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A note on the lattice properties of the linear maps of finite rank

John W. Chaney

It is shown that if E is a barreled locally convex lattice and F is a quasi-complete and order complete locally convex lattice then $E' \otimes F$ equipped with the cone of positive continuous linear maps of finite rank is a lattice if and only if E' or F has finite dimensional order intervals.

It is known that the space $L^b(E, F)$ of order bounded linear maps from a barreled locally convex lattice E into a quasi-complete and order complete locally convex lattice F is a lattice when it is equipped with the cone of positive linear maps. Since the space of continuous linear maps of finite rank from E into F, the space represented by $E' \otimes F$, is a subspace of $L^b(E, F)$ then it seems natural to determine when $E' \otimes F$ equipped with the cone of positive continuous linear maps of finite rank is a lattice. Our main result will show that $E' \otimes F$ is a lattice if and only if E' or F has finite dimensional order intervals. Note that if a Banach lattice has finite dimensional order intervals then it is finite dimensional. Although it is perhaps expected that $E' \otimes F$ is seldom a lattice, it seems to require some effort to show this even in concrete situations such as $l_2 \otimes l_2$. Our main result provides a complete solution to the general problem.

The *biprojective cone* in $E \otimes F$ is defined by

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$$K_b = \Big\{u = \sum x_i \otimes y_i \in E \otimes F : \sum \langle x_i, x' \rangle \langle y_i, y' \rangle \geq 0 \text{ for all } x' \geq 0 \text{ in } E' \text{ and } y' \geq 0 \text{ in } F' \Big\} .$$

The biprojective cone in $E' \otimes F$ coincides with the cone of positive continuous linear maps of finite rank from E into F. Also, the image of K_b under the canonical map from $C(X) \otimes C(Y)$ into $C(X \times Y)$ is the set of positive functions which lie in the range of this map.

Another tensor product ordering is given by the *projective cone* K_p which is defined by

$$K_p = \Big\{u \in E \otimes F : u = \sum x_i \otimes y_i,\ x_i \geq 0 \text{ in } E,\ y_i \geq 0 \text{ in } F\Big\} .$$

$E \otimes_p F$ will denote $E \otimes F$ equipped with K_p, while $E \otimes_b F$ will denote $E \otimes F$ equipped with K_b. We will show that if E and F are quasi-complete and order complete locally convex lattices then any one of the three conditions:

(1) $E \otimes_p F$ is a lattice,

(2) $E \otimes_b F$ is a lattice,

(3) $K_b = K_p$,

is equivalent to the condition that E or F have finite dimensional order intervals.

We refer the reader to [1] for background material and notation concerning vector lattices. The following construction will appear frequently in this work. If G is a vector lattice and x is a positive element of G then define G_x to be the linear hull of $[-x, x]$. If G is a quasi-complete and order complete locally convex lattice then G_x is lattice isomorphic to a $C(X)$ where X is an extremally disconnected compact Hausdorff space (see pages 16, 114, and 109 in [1]). Observe that if E is barreled then E' is quasi-complete for every γ topology by (6.1) of Chapter IV in [2]. Also note that a Banach lattice with finite dimensional order intervals is finite dimensional.

PROPOSITION 1. *Let X and Y be infinite compact metric spaces. Then:*

(1) *$C(X) \otimes_p C(Y)$ is not a lattice;*

(2) *$C(X) \otimes_b C(Y)$ is not a lattice;*

(3) *if f in $C(X)$ and g in $C(Y)$ have the property that (range f) $\cap$ (range g) is an infinite set then*
$u = f^2 \otimes 1 - 2f \otimes g + 1 \otimes g^2$ *is in* $K_b \backslash K_p$.

