes the one-parametric family consisting of π_0 and those parabolas of τ which meet π_0 exactly once outside P; $\overline{\phi(\pi_0)}$ denotes its compactification; cf. **(2**, 2.16**)**.

An *arc* A in the real affine plane is the continuous image of a real interval. The images of distinct points of the parameter interval are to be considered to be different points of A even though they may coincide in the plane. The letters p, s, q, . . . denote points of arcs. The parameters s and p are supposed to be distinct and s will usually be "sufficiently close" to p. We shall restrict our attention to arcs A with the property that each point p has a neighbourhood N on the parameter interval such that for every $s \in N$, $s \neq p$, the image points s and p in the affine plane are distinct. This condition is automatically satisfied if p has finite linear order. It is also satisfied if A is *differentiable* at p, i.e., if A satisfies the following condition.

(i) CONDITION I. *If the parameter s is sufficiently close to the parameter p, $s \neq p$, the line ps is uniquely determined. It converges if s tends to p;* cf. **(2**, 3.2**)**.

The limit straight line $\mathfrak{T}$ is the ordinary tangent of A at p.

(ii) If A is differentiable, the non-degenerate, non-tangent parabolas through an interior point p of A all intersect A at p or all of them support **(**2, Theorem 2**)**.

(iii) Let $A - p \subset \mathfrak{T}_*$, $s \in A - p$. The two parabolas $\pi_1(\phi; s)$ and $\pi_2(\phi; s)$ of $\phi \subset \tau$ at an end point p which pass through s can be numbered in such a way that

$$\lim_{s \to p} \pi_2(\phi; s)$$

exists and is equal to the limit of the double ray through s with the vertex p **(2**, Lemma 17**)**.

(iv) CONDITION II. *Let A be differentiable at p and let all the points of $A - p$ close to p lie in one of the closed half-planes bounded by $\mathfrak{T}$, say in $\mathfrak{T}_* \cup \mathfrak{T}$. If $Q \in \mathfrak{T}_*$, then the two tangent parabolas of A at p through Q and s converge when s tends to p;* cf. **(2**, 4.1**)**.

The limit parabolas of Condition II are called *osculating parabolas* of A at p through Q. They are denoted by $\pi_i(\sigma; Q)$, $i = 1, 2$. The family of all the osculating parabolas of A at p is denoted by σ and the compactified family by $\bar{\sigma}$.

(v) If Condition II holds for one point $Q \in \mathfrak{T}_*$, then it holds for every such point **(2**, Theorem 3**)**.

Condition II implies the following statements.

(vi) If A is twice parabolically differentiable at p, the set σ of the osculating parabolas of A at p is one of the following three subsets of $\bar{\tau}$:

Type 1. σ is one of the one-parameter families ϕ described above;

Type 2. σ consists of all the double rays of $\bar{\tau}$ with the common vertex p which lie in $\mathfrak{T}_* \cup p$;

Type 3. σ consists of all the pairs of parallel lines of $\bar{\tau}$ which lie in $\mathfrak{T}_* \cup \mathfrak{T}$; cf. (**2**, Theorem 4).

(vii) Let $\delta \subset \tau$ be a diametral pencil of parabolas (cf. **2**, 1.5) and let $\pi(\delta; s)$ denote the unique member of δ through s. If the osculating parabolas of A at p are non-degenerate or double rays [pairs of parallel lines], then as $s \to p$ the parabola $\pi(\delta; s)$ converges to the unique osculating parabola in δ [the single line $\mathfrak{T}$] (**2**, Theorem 5).

(viii) The non-osculating tangent parabolas of A at an interior point all support A there (**2**, Theorem 6).

(ix) Let p be an end point of A. If ϕ is one of the subfamilies of τ defined above and $\phi \neq \sigma$, then the parabolas $\pi_i(\phi; s)$ of ϕ through $s \in A$, $i = 1, 2$, converge to a double ray on $\mathfrak{T}$ with the vertex p as $s \to p$ (**2**, Theorem 7).

(x) Let p be of Type 1. By (iii), the one-sided limits

$$\lim_{s \to p} \pi_2(\sigma; s)$$

are double rays on $\mathfrak{T}$ with the vertex p.

CONDITION III.

$$\pi(p) = \lim_{s \to p} \pi_1(\tau; s)$$

exists; cf. (**2**, 5.1).

$\pi(p)$ is called the *superosculating parabola* of A at p. It is either non-degenerate (Type 1(a)), or a double ray on $\mathfrak{T}$ with the vertex p. If p is an end point of A, p is of Type 1(b) [Type 1(c)] if this double ray is equal [opposite] to

$$\lim_{s \to p} \pi_2(\sigma; s).$$

Condition III implies the following.

(xi) If the end point p of A is of Type 1(b) and π is any osculating parabola of A at p which does not meet A outside p, then $A - p \subset \pi^*$ (**2**, Corollary of Lemma 18).

(xii) If p is an interior point of A, then the osculating parabolas of $\sigma - \pi(p)$ all support A at p or all intersect according as A has or has not a cusp at p (**2**, Theorem 8).

2. Generalized parabolic differentiability.

2.1. General tangent parabolas. Let p, Q, R be non-collinear points. We call π a *general tangent parabola* of A at p if there exists a sequence of quadruples of mutually distinct points u_n, v_n, Q_n, R_n such that u_n and v_n converge on A to p, $Q_n \to Q$, $R_n \to R$, and there is a parabola $\pi(u_n, v_n, Q_n, R_n)$ through these points which converges to π.

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Suppose that p, Q', R' are not collinear, Q' and R' lie on the same side of the tangent of π at p, and $Q_n' \to Q'$, $R_n' \to R'$. Using the same sequence u_n, v_n as above, the quadrangle u_n, v_n, Q_n', R_n' is convex when u_n and v_n are close to p on A. Thus, there are two parabolas $\pi_i(u_n, v_n, Q_n', R_n')$, $i = 1, 2$, through these points. Any limit parabola π_i' of the $\pi_i(u_n, v_n, Q_n', R_n')$ will touch π at p and will pass through Q' and R'. As in **(2**, 3.4**)**, we can verify that $\pi_1' \neq \pi_2'$. Thus π_1' and π_2' are the two parabolas through Q' and R' which touch π at p. Every non-degenerate parabola which touches π at p can be constructed in this way. From these remarks, we readily obtain the following lemma.

LEMMA 1. *If π is a general tangent parabola of A at p, then every non-degenerate member of the two-parameter family of parabolas which touch π at p is also a general tangent parabola of A at p.*

We agree to compactify this family by considering any parabola of its closure to be a general tangent parabola of A at p.

