Canad. Math. Bull. Vol. 15 (3), 1972.

DISCONTINUITY CONDITIONS ON TRANSFORMATION GROUPS

BY
D. V. THOMPSON

1. Throughout this paper, (X, T, π) is a topological transformation group [1], $L=\{x \in X : xt=x$ for some $t \in T-\{e\}\}$ and $0=X-\bar{L}$ is nonempty; standard topological concepts are used as defined in [2].

The problem to be considered here has been studied in [3] and [6]. In [3], X is assumed to be a compact metric space, and each $t \in T$ satisfies a convergence condition on certain subsets of X. Under these conditions, Kaul proved that if T is equicontinuous on 0, then the group properties of discontinuity, proper discontinuity, and Sperner's condition (see Definition 1) are equivalent.

This paper obtains Kaul's result, while admitting weaker conditions on X, and a condition on T which is a generalization of equicontinuity (see Definition 2).

2. Definition 1. (1) T is *discontinuous* if, for any $x \in 0$, all the accumulation points of $xT=\{xt : t \in T\}$ lie in $\bar{L}$.

(2) T is *properly discontinuous* if, for any $x \in 0$, there is an open set U in 0 containing x such that $U(T-\{e\}) \cap U=\emptyset$.

(3) T satisfies *Sperner's condition* if, for any compact subset C of 0,

$$\{t \in T : Ct \cap C \neq \emptyset\}$$

is finite.

Definition 2. (1) T is *regular at* x if, for any $S \subset T$, and any open set V containing $\overline{xS}$, there is an open set U containing x such that $US \subset V$.

(2) If $Y \subset X$, then T is *regular on* Y if T is regular at y for each $y \in Y$.

Remark. Kaul has proved in [5] that if X is a metric space, then given any $x \in X$ such that $\overline{xT}$ is compact, T is equicontinuous at x if and only if T is regular at x.

From the lemma to follow, we easily obtain the desired

Theorem. *If X is a locally compact T_2 space, and if T is regular on* 0, *then the conditions of discontinuity, proper discontinuity, and Sperner's condition are equivalent.*

Received by the editors November 13, 1970 and, in revised form, March 22, 1971.

https://doi.org/10.4153/CMB-1972-075-3 Published online by Cambridge University Press

Remark. Note that if X is a compact metric space, then for any $x \in X$, $\overline{xT}$ is compact; thus by the preceding remark, if T is regular on 0, then T is equicontinuous on 0. As X is clearly locally compact and T_2, the result quoted in 1 is a corollary of the theorem to be proved.

3. We prove the required

Lemma. (1) *If* 0 *is* T_1 *and* T *is properly discontinuous then* T *is discontinuous.*

(2) *If* X *is regular and if* T *is discontinuous and regular on* 0, *then* T *is properly discontinuous.*

(3) *If* 0 *is* T_1 *and locally compact, and if* T *satisfies Sperner's condition, then* T *is discontinuous.*

(4) *If* X *is* T_2 *and* 0 *is locally compact, and if* T *is discontinuous and regular on* 0, *then* T *satisfies Sperner's condition.*

Proof. (1) If $x \in 0$, then for any $y \in 0$, by proper discontinuity, there is an open set U in 0 containing y such that $U(T-\{e\}) \cap U=\emptyset$; if there is an $s \in T$ such that $xs \in U$, then $xs(T-\{e\}) \cap U=\emptyset$, which implies that $xT \cap U=\{xs\}$. Because 0 is T_1, then, y is not an accumulation point of xT.

We have shown that if $x \in 0$, then for any $y \in 0$, y is not an accumulation point of xT; therefore, for any $x \in 0$, the accumulation points of xT lie in $\bar{L}$, and T is discontinuous.

(2) If T fails to be properly discontinuous at $x \in 0$, then given any open set U_α in 0 containing x, there is an $x_\alpha \in U_\alpha$ and a $t_\alpha \in T-\{e\}$ with $x_\alpha t_\alpha \in U_\alpha$.