Proof. (1) Choose a sequence (x_n) in X of distinct points with limit point x_0 such that $x_n \neq x_0$ for all n. Similarly choose (y_n) and y_0 in Y. Let 1 be the identically 1 function in $C(Y)$, and define three other positive functions f and h in $C(X)$ and g in $C(Y)$ such that

$$f(x_n) = \begin{cases} 1 & \text{if } n = 0 , \\ 0 & \text{if } n = 1 , \\ 2^{n-1}/(2^{n-1}+1) & \text{if } n > 1 , \end{cases}$$

$$g(y_n) = \begin{cases} 0 & \text{if } n = 0 , \\ 2^{1-n} & \text{if } n > 0 , \end{cases}$$

$$h(x_n) = 1 - f(x_n) .$$

Let $\rho : C(X) \otimes C(Y) \to C(Y; C(X))$ be the canonical map. Let $H = \rho(f \otimes g) \vee \rho(h \otimes 1)$, then for $m = 1, 2, \ldots$,

$$H(y_m)(x_n) = \begin{cases} 1 & \text{if } n = 1 ; \\ 1/(2^{n-1}+1) & \text{if } 1 < n \leq m , \\ 2^{n-m}/(2^{n-1}+1) & \text{if } n > m . \end{cases}$$

It will now be shown that $\{H(y_m) : m = 1, 2, \ldots\}$ is linearly independent in $C(X)$. If $H(y_m) = \alpha_1 H(y_1) + \ldots + \alpha_{m-1} H(y_{m-1})$ in $C(X)$ then evaluating this equation at x_1, x_2, and x_i $(i \leq m)$ yields

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$$1 = \alpha_1 + \ldots + \alpha_{m-1} ,$$

$$1/3 = (2\alpha_1 + \alpha_2 + \ldots + \alpha_{m-1})/3 ,$$

$$1/(2^{i-1}+1) = \left[2^{i-1}\alpha_1 + 2^{i-2}\alpha_2 + \ldots + \alpha_i + \alpha_{i+1} + \ldots + \alpha_{m-1}\right]/(2^{i-1}+1) .$$

These equations clearly imply that $\alpha_i = 0$ for $1 \le i \le m-1$, which is impossible. Therefore, the range of H is infinite dimensional in $C(X)$.

Since $C(Y;\, C(X))$ and $C(X\times Y)$ are canonically norm and lattice isomorphic then we may consider the range of ρ to be contained in $C(X\times Y)$ and consider H as an element of $C(X\times Y)$. We have just seen that H is not in the range of ρ .

Suppose u is an element of $C(X) \otimes_p C(Y)$ such that $u \ge f \otimes g$ and $u \ge h \otimes 1$. Write $u - f \otimes g = \sum_{j=1}^{n} p_j \otimes q_j$ and $u - h \otimes 1 = \sum_{k=1}^{m} s_k \otimes t_k$ where p_j and s_k are positive elements of $C(X)$ and q_j and t_k are positive elements of $C(Y)$. Since $H \le \rho(u)$ and $H \ne \rho(u)$ then there exist indices e and r and $(x, y) \in X \times Y$ such that $p_e(x)q_e(y)$ and $s_r(x)t_r(y)$ are not zero. Find two non-zero positive functions p in $C(X)$ and q in $C(Y)$ such that p_e and s_r are greater than p in $C(X)$ and q_e and t_r are greater than q in $C(Y)$. Since $u - p \otimes q \ge f \otimes g$ and $u - p \otimes q \ge h \otimes 1$ then $C(X) \otimes_p C(Y)$ is not a lattice.

(2) Let ρ, f, g, h, 1, H be defined as in part (1). Suppose that $u \ge f \otimes g$ and $u \ge h \otimes 1$ in $C(X) \otimes_b C(Y)$. Since $\rho(u) \ne H$ and $\rho(u) \ge H$ in $C(X\times Y)$ then choose non-zero positive functions p in $C(X)$ and q in $C(Y)$ such that $\rho(u)(x, y) - H(x, y) \ge p(x)q(y)$ on $X \times Y$. Since $u - p \otimes q \ge f \otimes g$ and $u - p \otimes q \ge h \otimes 1$ in $C(X) \otimes_b C(Y)$ then $C(X) \otimes_b C(Y)$ is not a lattice.