Remark. If A is differentiable at p, every tangent parabola of A at p is a general tangent parabola. The converse need not be true. For example, a differentiable cusp point has general tangent parabolas other than the ordinary tangent parabolas; c.f. 2.7.

2.2. General osculating parabolas. We call π a *general osculating parabola* of A at p if there exists a sequence of quadruples of mutually distinct points u_n, v_n, w_n, R_n, such that u_n, v_n, w_n converge to p on A, $R_n \to R$, $R \neq p$, and there exists a parabola $\pi(u_n, v_n, w_n, R_n)$, through these points which converges to π.

We observe that *every general osculating parabola is a general tangent parabola.*

Let $\pi_1 = \lim \pi_1(u_n, v_n, w_n, R_n)$ be a non-degenerate general osculating parabola of A at p. There is associated with $\pi_1(u_n, v_n, w_n, R_n)$, a second parabola $\pi_2(u_n, v_n, w_n, R_n)$, and any limit parabola π_2 of the $\pi_2(u_n, v_n, w_n, R_n)$ is also a general osculating parabola of A at p. The parabolas π_1 and π_2 intersect at R and p and meet nowhere else. Thus, π_2 is the other parabola through R of the family $\phi(\pi_1)$ determined by π_1 described in 1.2; cf. **(2**, 2.16**)**.

If $Q \in \pi_1$, $Q \neq p$, $Q_n \in \pi_1(u_n, v_n, w_n, R_n)$, $Q_n \to Q$, then

$$\pi_1(u_n, v_n, w_n, Q_n) = \pi_1(u_n, v_n, w_n, R_n)$$

and

$$\lim \pi_1(u_n, v_n, w_n, Q_n) = \lim \pi_1(u_n, v_n, w_n, R_n) = \pi_1.$$

From the above, $\lim \pi_2(u_n, v_n, w_n, Q_n)$ is also a general osculating parabola belonging to $\phi(\pi_1)$. This construction is readily seen to yield any $\pi \in \phi(\pi_1)$.

We now compactify the family of general osculating parabolas of A at p. Then our discussion implies the following lemma.

LEMMA 2. *If π_1 is a non-degenerate general osculating parabola of A at p, then every $\pi \in \phi(\pi_1)$ is also a general osculating parabola of A at p.*

Remark. If A is twice differentiable at p, every osculating parabola of A at p is a general osculating parabola. A twice differentiable cusp point, however, has general osculating parabolas other than the ordinary osculating parabolas.

2.3. General superosculating parabolas. If there exists a sequence of quadruples of mutually distinct points t_n, u_n, v_n, w_n which converge on A to p, such that π is the limit of a parabola $\pi(t_n, u_n, v_n, w_n)$ through these points, we call π a *general superosculating parabola* of A at p.

Every general superosculating parabola is a general osculating parabola.

Let $\pi_1 = \lim \pi_1(t_n, u_n, v_n, w_n)$ be a general non-degenerate superosculating parabola of A at p. There is associated with $\pi_1(t_n, u_n, v_n, w_n)$, a second parabola $\pi_2(t_n, u_n, v_n, w_n)$ (which may be a pair of parallel lines) and any limit parabola π_2 of the $\pi_2(t_n, u_n, v_n, w_n)$ is also a general superosculating parabola of A at p. π_1 and π_2 support at p and meet nowhere else. Let $\mathfrak{T}_1$ be the tangent of π_1 at p and suppose that $\pi_1 \subset \mathfrak{T}_{1*} \cup p$. Then π_2 is either the double line on $\mathfrak{T}_1$, or a double ray on $\mathfrak{T}_1$ through p, or a double ray in $\mathfrak{T}_1^* \cup p$ with the vertex p. By **(2**, Lemma 8**)**, the line uv tends to $\mathfrak{T}_1$, however, and this result can be used to exclude the third case.

Remark. If A satisfies Condition III at p, then $\pi(p)$ is a general superosculating parabola of A at p. A cusp point which satisfies Condition III, however, has general superosculating parabolas other than $\pi(p)$.

If A has a differentiable cusp at p, every tangent parabola of A at p which supports A at p is a general superosculating parabola of A at p.

If π_0 and $\pi_1 \in \phi(\pi_0)$ are general superosculating parabolas of A at p which support A at p, then every π between π_0 and π_1 in ϕ is a general superosculating parabola.

2.4. The following statements are readily verified.

LEMMA 3. *Let p be an interior point of A.*

(i) *Any parabola through p which supports A at p is a general tangent parabola of A at p.*

(ii) *Any general tangent parabola of A at p which intersects A at p is a general osculating parabola.*

(iii) *Any general osculating parabola of A at p which supports A at p is a general superosculating parabola.*

2.5. *Let π be the limit of a sequence of parabolas $\{\pi_n\}$.* We provide the points of $\pi_n \cap A$ with the following multiplicities: A point $s_n \in \pi_n$ is counted twice if π_n is a general tangent parabola at s_n, but not a general osculating parabola; s_n is written down three times if π_n is a general osculating parabola, but not a general superosculating parabola; s_n counts four times if π_n is a general superosculating parabola at s_n.

LEMMA 4. *Suppose that k (not necessarily distinct) points of $\pi_n \cap A$ converge to p as $n \to \infty$. Then π is a general tangent parabola if $k = 2$; π is a general osculating parabola if $k = 3$; π is a general superosculating parabola if $k = 4$.*

2.6. Let p be an end point of A. Suppose that

$$\mathfrak{T}_1 = \lim_{s_n \to p} (ps_n), \qquad \mathfrak{T}_2 = \lim_{r_n \to p} (pr_n)$$

exist. Then $\mathfrak{T}_1$ and $\mathfrak{T}_2$ divide the pencil of lines through p into two open intervals. At least one of them has the property that every line $\mathfrak{T}$ belonging to it meets every neighbourhood of p on A. Thus, there exists a sequence of points $u_n \in \mathfrak{T} \cap A$ converging to p. In particular, $\mathfrak{T}$ will be a general tangent of A at p.

Similarly, to every $\pi \in \tau(\mathfrak{T})$ there exists a sequence of points $v_n \in \pi \cap A$ converging to p. Thus π will be a general superosculating parabola of A at p.

2.7. Convex arcs. Let A be a convex arc with the end point p. Thus A has linear order two and it is well known that A *satisfies Condition* I *at p;* cf. 1.2 (i). Thus A has a well-defined tangent $\mathfrak{T}$ at p.

The line tu converges to $\mathfrak{T}$ as the distinct points t and u converge on A to p. For the reader's convenience, we include a proof. The line tu is denoted by $\mathfrak{L}(t, u)$.

