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ON THE PONTRJAGIN ALGEBRA OF A CERTAIN CLASS OF FLAGS OF FOLIATIONS

BY

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ABSTRACT. Let $(\mathcal{M}, g)$ be a Riemannian manifold and let $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ be mutually orthogonal distributions on $\mathcal{M}$ of dimensions p_1, p_2, p_3 such that $p_1 + p_2 + p_3 = \dim \mathcal{M}$. We assume that $\mathcal{V}_1$ and $\mathcal{V}_1 \oplus \mathcal{V}_2$ are integrable and that all the geodesics of $\mathcal{M}$ with initial tangent vector in $\mathcal{V}_2$ remain tangent to $\mathcal{V}_2$. Then, we prove that $\mathrm{Pont}^k(\mathcal{V}_2 \oplus \mathcal{V}_3) = 0$ for $k > p_2 + 2p_3$, where $\mathrm{Pont}^k(\mathcal{V}_2 \oplus \mathcal{V}_3)$ is the subspace of the Pontrjagin algebra of $\mathcal{V}_2 \oplus \mathcal{V}_3$ generated by forms of degree k.

1. **Introduction**. Let $(\mathcal{M}, g)$ be a Riemannian manifold and $\mathcal{V}$ a distribution on $\mathcal{M}$. If $\mathcal{V}$ is integrable, then Bott's theorem [1] says that $\mathrm{Pont}^k(\mathcal{V}^\perp) = 0$ for $k > 2 \dim (\mathcal{V}^\perp)$, where $\mathrm{Pont}^k(\mathcal{V}^\perp)$ is the subspace of the Pontrjagin algebra of $\mathcal{V}^\perp$ generated by forms of degree k. On the other hand, Pasternack [7] has proved that if $\mathcal{V}$ is integrable and the metric is bundle-like with regard to the corresponding foliation, then $\mathrm{Pont}^k(\mathcal{V}^\perp) = 0$ for $k > \dim (\mathcal{V}^\perp)$. In this paper we prove a result on flags of two foliations that turns out to be a generalization of the foregoing ones. In fact, let $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ be mutually orthogonal distributions of dimensions p_1, p_2, p_3 on $(\mathcal{M}, g)$, such that the tangent bundle $T\mathcal{M} = \mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3$, where we identify each distribution with the vector bundle determined by it. We assume that $\mathcal{V}_1$ and $\mathcal{V}_1 \oplus \mathcal{V}_2$ are integrable and that $\mathcal{V}_2$ is a totally geodesic distribution, in the sense that any geodesic of $\mathcal{M}$ with initial tangent vector in $\mathcal{V}_2$ remains tangent to $\mathcal{V}_2$. Then, $\mathrm{Pont}^i(\mathcal{V}_2)\,\mathrm{Pont}^j(\mathcal{V}_3) = 0$ for $j > 2p_3$ or $i + j > p_2 + 2p_3$. This condition is stronger than the one obtained by L. A. Cordero and X. Masa in [2] for flags of two foliations (subfoliations) without the additional hypothesis on $\mathcal{V}_2$. In [2] an extensive study of the characteristic classes of subfoliations is also carried out.

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2. **The generalized second connection**. In order to find a suitable expression for the Pontrjagin polynomials of the bundle $\mathcal{V}_2 \oplus \mathcal{V}_3$, we define what we call the generalized second connection $\tilde{\nabla}$. Since its construction is valid for an arbitrary number of distributions, we shall consider, within this section, k mutually orthogonal distributions $\mathcal{V}_1, \ldots, \mathcal{V}_k$ on a Riemannian manifold $(\mathcal{M}, g)$ such that $\mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_k = T\mathcal{M}$.

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If ν_i is the projector of $T\mathcal{M}$ onto $\mathcal{V}_i$, $1 \le i \le k$, and ∇ is the Levi-Civita connection, we set

$$\tilde{\nabla}_{A_i} B_i = \nu_i(\nabla_{A_i} B_i) \text{ for } 1 \le i \le k$$

and

$$\tilde{\nabla}_{A_i} B_j = \nu_j [A_i, B_j] \text{ for } 1 \le i, j \le k \text{ and } i \neq j.$$

Throughout the paper the subscripts of the vector fields, if any, indicate the distribution to which they belong.

It is clear that $\tilde{\nabla}$ is a well-defined connection on $T\mathcal{M}$ which satisfies

(i) For all i, $1 \le i \le k$, $\tilde{\nabla} \nu_i = 0$, or, equivalently, the restriction of $\tilde{\nabla}$ to each $\mathcal{V}_i$ is a connection in $\mathcal{V}_i$.

(ii) $A_i(g(B_i, C_i)) = g(\tilde{\nabla}_{A_i} B_i, C_i) + g(B_i, \tilde{\nabla}_{A_i} C_i)$, since the $\mathcal{V}_i$'s are mutually orthogonal and $\nabla g = 0$.