(3) Suppose (range f) $\cap$ (range g) is infinite. Choose sequences (x_n) in X and (y_n) in Y of distinct points having distinct limit

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points x_0 and y_0 such that $f(x_n) = g(y_n)$ and $\{f(x_n) : n = 0, 1, \ldots\}$ is a sequence of distinct real numbers. Let $u = f^2 \otimes 1 - 2f \otimes g + 1 \otimes g^2$ in $C(X) \otimes C(Y)$. Since $\rho(u) = (f-g)^2$ in $C(X \times Y)$ then u is in K_b.

However, suppose $\rho(u)(x, y) = \sum_{j=1}^{k} f_j(x)g_j(y)$ where f_j and g_j are positive non-zero in $C(X)$ and $C(Y)$, respectively. We can choose an index r such that $f_r(x_0) > 0$ since $\{g(y_n)\}$ is a sequence of distinct real numbers. Consequently, there exists an integer n_r such that if $n \geq n_r$ then $f_r(x_n) > 0$. Since $0 = (f(x_n)-g(y_n))^2 = \sum_{j=1}^{k} f_j(x_n)g_j(y_n)$ then $f_j(x_n)g_j(y_n) = 0$ for each j because f_j and g_j are non-negative. But $f_r(x_n) > 0$ for $n \geq n_r$ so that $g_r(y_n) = 0$ for $n \geq n_r$. Let $p = \max\{n_r : f_r(x_0) > 0\}$. If $n \geq p$ then

$(f(x_0) - g(y_n))^2 = \sum_{j=1}^{k} f_j(x_0)g_j(y_n) = 0$. This contradicts the fact that $\{g(y_n)\}$ is a sequence of distinct real numbers. Therefore u is not in K_p.

PROPOSITION 2. *If X and Y are infinite extremally disconnected compact Hausdorff spaces then:*

(1) $C(X) \otimes_b C(Y)$ *is not a lattice;*

(2) $C(X) \otimes_p C(Y)$ *is not a lattice;*

(3) $K_p \neq K_b$.

Proof. Let $\{C_i\}$ be an infinite countable collection of distinct clopen subsets of X, where $C_i \neq X$. Let G be the closed linear hull of the characteristic functions χ_{C_i} equipped with the norm of $C(X)$. Let X_1 be the quotient of X obtained by identifying x and y when $f(x) = f(y)$ for all f in G. If A is the quotient map from X into

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X_1 and x and y are elements of X such that $Ax \neq Ay$ then there exists a function $f \in G$ such that $f(x) \neq f(y)$. Therefore, there exists a function $\phi \in LH\left(\chi_{C_i}\right)$ such that $\phi(x) \neq \phi(y)$, and so there exists a clopen set C in the algebra of sets generated by $\left(C_i\right)$ such that $x \in C$ and $y \notin C$. Thus, $A(C)$ and $A(-C)$ are disjoint neighborhoods of $A(x)$ and $A(y)$, respectively. Hence, the quotient topology on X_1 is Hausdorff. G is canonically norm and lattice isomorphic to $C\left(X_1\right)$ by the Stone-Weierstrass Theorem. Since G is separable then X_1 is an infinite compact metrizable space. Similarly, define an infinite compact metrizable space Y_1 as a quotient space of Y.