Proof. Let p, t, u, v lie on A in that order. From the convexity of A we may assume that

(1) $$\mathfrak{L}(t, u) \subset [\mathfrak{L}(p, t)_* \cap \mathfrak{L}(t, v)^*] \cup [\mathfrak{L}(p, t)^* \cap \mathfrak{L}(t, v)_*] \cup t.$$

Letting t and u tend to p, we conclude that $\lim \mathfrak{L}(t, u)$ lies in the closure of

$$[\mathfrak{T}_* \cap \mathfrak{L}(p, v)^*] \cup [\mathfrak{T}^* \cap \mathfrak{L}(p, v)_*]$$

for each choice of v on A. Letting v tend to p, we get $\lim \mathfrak{L}(t, u) = \mathfrak{T}$.

Any general tangent parabola of A at p will touch $\mathfrak{T}$ at p. Thus, in this case, *the set of general tangent parabolas of A at p will coincide with $\bar{\tau}$.*

2.8. Notations. We use the notation of the preceding section. Thus A will denote a convex arc.

Suppose that $A - p \subset \mathfrak{T}_*$. Let R be a fixed point in $\mathfrak{T}_*$. If t, u, v are three mutually distinct points sufficiently close to p on A, then the quadrangle t, u, v, R will be strictly convex and there will be two parabolas through t, u, v, R.

If t, u, v lie on A in that order, we denote by $\pi_1(t, u, v, R)$ $[\pi_2(R, t, u, v)]$ the parabola through these four points such that they lie on it in the order t, u, v, R $[R, t, u, v]$.

Suppose that four mutually distinct points t, u, v, w lie on A in that order. Then there are two parabolas through these points. If t, u, v, w are sufficiently close to p, one of them will be non-degenerate and the points t, u, v, w will lie on it in the same order. The other parabola is either a pair of parallel lines or a

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non-degenerate parabola and, in the latter case, the points will lie on it in the order w, t, u, v or u, v, w, t.

We shall use the notation $\pi_1 = \pi_1(t, u, v, w)$ to indicate that the order in which the four points lie on π_1 is the same as their order on A. The other parabola through these points will be denoted by $\pi_2(w, t, u, v)$ or $\pi_2(u, v, w, t)$.

Let $\pi(P; QR)$ denote the (degenerate) parabola consisting of QR and the line through P parallel to QR. Thus, $\pi_2 = \pi_2(w, t, u, v)$ if $w \in \pi(t; uv)^*$ but $\pi_2 = \pi_2(u, v, w, t)$ if $w \in \pi(t; uv)_*$.

3. Strong differentiability.

3.1. We call an arc A *strongly differentiable* at p if the following condition is satisfied.

CONDITION I′. *If two distinct points u and v converge on A to p, the line uv always converges.*

In particular, if we take $u = p$, we see that Condition I′ implies Condition I and the line uv converges to $\mathfrak{T}$.

By 2.7, a convex arc satisfies Condition I′ at an end point.

LEMMA 5. *A convex arc A which satisfies Condition I at an interior point p also satisfies Condition I′ at p.*

Proof. By 2.7,

$$\lim_{t,u \to p} \mathfrak{L}(t, u) = \mathfrak{T}$$

if t and u lie on the same side of p on A.

Let t, p, u, v lie on A in that order. From the convexity of A, we may again assume (1), and the proof follows the lines of 2.7.

Remarks. If A is strongly differentiable at p, the family of general tangent parabolas of A at p coincides with $\bar{\tau}$; cf. 2.7.

If A is differentiable everywhere, then A is strongly differentiable at p if and only if the family of tangent parabolas of A is continuous at p.

3.2. We call A *strongly twice parabolically differentiable* at p if A satisfies the following condition.

CONDITION II′. *There exists a point $R \neq p$ with the following properties. If t, u, v are mutually distinct and lie sufficiently close to p, the parabola $\pi_1(t, u, v, R)$ exists. It converges as t, u, v converge to p.*

We note the following implications of Condition II′:

(i) *A sufficiently small neighbourhood of p on A is convex.*

(ii) *A satisfies Condition I and hence Condition I′ at p.*

(iii) *$A - p$ and R will lie on the same side of the tangent $\mathfrak{T}$ of A at p.*

(iv) *A satisfies Condition II at p.*

(v) *If R' lies sufficiently close to R and the t, u, v are close to p, $\pi_1(t, u, v, R')$ and $\pi_2(R', t, u, v)$ exist. They converge as R' tends to R and t, u, v, to p.*

(vi) $\lim \pi_1(t, u, v, R) = \lim \pi_1(p, u, v, R) = \lim \pi_1(\tau; v, R) = \pi_1(\sigma; R)$, *and similarly*, $\lim \pi_2(R, t, u, v) = \lim \pi_2(R, p, u, v) = \lim \pi_2(\tau; v, R) = \pi_2(\sigma; R)$, *where the $\pi_i(\tau; v, R)$ are the tangent parabolas of A at p through v and R and the $\pi_i(\sigma; R)$ are the osculating parabolas of A at p through R; $i = 1, 2$.*

Let A satisfy Condition II everywhere. If Condition II′ holds at p, then the family of osculating parabolas of A is continuous at p. (The converse of this statement is also true. But the author possesses only a fairly lengthy analytical proof.)

3.3. THEOREM 1. *If Condition II′ is satisfied for one point R, then it is satisfied for every point $Q \in \mathfrak{T}_*$.*

Proof. Let $\pi_1 = \lim \pi_1(t, u, v, R)$ exist. As in 3.2, A satisfies Condition II and $\pi_1 \in \sigma$. Let $Q' \to Q$ and let π be any accumulation parabola of either the sequence $\{\pi_1(t, u, v, Q')\}$ or the sequence $\{\pi_2(Q', t, u, v)\}$. We may assume, for example, that $\pi = \lim \pi_1(t, u, v, Q')$. Thus π and π_1 meet at p with a multiplicity $\geqslant 3$. By the remark near the end of 3.1, $\pi \in \bar{\tau}$.

If p is of Type 1, π cannot be a pair of parallel lines or a double ray, otherwise $\pi_1(t, u, v, R)$ and $\pi_1(t, u, v, Q')$ would have more than four points in common. Hence π is non-degenerate and either $\pi = \pi_1$ or π and π_1 meet at p with the exact multiplicity three. Thus $\pi \in \phi(\pi_1)$.

Similarly, if p is of Type 2, π cannot be non-degenerate or a pair of parallel lines. Hence π is the unique double ray of σ through Q.

If p is of Type 3, π cannot be non-degenerate or a double ray. Hence π is the unique pair of parallel lines of σ through Q.