If x is not an accumulation point of xT, then there is an open set V in 0 containing x such that $(V-\{x\}) \cap xT=\emptyset$; setting $F=T-\{e\}$, $xF \subset X-V$, which is closed in X, so $\overline{xF} \subset X-V$. Because X is regular, there are disjoint open sets V_1 and V_2 containing x and $X-V$, respectively. Now T is regular at x, and V_2 is an open set containing $\overline{xF}$, so there is an open set W containing x and lying (w log) in $V_1 \cap 0$ such that $WF \subset V_2$. Because $W=U_\beta$ for some index β, we have $x_\beta t_\beta \in U_\beta$; however, $t_\beta \in F$ and $x_\beta \in W$, so $WF \subset V_2$ implies that $x_\beta t_\beta \in V_2$. As $W \cap V_2=\emptyset$, we have a contradiction; then x is an accumulation point of xT lying in 0.

We have shown that if T fails to be properly discontinuous at $x \in 0$, then T fails to be discontinuous at x; this completes the proof.

(3) If $x \in 0$, then for any $y \in 0$, by local compactness, there is an open set U containing y such that $\bar{U} \cap 0$ is compact; the fact that T satisfies Sperner's condition then implies that $xT \cap \bar{U} \cap 0$ is a finite set. But since 0 is T_1, this implies that y is not an accumulation point of x.

Therefore, as in (1), T is discontinuous.

https://doi.org/10.4153/CMB-1972-075-3 Published online by Cambridge University Press

(4) If Sperner's condition fails, then there is a compact set C in 0 such that $F=\{t \in T: Ct \cap C \neq \emptyset\}$ is infinite. For any $t_\alpha \in F$, there is an $x_\alpha \in C$ such that $x_\alpha t_\alpha \in C$; let $C_0=\{x_\alpha : t_\alpha \in F\}$ and let $C_1=\{x_\alpha t_\alpha : t_\alpha \in F\}$.

If both C_0 and C_1 are finite, then there is an $x_\alpha \in C_0$ and $t_{\alpha_1}, t_{\alpha_2} \in F$ such that $x_\alpha t_{\alpha_1}=x_\alpha t_{\alpha_2}$; this implies that $x_\alpha(t_{\alpha_1} t_{\alpha_2}^{-1})=x_\alpha$, that is, $x_\alpha \in L$, a contradiction. Therefore, at least one of C_0, C_1 is infinite.

We can assume (*w*log) that C_0 is infinite; because $C_0 \subset C$, C_0 admits an accumulation point $x \in C$.

Let $F_0=\{t_\alpha \in F: xt_\alpha \in C\}$. If F_0 is infinite then xF_0 finite implies, as before, that $x \in L$, a contradiction. However, if xF_0 is infinite, then $xF_0 \subset C$ implies that xF_0 admits an accumulation point in $C \subset 0$; but then xT admits an accumulation point in 0, and T fails to be discontinuous at x, again a contradiction.

If F_0 is finite, let $F_1=F-F_0$; note that because 0 is T_1, $\{x_\alpha : t_\alpha \in F_1\}$ admits x as an accumulation point.

Because C is compact and 0 is locally compact, C admits open sets U containing C such that $\bar{U} \cap 0$ is compact.

Assume that, for any such U, $F_1(U)=\{t_\alpha \in F_1: xt_\alpha \in \bar{U}\}$ is finite; let $F_2=F_1-F_1(U)$, and note that because 0 is T_1, $\{x_\alpha : t_\alpha \in F_2\}$ admits x as an accumulation point. Now $xF_2 \subset X-U$, which is closed in X, so $\overline{xF_2} \subset X-U$; further, since X is T_2, $X-C$ is an open set containing $X-U$, (and therefore $\overline{xF_2}$). As T is regular at x, there is an open set V containing x and lying (*w* log) in U such that $VF_2 \subset X-C$. But V, being an open set containing x, contains some x_α with $t_\alpha \in F_2$. We have just shown that $x_\alpha t_\alpha \in X-C$; by the original definition $x_\alpha t_\alpha \in C$, a contradiction.

Therefore, there is an open set U containing C such that $\bar{U} \cap 0$ is compact and $F_1(U)$ is infinite. As before, this implies that either $x \in L$ or T fails to be discontinuous at x, both of which are contradictions.

Therefore, under the given conditions, if Sperner's condition fails, we are led to a contradiction. The proof is complete.

REMARK. The following result, connecting proper discontinuity and Sperner's condition, may be shown in a similar manner:

(5) If 0 is T_2 and locally compact, and if T satisfies Sperner's condition, then T is properly discontinuous.

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UNIVERSITY OF SASKATCHEWAN,
REGINA, SASKATCHEWAN