(1) By Proposition 1, there exists an element u in $C\left(X_1\right) \otimes_b C\left(Y_1\right)$ which has no positive part. Identify u with its image in $C(X) \otimes C(Y)$ and suppose that the positive part of u, denoted by u^+, exists in $C(X) \otimes_b C(Y)$. By the argument given in the last part of the proof of part (2) in Proposition 1 the map I from $C(X) \otimes_b C(Y)$ into $C(X \times Y)$ preserves the supremum of a finite set. Therefore, $I\left(u^+\right)$ is the positive part of $I(u)$ in $C(X \times Y)$. Since $X_1 \times Y_1$ is a quotient of $X \times Y$ then the canonical map J from $C\left(X_1 \times Y_1\right)$ into $C(X \times Y)$ is a norm and lattice isomorphism. Let K be the canonical map from $C\left(X_1\right) \otimes C\left(Y_1\right)$ into $C\left(X_1 \times Y_1\right)$. Let v denote the positive part of $K(u)$ where we consider u as an element of $C\left(X_1\right) \otimes C\left(Y_1\right)$. Since $J(v) = u^+$ then, when we consider v and u^+ as compact linear maps from $C\left(X_1\right)'$ into $C\left(Y_1\right)$ and from $C(X)'$ into $C(Y)$, respectively, the diagram of Figure 1 commutes. The unidentified maps in Figure 1 are canonical maps. Since the canonical map from $C\left(X_1\right)$ into $C(X)$ is a norm isomorphism then the adjoint maps $C(X)'$ onto $C\left(X_1\right)'$. Therefore, since the range of u^+ is finite dimensional then the range of v is finite dimensional. Hence, v is in the

biprojective cone and must be the positive part of u in $C(X_1) \otimes_b C(Y_1)$, a contradiction.

$$\begin{array}{ccc} C(X_1)' & \longleftarrow & C(X)' \\ \downarrow v & & \downarrow u^+ \\ C(Y_1) & \longrightarrow & C(Y) \end{array}$$

Figure 1

(2) By Proposition 1, there exists u in $C(X_1) \otimes_b C(Y_1)$ for which no positive part exists. Identify u with its image in $C(X) \otimes C(Y)$ and suppose u^+ exists in $C(X) \otimes_p C(Y)$. By the argument given in the last half of part 1 in the proof of Proposition 1, u^+ is also the positive part of u in $C(X \times Y)$. Let v be the positive part of u in $C(X_1 \times Y_1)$. By the argument just given in part 1 above, v is an element of the biprojective cone and also the positive part of u in $C(X_1) \otimes_b C(Y_1)$, a contradiction.

(3) Let f and g be functions in $C(X_1)$ and $C(Y_1)$ such that (range f) $\cap$ (range g) is infinite and let $u = f^2 \otimes 1 - 2f \otimes g + 1 \otimes g^2$. By Proposition 1, u is in the biprojective cone and not in the projective cone in $C(X_1) \otimes C(Y_1)$. Suppose $K_b = K_p$ in $C(X) \otimes C(Y)$, then $u = \sum_{i=1}^{n} f_i \otimes g_i$ where the f_i's are positive elements in $C(X)$ and the g_i's are positive elements in $C(Y)$. Since X is extremally disconnected then the linear hull of the characteristic functions of clopen sets is dense in $C(X)$. Therefore, we can find an infinitely countable number of clopen subsets (B_i) of X such that $(C_j) \subseteq (B_i)$ and the closed linear hull H of the characteristic functions of the B_i contains the functions $f_1, \ldots, f_n$. H is canonically norm and lattice isomorphic to $C(X_2)$ where the infinite compact metric space X_2 is a quotient of X and X_1 is a quotient of X_2. Likewise, construct an infinite compact metric space Y_2 such that $C(Y_2)$ contains $g_1, \ldots, g_n$, Y_2 is a

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quotient of Y, and Y_1 is a quotient of Y_2. Since X_1 is a quotient of X_2 then the canonical map I from $C(X_1)$ into $C(X_2)$ preserves the range of an element. Likewise, the canonical map K from $C(Y_1)$ into $C(Y_2)$ preserves range. In particular, (range If) $\cap$ (range Kg) is an infinite set, and so by Proposition 1, $v = (If)^2 \otimes 1 - 2If \otimes Kg + 1 \otimes (Kg)^2$ is not in the projective cone in $C(X_2) \otimes C(Y_2)$. However, $v = \sum_{i=1}^{n} f_i \otimes g_i$ in $C(X_2) \otimes C(Y_2)$, a contradiction.