3.4. We call A *strongly parabolically differentiable* at p if A satisfies the following condition.

CONDITION III′. *Suppose that the points t, u, v, w are mutually distinct and lie on A in that order. If they are sufficiently close to p, the parabola $\pi_1(t, u, v, w)$ exists. It converges as these four points converge to p.*

It can be proved that *Condition* III′ *implies Condition* II′ *and Condition* III; cf. 1.2 (x). We readily prove that

$$\lim \pi_1(t, u, v, w) = \lim \pi_1(p, u, v, w) = \lim \pi_1(\tau; v, w) = \lim \pi_1(\sigma; w) = \pi(p),$$

where $\pi(p)$ is the superosculating parabola of A at p defined in 1.2 (x).

If Condition III holds at each point u of A and Condition III′ holds at p, then $\pi(u)$ tends to $\pi(p)$ as $u \to p$.

We observe that for every arc A and point p which satisfies Condition III′, $\pi_2(t, u, v, w)$ does not converge and each of the alternatives mentioned at the end of 2.3 will actually occur.

4. Arcs of finite parabolic order.

4.1. An arc A is said to be of *finite parabolic order* if every parabola meets A at a finite number of points only. (We do not require this number to be bounded.) In this section, we assume that p is an end point of such an arc. It is well known that A satisfies Condition I at p. Thus the tangent $\mathfrak{T}$ and the family τ of the tangent parabolas of A at p are defined.

4.2. The pencil τ_Q. Let $Q \in \mathfrak{T}_*$, say. Let $\mathfrak{L}$ denote the line pQ and let $\mathfrak{M}$ denote the line through Q parallel to $\mathfrak{T}$.

The parabolas of τ through Q constitute a one-parameter subfamily τ_Q of τ. To each line $\mathfrak{D} \neq \mathfrak{L}$ through p there corresponds exactly one member of τ_Q with the diameter $\mathfrak{D}$. Through each point $R \in \mathfrak{T}_* - \mathfrak{L}$ there pass two parabolas of τ_Q, viz., $\pi_i(\tau; Q, R)$, $i = 1, 2$. We note that the pair $\pi_i(\tau; Q, R)$ depends continuously on R.

If $\mathfrak{M}$ separates [does not separate] R and $\mathfrak{T}$, then the diameters

$$\mathfrak{D}\{\pi_1(\tau; Q, R)\} \text{ and } \mathfrak{D}\{\pi_2(\tau; Q, R)\}$$

through p are not separated [are separated] by $\mathfrak{T}$ and $\mathfrak{L}$.

We verify these statements by choosing an affine co-ordinate system such that Q has the co-ordinates $(0, 1)$, and τ touches the x-axis at the origin. Then, the parabolas of τ_Q have equations of the form $(y - \lambda x)^2 = y$. If R has the co-ordinates (x_0, y_0), then the λ_i which are associated with the $\pi_i(\tau; Q, R)$ must satisfy the equation

$$\lambda^2 x_0^2 - 2\lambda x_0 y_0 + y_0(y_0 - 1) = 0,$$

and λ_1 and λ_2 have opposite signs or the same sign according as $y_0 < 1$ or $y_0 > 1$.

We may assume that $A - p \subset \mathfrak{T}_* \cap \mathfrak{L}^*$. Thus, we can number the $\pi_i(\tau; Q, s)$ such that $\mathfrak{D}\{\pi_1(\tau; Q, s)\} \cap \mathfrak{T}_*$ lies in $\mathfrak{L}_*$ and $\mathfrak{D}\{\pi_2(\tau; Q, s)\} \cap \mathfrak{T}_*$ lies in $\mathfrak{L}^*$. It may be observed that this numbering is consistent with the numbering of the $\pi_i(\sigma; s)$ in 1.2 (iii).

Let $Q \in \pi \in \tau$. If $s \in \pi_*$, then $\mathfrak{D}\{\pi_1(\tau; Q, s)\}$ and $\mathfrak{T}$ separate $\mathfrak{D}\{\pi\}$ and $\mathfrak{L}$.

4.3. *We verify that A satisfies Condition* II *at p.* Let s_{2n} $[s_{2n+1}]$ be a sequence of points converging to p on $A - p$ and suppose that π' $[\pi'']$ is a limit parabola of the sequence of parabolas $\pi_1(\tau; Q, s_{2n})$ $[\pi_1(\tau; Q, s_{2n+1})]$ and that $\pi' \neq \pi''$. We may assume that $A - p$ does not meet π' or π'' and, in particular, it does not meet the arcs of π' or π'' between $\mathfrak{M}$ and $\mathfrak{T}$. Thus $A - p$ lies in $\mathfrak{T}_* \cap \mathfrak{M}^* \cap \mathfrak{L}^*$ between these arcs, say, in $\pi_*' \cap \pi''^*$. Let π be any parabola of τ_Q through $\pi_*' \cap \pi''^* \cap \mathfrak{L}_*$. Since s_{2n} $[s_{2n+1}]$ lies in π^* $[\pi_*]$, the arc $s_{2n}s_{2n+1}$ of A meets π for all n. This contradicts the assumption that A has finite parabolic order.

Similarly, $\pi_2(\tau; Q, s)$ has a unique limit parabola as s tends to p.

4.4. Let p be of Type 1; cf. 1.2 (vi). *We verify that A satisfies Condition* III *at p.*

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Let the $\pi_i(\sigma; s)$ be numbered as in 1.2 (iii). Thus $\pi_2(\sigma; s)$ converges to a double ray on $\mathfrak{T}$ with the vertex p as s tends to p. Suppose that $\pi_1(\sigma; s)$ has two distinct limit parabolas π' and π'', and let $\pi_1(\sigma; s_{2n}) \to \pi'$, $\pi_1(\sigma; s_{2n+1}) \to \pi''$. Let π be any parabola of σ between π' and π''. We may assume that $A - p$ lies in $\mathfrak{D}(\pi)^*$. Then $\mathfrak{D}\{\pi_1(\sigma; s)\} \cap \mathfrak{T}_*$ lies in $\mathfrak{D}(\pi)_*$ [$\mathfrak{D}(\pi)^*$] if and only if s lies in π_* [π^*]. In particular, $\mathfrak{D}\{\pi(\sigma; s_{2n})\}$ and $\mathfrak{D}\{\pi(\sigma; s_{2n+1})\}$ are separated by $\mathfrak{D}(\pi)$ and $\mathfrak{T}$, if s_{2n} and s_{2n+1} are sufficiently close to p. Hence s_{2n} and s_{2n+1} lie on opposite sides of π. Thus the arc $s_{2n}s_{2n+1}$ of A meets π for all n. This is impossible.

The results of Section 4 are summarized in the following.