THEOREM 3. *If E and F are quasi-complete and order complete locally convex lattices then the following conditions are equivalent:*

(1) *$E \otimes_b F$ is a lattice;*

(2) *$E \otimes_p F$ is a lattice;*

(3) *$K_b = K_p$;*

(4) *E or F has finite dimensional order intervals.*

Proof. Let $K_{E'}$ denote the positive cone in E'. The symbol $\leq_p$ will be used to denote the order relation determined by K_p in $E \otimes F$.

a. If x and u are positive in E and y and v are positive in F and $0 \leq_p u \otimes v \leq_p x \otimes y$ then either $u \leq x$ or $v \leq y$.

Suppose there is an x' in $K_{E'}$ such that $\langle x', x-u \rangle < 0$, then for any y' in $K_{F'}$,

$$\begin{aligned} 0 &\leq \langle x' \otimes y', x \otimes y - u \otimes v \rangle \\ &= \tfrac{1}{2}(\langle x', x-u \rangle \langle y', y+v \rangle + \langle x', x+u \rangle \langle y', y-v \rangle) . \end{aligned}$$

Since $\langle y', y-v \rangle \geq -\frac{\langle x', x-u \rangle}{\langle x', x+u \rangle} \langle y', y+v \rangle \geq 0$ then $y \geq v$.

b. $K_p \cap (E \otimes_p F)_{x \otimes y} \subseteq K_p^{xy}$, where K_p^{xy} is the projective cone from $E_x \otimes_p F_y$.

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If $0 \leq_p \sum_{i=1}^{n} u_i \otimes v_i \leq_p x \otimes y$, where $u_i \geq 0$ and $v_i \geq 0$, then $0 \leq_p u_i \otimes v_i \leq_p x \otimes y$. By a, suppose $0 < u_i \leq x$ and choose x' in $K_{E'}$ such that $\langle x, u_i \rangle > 0$. For y' in K_{F} we have

$\langle x', u_i \rangle \langle y', v_i \rangle \leq \langle x', x \rangle \langle y', y \rangle$ and $0 \leq \left\langle y', y - \frac{\langle x', u_j \rangle}{\langle x', x \rangle} v_i \right\rangle$. Thus $v_i \in F_y$ and so $u_i \otimes v_i \in K_p^{xy}$ and $\sum_{i=1}^{n} u_i \otimes v_i$ is in K_p^{xy} , and $K_p \cap (E \otimes_p F)_{x\otimes y} \subseteq K_p^{xy}$.

c. $E_x \otimes_p F_y = (E \otimes_p F)_{x\otimes y}$, and $K_p^{xy} = K_p \cap (E_x \otimes F_y)$.

Clearly $K_p^{xy} \subseteq K_p \cap (E \otimes_p F)_{x\otimes y}$. By b, $K_p^{xy} = K_p \cap (E \otimes_p F)_{x\otimes y}$. If u is in $(E \otimes_p F)_{x\otimes y}$ then $-\alpha(x \otimes y) \leq_p u \leq_p \alpha(x \otimes y)$ for some $\alpha > 0$. Therefore $u + \alpha(x \otimes y)$ is in $K_p \cap (E \otimes_p F)_{x\otimes y} = K_p^{xy}$. Since u is in $E_x \otimes F_y$ then $(E \otimes_p F)_{x\otimes y} \subseteq E_x \otimes_p F_y$. Since $E_x \otimes F_y \subseteq (E \otimes_p F)_{x\otimes y}$ then they are equal.

d. $K_b^{xy} = (E_x \otimes F_y) \cap K_b$, where K_b^{xy} is the biprojective cone in $E_x \otimes_b F_y$.