THEOREM 2. *An end point p of an arc A of finite parabolic order is twice parabolically differentiable. If p is of Type* 1, *then A satisfies Condition* III.

Remark. Finite parabolic order does not imply strong parabolic differentiability. The arc given by

$$y = 0 \qquad \text{when } x = 0$$

and

$$y = 2^{-n}x^2\{2 - \sin^2 \pi[2^{n-1}(2n + 1)x - n]\}$$

when $2^{-n} \leqslant x \leqslant 2^{-n+1}$, $n = 1, 2, \ldots$, is of finite (but not of bounded) parabolic order. No neighbourhood of the origin, however, is convex, and hence the origin does not satisfy Condition II′.

5. Arcs of parabolic order four.

5.1. Multiplicities. Let A_4 be an open arc of parabolic order four. A point of A_4 will converge if its parameter tends to one of the end points of the parameter interval. Thus A_4 has two well-defined end points p and p'.

It is readily verified that A_4 has linear order two. Thus, A_4 is convex and there are always two parabolas through four distinct points of A_4.

Using the methods of (**1**, 3.3), the following theorem can be proved.

THEOREM 3. *The parabolic order of A_4 is not changed by*

(i) *the addition of one end point p to A_4;*

(ii) *the introduction of multiplicities at p, such that p is counted once, twice, three times, four times, respectively, on a non-tangent parabola through p, a non-osculating tangent parabola at p, an osculating parabola $\neq \pi(p)$, and $\pi(p)$;*

(iii) *the introduction of multiplicities at interior points q of A_4, such that q is counted once* [*twice*] *on any parabola which intersects* [*supports*] *A_4 at q.*

Proof. Suppose that a parabola π through p intersects A_4 at q and meets A_4 at three other mutually distinct points r, s, and t. Choose disjoint neighbourhoods N of p and M of q which do not contain r, s, or t. If v converges to p on N, then one of the parabolas through r, s, t, v, say, $\pi_1(r, s, t, v)$, will converge to π; cf. (**2**, Lemma 3). Hence $\pi_1(r, s, t, v)$ will intersect M if v is sufficiently close to p. Thus, we conclude that *if a parabola through p meets A_4 at four points, then all of them are points of support.*

Similarly, by approximating a tangent [an osculating] parabola by a non-tangent parabola through p [a non-osculating tangent parabola] we can verify, in turn, that *if a tangent [an osculating] parabola of A_4 at p meets A_4 at three [two] points, then they are points of support.* We can then prove that $\pi(p)$ *does not intersect* A_4.

If a parabola π supports A_4 at q and also meets $A_4 \cup p$ at r, s, and t, then a suitable parabola near π through r, s, and t will intersect A_4 twice near q. This is excluded by the above discussion and the definition of A_4. Hence, *a parabola through four points of $A_4 \cup p$ does not support A_4 at any of them*, and it follows that *no parabola through p meets $A_4 \cup p$ in five points.*

Similar arguments can be used to show that *a parabola through three points of $A_4 \cup p$ does not support A_4 at two of them.*

Using the same methods, we then verify, in turn, that *a tangent [an osculating] parabola of A_4 at p through two points [a point] of A_4 intersects A_4 at each of them [at that point]*. Thus, *no tangent [osculating] parabola of A_4 at p meets A_4 at more than two points [more than one point]*. Finally, we conclude that $\pi(p)$ *does not meet* A_4.

From now on, we may assume that p belongs to A_4, whenever it is convenient to do so.

Remark. It is well known that there exist open arcs A of linear order two whose linear order is increased when both of their end points are added and each of them is counted twice on its tangent. This occurs when the end points of A are distinct and the tangent of one passes through the other.

Similarly, it can happen that an open arc A_4 no longer retains the parabolic order four when both of the (distinct) end points are added and the multiplicities described in Theorem 3 (ii) are introduced. This will occur if and only if an osculating parabola at one end point of A_4 is also a tangent parabola of A_4 at its other end point.

5.2. The following lemmas can be verified.

LEMMA 6. *No general superosculating [osculating; tangent] parabola at an interior point q of A_4 intersects A_4 at one [two; three] other point(s).*

Let q be an interior point of A_4. We say that π *meets A_4 exactly two [three] times at q* if π is a general tangent parabola [a general osculating parabola] at q but not a general osculating parabola [not a general superosculating parabola].

LEMMA 7. *A general superosculating parabola at an interior point q of A_4 supports A_4 at q.*

A general tangent [osculating] parabola π at q will support [intersect] A_4 at q in either of the following cases:

(i) *π intersects A_4 at two distinct points [a point] different from q;*

(ii) $$\pi = \lim_{t,u\to q} \pi(t, u, d, e) \quad [\pi = \lim_{t,u,v\to q} \pi(t, u, v, e)],$$

where d and e are fixed points of A_4.

Thus, π *meets* A_4 *exactly twice [three times] at q if* (i) *or* (ii) *holds.*

5.3. By Theorem 2, A_4 satisfies Condition II at p. If p is of Type 1, A_4 also satisfies Condition III at p.

THEOREM 4. *An end point p of* A_4 *is of Type* 1(*a*), *Type* 1(*c*), *or Type* 3; cf. 1.2 (vi), (x).

Proof. (i) Suppose that p is an end point of Type 2 of the open arc A_4. Thus $\pi(\sigma; r)$ is the double ray with the vertex p through the point r of A_4. By Theorem 3, A_4 intersects $\pi(\sigma; r)$ at r. Hence, there are tangent parabolas of A_4 at p close to $\pi(\sigma; r)$ which meet A_4 near p and intersect A_4 twice near r. This is excluded by Theorem 3.

(ii) Suppose that p is of Type 1(*b*). Thus, the osculating parabolas of A_4 at p are non-degenerate and $\pi(p)$ is the double ray on $\mathfrak{T}$ with the vertex p which coincides with

$$\lim_{s\to p} \pi_2(\sigma; s).$$

We number the $\pi_i(\sigma; s)$, $i = 1, 2$, as in 1.2 (iii).

Now $\mathfrak{D}\{\pi_2(\sigma; s)\} \cap \mathfrak{T}_*$ will meet A_4 at a point q which lies between s and p on A_4. Thus q lies in $\pi_2(\sigma; s)_*$. By 1.2 (xi), a small neighbourhood of p on A_4 lies in $\pi_2(\sigma; s)^*$. Hence $\pi_2(\sigma; s)$ also meets A_4 between q and p. This is excluded by Theorem 3.

We observe that the three types exist. For example, an arc of an ellipse [a hyperbola] has parabolic order four; an end point p is of the Type 1(*a*) and $A_4 \subset \pi(p)_*$ [$A_4 \subset \pi(p)^*$].

The arc

$$x = s^m, \qquad y = s^{2m}(1 + s^n), \qquad 0 \leqslant s \leqslant 1, m > n,$$

is of parabolic order four, and the origin, given by $s = 0$, is of Type 1(*c*).

The arc

$$x = s, \qquad y = s^3, \qquad 0 \leqslant s \leqslant 1,$$

has parabolic order four and the origin is of Type 3; cf. (**2**, 5.4).

Remarks. The arc

$$x = s^2, \qquad y = s^3, \qquad s \geqslant 0,$$

is of parabolic order five and the origin is of Type 2.

The arc

$$x = s^2, \qquad y = s^4 - s^5, \qquad 0 \leqslant s \leqslant 1/10,$$

is of parabolic order five and the origin is of Type 1(*b*).

5.4. THEOREM 5. *An interior point p of A_4 is of Type* 1(*a*) *with respect to each of the subarcs into which A_4 is decomposed by p.*

Proof. Let $A_4 = B_4 \cup p \cup B_4'$ and let σ denote the family of osculating parabolas of B_4 at p.

Suppose that p is of Type 1(*c*) [Type 3] with respect to B_4. If $s \in B_4$ is sufficiently close to p, then $\pi_1(\sigma; s)$ [$\pi(\sigma; s)$] will intersect B_4'. This is excluded by Lemma 6.

Remarks. If an arc A_5 of parabolic order five is parabolically differentiable at an interior point p, then p is not of Type 1(*c*) or Type 3.

The arc

$$x = s^3, \qquad y = s^6 + s^8, \qquad -\infty < s < \infty,$$

is of parabolic order six; the origin, given by $s = 0$, is of Type 1(*c*).

The arc

$$x = s, \qquad y = s^4, \qquad -\infty < s < \infty,$$

is of parabolic order six and the origin is of Type 3.

5.5. LEMMA 8. *Every point of A_4 satisfies Condition* I′.

Proof. An interior point $p \in A_4$ decomposes a small neighbourhood M of p on A_4 into two subarcs N and N'. Thus $M = N \cup p \cup N'$. *By* 2.7, N and N' *are strongly differentiable at p.*

Suppose that the tangents $\mathfrak{T}$ and $\mathfrak{T}'$ of N and N' respectively at p are distinct. Because of the convexity of A_4, we may assume that $N \cup N' \subset \mathfrak{T}_* \cap \mathfrak{T}'_*$.

By Theorem 5, the family σ of the osculating parabolas of N at p is of Type 1. If $s \in N$, $\pi_2(\sigma; s)$ is a non-degenerate parabola which intersects $\mathfrak{T}'$ at p and is close to the double ray on $\mathfrak{T}$ with the vertex p in $\mathfrak{T}'_* \cup p$ Thus, $\pi_2(\sigma; s)$ intersects N at s, supports A_4 at p, and hence intersects N' if s is sufficiently close to p. This is excluded by Lemma 6.

Hence $\mathfrak{T}$ and $\mathfrak{T}'$ must coincide, and Lemma 5 then implies that A_4 satisfies Condition I′.

5.6. LEMMA 9. *An end point of an arc A_4 of parabolic order four satisfies Condition II′.*

Proof. Let p, q, r, s, t, e be mutually distinct points on A_4 in that order. By Theorem 3, a parabola through four points of A_4 intersects A_4 at each of these points $\neq p$. We consider first the case in which p is of Type 1.

By Theorem 3, the arc $A_4 - p$ lies in $\pi(p)_*$ or $\pi(p)^*$. In the former case, it also follows from Theorem 3 that a small open arc of A_4 with the end point p lies in

$$\pi_1(\sigma; e)^* \cap \pi_1(\tau; t, e)_* \cap \pi_1(p, s, t, e)^*.$$

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Thus, in this case,

$$\text{(2)} \qquad \pi_1(q, r, s, e) \text{ passes through the region of } \pi_1(p, q, s, e)^* \cap \pi_1(q, s, t, e)_* \text{ which contains } t \text{ on its boundary.}$$

Letting first q, then r and s together, and finally t, tend to p on A_4, we get, from (2)

$$\text{(3)} \qquad \lim_{r,s\to p} \pi_1(p, r, s, e) = \pi_1(\sigma; e).$$

From (2) and (3), we get, by first letting q, r, s tend to p together, and then letting $t \to p$,

$$\text{(4)} \qquad \lim_{q,r,s\to p} \pi_1(q, r, s, e) = \pi_1(\sigma; e).$$

A similar argument holds in the case when A_4 lies in $\pi(p)^*$. Thus A_4 satisfies Condition II$'$ at p. In particular,

$$\lim_{r,s\to p} \pi_2(e, p, r, s) = \lim_{q,r,s\to p} \pi_2(e, q, r, s) = \pi_2(\sigma; e).$$

The case where p is of Type 3 may be dealt with using the same method, and we have

$$\lim_{q,r,s\to p} \pi_1(q, r, s, e) = \lim_{q,r,s\to p} \pi_2(e, q, r, s) = \lim_{r,s\to p} \pi_1(p, r, s, e)$$
$$= \lim_{r,s\to p} \pi_2(e, p, r, s) = \pi(\sigma; e),$$

where, in this case, $\pi(\sigma; e)$ is a pair of parallel lines.

5.7. Lemma 10. *An end point p of Type 1 of an arc A_4 of parabolic order four satisfies Condition* III$'$.

Proof. Let p, q, r, s, t, u be mutually distinct points on A_4 in that order.

Let p be of Type 1(a) and assume, for example, that $A_4 \subset \pi(p)_*$; cf. 5.3, Example 1. Thus, the open arc of $\pi_1(q, r, s, t)$ bounded by r and t passes through the region of

$$\pi_1(p, q, r, t)^* \cap \pi_1(q, r, t, u)_*$$

which contains s. Letting q, then r, then s and t together, and finally u, tend to p, we get

$$\text{(5)} \qquad \lim_{s,t\to p} \pi_1(\tau; s, t) = \pi(p).$$

Now, $\pi_1(q, r, s, t)$ passes through the region of

$$\pi_1(p, q, s, t)_* \cap \pi_1(q, s, t, u)^*$$

which contains r. Let q tend to p. Then let r, s, and t tend to p together (using (3) and (5)). Finally, let u tend to p. This yields

$$\text{(6)} \qquad \lim_{r,s,t\to p} \pi_1(p, r, s, t) = \pi(p).$$

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Finally, the arc of $\pi_1(q, r, s, t)$ between s and t lies in the region of

$$\pi_1(p, r, s, t)^* \cap \pi_1(r, s, t, u)_*$$

which has s and t on its boundary. Let q, r, s, t tend to p together (using (4) and (6)). Then let $u \to p$. This yields

$$\lim_{q,r,s,t \to p} \pi_1(q, r, s, t) = \pi(p). \tag{7}$$

A similar proof gives (7) in the cases where p is of Type 1(a) with $A_4 \subset \pi(p)^*$, or p is of Type 1(c).