If u is in $K_b \cap (E_x \otimes F_y)$ then u is in $K_b \cap (E_x \otimes F)$; since u is a positive linear map from F' into E and $u(F') \subseteq E_x$ then u is a positive linear map from F' into E_x . By transposition, we may consider u as a positive map from $(E_x)'$ into F . Since $u((E_x)') \subseteq F_y$ then u is a positive map from $(E_x)'$ into F_y . Therefore, u is in K_b^{xy} and $K_b \cap (E_x \otimes F_y) \subseteq K_b^{xy}$. Since K_b^{xy} is contained in K_b then $K_b^{xy} = K_b \cap (E_x \otimes F_y)$.

e. $K_b = \bigcup\left\{K_b^{xy} : x \geq 0,\ y \geq 0\right\}$.

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This follows directly from d.

(1) implies (4). Suppose $[-x, x]$ and $[-y, y]$ are infinite dimensional order intervals in E and F. By Proposition 2, choose u in $E_x \otimes_b F_y$ such that u^+ does not exist. If $E \otimes_b F$ is a lattice then u^+ exists in $E \otimes_b F$. Find $x_1 \geq x$ and $y_1 \geq y$ such that u^+ is in $E_{x_1} \otimes_b F_{y_1}$. By d, u^+ is the positive part of u in $E_{x_1} \otimes_b F_{y_1}$. Also, by straightforward computation u^+ is the positive part of u when they are considered as compact maps from $(E_{x_1})'$ into F_{y_1}. Since $u((E_{x_1})') \subset F_y$ then for $x' \geq 0$ in $(E_{x_1})'$, $u([0, x'])$ is bounded and finite dimensional in F_y, hence order bounded. Therefore, $\sup u([0, x'])$, which exists in F_{y_1}, must also be an element of F_y. Therefore, $u^+((E_{x_1})') \subset F_y$ and u^+, when considered as an element of $E_{x_1} \otimes F_{y_1}$, is the positive part of u in $E_{x_1} \otimes_b F_y$. Now consider u and u^+ as compact linear maps from $(F_y)'$ into E_{x_1}. By repeating the above procedure it can be shown that u^+ can be considered as an element of $E_x \otimes F_y$ and that u^+ is the positive part of u in $E_x \otimes_b F_y$.

(2) implies (4). Let $[-x, x]$ and $[-y, y]$ be infinite dimensional order intervals in E and F. If $E \otimes_p F$ is a lattice then so is $(E \otimes_p F)_{x \otimes y}$. Since $(E \otimes_p F)_{x \otimes y} = E_x \otimes_p F_y$ then we have a contradiction of Proposition 2.

(3) implies (4). Let. $[-x, x]$ and $[-y, y]$ be infinite dimensional order intervals in E and F. If $K_b = K_p$ then by c and d,

$K_b^{xy} = (E_x \otimes F_y) \cap K_b = (E_x \otimes F_y) \cap K_p = K_p^{xy}$, a contradiction of Proposition 2.

(4) implies (1), (2), (3). Suppose each order interval in E is finite dimensional. Since $E_x \otimes_p F_y$ is lattice isomorphic to a finite

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product $\prod_{i=1}^{n} (F_y)$, $E_x \otimes_p F_y$ is a solid subset of $E \otimes_p F$ (by c), and $E \otimes_p F = \cup\{E_x \otimes_p F_y : x \geq 0, y \geq 0\}$ then $E \otimes_p F$ is a lattice. Since $E_x \otimes_b F_y$ is also lattice isomorphic to the same finite product $\prod_{i=1}^{n} (F_y)$, then $K_b^{xy} = K_p^{xy}$ and, by e, $K_b = \cup K_b^{xy} = \cup K_p^{xy} = K_p$. Since $E \otimes_p F$ is a lattice and $K_b = K_p$ then $E \otimes_b F$ is a lattice.

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Department of Mathematics,
University of Illinois at Urbana-Champaign,
Illinois,
USA.

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