Lemmas 9 and 10 yield the following theorem.

THEOREM 6. *An end point p of an arc A_4 of parabolic order four is twice strongly parabolically differentiable. If p is of Type* 1, *then A_4 is three times strongly parabolically differentiable at p.*

Remark. If $p \in A_4$ is of Type 3, then $\pi_1(q, r, s, t)$ does not converge. In particular, $\pi(\sigma; t)$ tends to the double line on $\mathfrak{T}$, while the parabola of τ which also touches A_4 at s tends to a double ray on $\mathfrak{T}$ with the vertex p as s tends to p. However, every accumulation parabola of the $\pi_1(q, r, s, t)$ is either a double ray on $\mathfrak{T}$ or the double line on $\mathfrak{T}$.

5.8. The Corollary of this section will be used in the proof of Theorem 7. *Let A be twice parabolically differentiable at its end point p.*

LEMMA 11. *Let $\pi \in \tau - \sigma$, $\pi' \in \phi(\pi)$. Let $s \in A - p$ lie sufficiently close to p. Then s does not lie between π and π'.*

Proof. Suppose that there exists a sequence $s \to p$, $s \in A - p$, $s \in \pi^* \cap \pi'_*$. Then any accumulation parabola of the $\pi(\phi; s)$ lies between π and π' in $\phi(\pi)$ and hence is non-degenerate. This is excluded by 1.2 (ix).

COROLLARY. *Let A satisfy Condition* I *at the interior point p and let A be one-sidedly twice parabolically differentiable at p with families of osculating parabolas σ and σ'. Let $\pi \in \tau$, $\pi \notin \sigma \cup \sigma'$. If π intersects* [*supports*] *A at p, then every member of $\phi(\pi)$ intersects* [*supports*] *A at p.*

5.9. THEOREM 7. *An interior point p of A_4 satisfies Condition* II$'$.

Proof. Let e and p be distinct points on A_4. Let N and N' be one-sided neighbourhoods of p such that $e \notin N \cup p \cup N'$.

Let π_1 and π_2 be two general osculating parabolas of A_4 at p, which are limits of parabolas through e. Suppose that $\pi_2 \notin \phi(\pi_1)$; thus π_1 and π_2 support at p. By Lemma 7, π_1 and π_2 both intersect A_4 at p. Hence there exist tangent parabolas of A_4 at p other than π_1 and π_2 which support π_1 and π_2 at p and intersect A_4 at p. Let π_3 be one of them.

By Lemma 9, the arcs $N \cup p$ and $N' \cup p$ satisfy Condition II′ at p and, in particular, Condition II. By Theorem 5, p is of Type 1(a) with respect to both N and N'. Let $\pi_i(\sigma; e)$ and $\pi_i(\sigma'; e)$, $i = 1, 2$, denote the pairs of osculating parabolas of N and N' respectively at p. At least one of π_1, π_2, π_3, say π, does not belong to $\sigma \cup \sigma'$. We may assume that $N \subset \pi^*$, $N' \subset \pi_*$. Let $\phi = \phi(\pi)$. By 1.2 (ix), each $\pi_i(\phi; s)$ is close to a double ray on $\mathfrak{T}$ with the vertex p if s is sufficiently close to p. Hence the end points of $N \cup p \cup N'$ lie in $\pi_i(\phi; s)^*$.

Let $s \in N$. Since $N \subset \pi^*$, Lemma 11 implies that a small open subarc of N with the end point p lies in $\pi_i(\phi; s)^*$. By the Corollary of Lemma 11, $\pi_i(\phi; s)$ intersects A_4 at p. Hence $\pi_i(\phi; s)$ also intersects N'. By Lemma 3, $\pi_i(\phi; s)$ is a general osculating parabola at p and also intersects N and N'. Since this is excluded by Lemma 6, we conclude that any two general osculating parabolas of A_4 at p through e belong to the same family ϕ. Thus A_4 satisfies Condition II′ at p.

5.10. The following result is analogous to Lemma 5.

THEOREM 8. *An arc of parabolic order four which satisfies Condition* III *at an interior point p also satisfies Condition* III′ *at p.*

The proof of Theorem 8 follows the lines of the proof of Lemma 10.

The author has shown in a forthcoming paper that A_4 need not satisfy Condition III at an interior point p.

6. A monotony property of the osculating parabolas of A_4.

6.1. If π is any non-degenerate parabola, $p \in \pi$, then $\mathfrak{N}_p\{\pi\}$ denotes the intersection of the diameter of π through p with π_*. In 6.2, we shall call $\mathfrak{N}_p\{\pi\}$ the *diametral ray* of π at p.

Let $p \in A_4$ be of Type 1. Let B_4 denote the open subarc of A_4 bounded by p and an end point of A_4. Let π be any general superosculating parabola of A_4 at p. Let $\pi(p)$ be the (unique) superosculating parabola of B_4 at p.

If p is an end point of A_4, Lemma 10 implies that $\pi = \pi(p)$.

Let p be an interior point of A_4. Then π and $\pi(p)$ both support A_4 at p; cf. Lemma 7. By Theorem 7, A_4 has a pencil σ of non-degenerate general osculating parabolas at p.

A general superosculating parabola π of A_4 at p is one of the following kinds: π can be non-degenerate, in which case $\pi \in \sigma$, or π can be a double ray on $\mathfrak{T}$ through p, or the double line on $\mathfrak{T}$. Only in the first case is $\pi = \lim \pi_1(t, u, v, w)$; cf. 2.8.

Suppose that the general superosculating parabola π lies in σ. Write $\mathfrak{N}_p\{\pi\} = \mathfrak{N}\{\pi\}$ and $\mathfrak{N}_p\{\pi(p)\} = \mathfrak{N}\{\pi(p)\}$. Thus, $\mathfrak{N}\{\pi\} = \mathfrak{D}\{\pi\} \cap \pi_*$, p is the end point of $\mathfrak{N}\{\pi\}$, and $\mathfrak{N}\{\pi\} \subset \mathfrak{T}_*$. Define $\mathfrak{N}\{\pi\}_* = \mathfrak{D}\{\pi\}_* \cap \mathfrak{T}_*$. We may assume that either $\mathfrak{N}\{\pi\}_* \subset \mathfrak{N}\{\pi(p)\}_*$ or $\mathfrak{N}\{\pi\}^* \subset \mathfrak{N}\{\pi(p)\}^*$.

LEMMA 12. *If*

$$B_4 \subset \pi(p)^* \cap \mathfrak{N}\{\pi(p)\}^* \quad [B_4 \subset \pi(p)_* \cap \mathfrak{N}\{\pi(p)\}^*],$$

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then

$$\mathfrak{R}\{\pi\}_* \subset \mathfrak{R}\{\pi(p)\}_* \quad \text{and} \quad B_4 \subset \pi^*$$
$$[\mathfrak{R}\{\pi\}^* \subset \mathfrak{R}\{\pi(p)\}^* \quad \text{and} \quad B_4 \subset \pi_*].$$

Proof. $B_4 \cap \pi = B_4 \cap \pi(p) = p$. Suppose that

$$\mathfrak{R}\{\pi\}^* \subset \mathfrak{R}\{\pi(p)\}^* \quad [\mathfrak{R}\{\pi\}_* \subset \mathfrak{R}\{\pi(p)\}_*].$$

Then $B_4 \subset \pi(p)^* \cap \pi_*$ $[B_4 \subset \pi(p)_* \cap \pi^*]$; otherwise $\pi_1(\sigma; s)$ could not converge to $\pi(p)$ as s tends to p on B_4. This implies, however, that $\pi(p)$ and π cannot both support A_4 at p.

COROLLARY. *If p is an interior point of A_4, then any non-degenerate general superosculating parabola of A_4 at p lies between the two one-sided superosculating parabolas of A_4 at p in the pencil $\sigma(p)$.*

We also observe that every parabola of σ which lies between the two one-sided superosculating parabolas of A_4 at p is a general superosculating parabola of A_4 at p; cf. Lemma 3.

6.2. THEOREM 9. ***Let** p **and** q be two distinct interior points of A_4. Then the diametral rays inside* [*outside*] *two non-degenerate general superosculating parabolas at p and q do not intersect if $A_4 \subset \pi(p)^*$* [*if $A_4 \subset \pi(p)_*$*].

Proof. Let B_4 be the open subarc of A_4 with the end points p and q and let $\pi(p)$ and $\pi(q)$ denote the superosculating parabolas of B_4 at p and q respectively. Let τ_q and σ_q denote the families of tangent and osculating parabolas of A_4 at q.

Suppose, for example, that $B_4 \subset \pi(p)^*$. We may assume that $B_4 \subset \mathfrak{R}_p\{\pi(p)\}^*$. Thus

$$B_4 \subset \pi(p)^* \cap \pi_1(\sigma; q)_* \cap \pi_1(\tau; \tau_q)^* \cap \pi_1(p; \sigma_q)_* \cap \pi(q)^*.$$

Since $q \in \mathfrak{R}_p\{\pi(p)\}^*$, we obtain

$$\mathfrak{R}_p\{\pi(p)\}^* \supset \mathfrak{R}_p\{\pi(\sigma; q)\}^* \supset \mathfrak{R}_p\{\pi(\tau; \tau_q)\}^*.$$

Put

$$\mathfrak{R}_q\{\pi\}^* = \mathfrak{D}_q\{\pi\}^* \cap \mathfrak{T}_{q*}.$$

Thus

$$\mathfrak{R}_p\{\pi(\tau; \tau_q)\} \,||\, \mathfrak{R}_q\{\pi(\tau; \tau_q)\}$$

and

$$\mathfrak{R}_p\{\pi(\tau; \tau_q)\}^* \supset \mathfrak{R}_q\{\pi(\tau; \tau_q)\}^*.$$

Also

$$\mathfrak{R}_q\{\pi(\tau; \tau_q)\}^* \supset \mathfrak{R}_q\{\pi(p; \sigma_q)\}^* \supset \mathfrak{R}_q\{\pi(q)\}^*.$$

Altogether

$$\mathfrak{R}_q\{\pi(q)\}^* \subset \mathfrak{R}_p\{\pi(p)\}^*.$$

Lemma 12 now yields our theorem.

The case $B_4 \subset \pi(p)_*$ is dealt with similarly.

We note that in both cases the diameter of $\pi(t)$ through a fixed point O rotates monotonically about O as t moves continuously and monotonically along A_4.

Remark. Our proof shows that Theorem 9 also holds if p or q are end points of Type $1(a)$ of A_4. By using Theorem 6, we readily verify that Theorem 9 remains valid if p or q are end points of Type $1(c)$ provided that $\Re\{\pi(p)\}$ is interpreted to be the ray incident with $\pi(p)$ if p, say, is of Type $1(c)$.

6.3. THEOREM 10. *All but a countable number of points of A_4 are strongly parabolically differentiable.*

Proof. Let p and q be the end points of A_4.

Case 1. *Let p and q be of Type* $1(a)$. Suppose that $A_4 \subset \pi(p)_*$. Let $s \in A_4$ be a point which does not satisfy Condition III′; then A does not satisfy Condition III at s; cf. Theorem 8. Let $\pi(s)$ and $\pi'(s)$ be the one-sided superosculating parabolas of A_4 at s. Let $\theta(s)$ be the angle between $\Re_s\{\pi(s)\}$ and $\Re_s\{\pi'(s)\}$. We may assume that $\Re_s\{\pi(s)\}^* \subset \Re_s\{\pi'(s)\}^*$. By Theorem 9, the regions

$$\Re_s\{\pi(s)\}^* \cap \Re_s\{\pi'(s)\}_* \text{ and } \Re_t\{\pi(t)\}^* \cap \Re_t\{\pi'(t)\}_* \text{ are disjoint if } s \neq t.$$

Thus there are not more than 2^n members in the class of points s for which

$$\pi/2^{n-1} \geqslant \theta(s) > \pi/2^n, \qquad n = 1, 2, 3, \ldots .$$

Since every s with $\theta(s) > 0$ is included in exactly one of these classes, there is only a countable set of points with $\theta(s) > 0$.

Case 2. Let p and/or q be of Type $1(c)$ or 3.

Consider two sequences of points $\{p_i\}$ and $\{q_i\}$ converging to p and q respectively such that on the parameter interval

$$p < p_{i+1} < p_i < q_i < q_{i+1} < q, \qquad i = 1, 2, \ldots .$$

By Case 1, each of the *arcs* with the end points

$$p_1, q_1; \quad p_2, p_1; \quad q_1, q_2; \quad p_3, p_2; \quad q_2, q_3; \quad \ldots$$

contains only a countable number of singular points. Hence the union A_4 of these countably many arcs also contains only a countable number of singular points.

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McMaster University