(iii) If T is the torsion tensor of $\tilde{\nabla}$, i.e., $T(M, N) = \tilde{\nabla}_M N - \tilde{\nabla}_N M - [M, N]$ for all M, $N \in \mathfrak{X}(\mathcal{M})$, then $g(T(A_i, M), B_i) = 0$ for all $M \in \mathfrak{X}(\mathcal{M})$ and A_i, $B_i \in \mathcal{V}_i$, $1 \le i \le k$, as a consequence of the symmetry of ∇.

Furthermore, we have

THEOREM 1. *Let $(\mathcal{M}, g)$ be a Riemannian manifold and $\mathcal{V}_1, \ldots, \mathcal{V}_k$ mutually orthogonal distributions on $\mathcal{M}$ such that $\mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_k = T\mathcal{M}$. There exists a unique connection in $T\mathcal{M}$ satisfying* (i), (ii) *and* (iii).

PROOF. Only uniqueness has to be proved. Assume ∇^* is such a connection in $T\mathcal{M}$ and let T^* be its torsion tensor; then, for any A_i, B_i, C_j with $i \neq j$, we have

$$\begin{aligned} 0 = g(T^*(A_i, C_j), B_i) &= g(\nabla^*_{A_i} C_j - \nabla^*_{C_j} A_i - [A_i, C_j], B_i) \\ &= -g(\nabla^*_{C_j} A_i + \nu_i[A_i, C_j], B_i), \end{aligned}$$

and since B_i is an arbitrary vector field in $\mathcal{V}_i$, we get

$$\nabla^*_{C_j} A_i = -\nu_i[A_i, C_j] = \tilde{\nabla}_{C_j} A_i.$$

On the other hand,

$$g(T^*(A_i, B_i), C_i) = 0$$

implies

$$\nabla^*_{A_i} B_i - \nabla^*_{B_i} A_i = \nu_i[A_i, B_i].$$

Now, the Riemannian connection satisfies

$$\begin{aligned} 2g(\nabla_{A_i} B_i, C_i) = A_i(g(B_i, C_i)) &+ B_i(g(A_i, C_i)) - C_i(g(A_i, B_i)) \\ &+ g([C_i, A_i], B_i) + g([C_i, B_i], A_i) + g([A_i, B_i], C_i). \end{aligned}$$

and a substitution in this formula gives $g(\nabla_{A_i} B_i, C_i) = g(\nabla^*_{A_i} B_i, C_i)$, whence

$$\nabla^*_{A_i} B_i = \nu_i(\nabla_{A_i} B_i) = \tilde{\nabla}_{A_i} B_i. \quad //$$

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It is easily seen that the semicanonical connection defined by Vaisman [9] satisfies the conditions of theorem 1, so that both connections are the same.

Let S be the curvature tensor of $\tilde{\nabla}$, i.e.

$$S(L, M, N, O) = g(\tilde{\nabla}_M \tilde{\nabla}_L N - \tilde{\nabla}_L \tilde{\nabla}_M N + \tilde{\nabla}_{[L,M]} N, O)$$

for all $L, M, N, O \in \mathfrak{X}(\mathcal{M})$.

As a consequence of properties (i), (ii) and (iii) of $\tilde{\nabla}$, we have

PROPOSITION 2. *Let $(\mathcal{M}, g)$ be a Riemannian manifold and $\mathcal{V}_1, \ldots, \mathcal{V}_k$ mutually orthogonal distributions on $\mathcal{M}$ such that $\mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_k = T\mathcal{M}$. If S is the curvature tensor of $\tilde{\nabla}$, then, for $i \neq j$:*

(a) $\quad S(M, N, A_i, B_j) = S(M, N, B_j, A_i) = 0,$

(b) $\quad S(A_i, B_i, C_j, D_j) = -g([A_i, v_j[B_i, C_j]], D_j) + g([B_i, v_j[A_i, C_j]], D_j) + g(\nabla_{C_j} v_j[A_i, B_i], D_j) + g([[A_i, B_i], C_j], D_j),$

(c) $\quad S(A_i, B_j, C_j, D_j) - S(A_i, C_j, B_j, D_j) = -\sum_{h \neq j} g([v_h[B_j, C_j], A_i], D_j),$

(d) $\quad S(A_i, B_i, C_i, D_i) + S(A_i, B_i, D_i, C_i) = \sum_{h \neq i} g(v_h[A_i, B_i], \nabla_{C_i} D_i + \nabla_{D_i} C_i),$

(e) $\quad \underset{A_i, B_i, C_i}{\mathfrak{S}} S(A_i, B_i, C_i, D_i) = 0.$ //

COROLLARY 3. *If one of the three following conditions is satisfied for $i \neq j$*

(a) $\mathcal{V}_j$ is integrable,

(b) $\oplus_{h \neq j} \mathcal{V}_h$ is integrable,

(c) For every $h \neq j$, $\mathcal{V}_h \oplus \mathcal{V}_i$ is integrable;

then,

$$S(A_i, B_j, C_j, D_j) = S(A_i, C_j, B_j, D_j). \; //$$

COROLLARY 4. *If $\mathcal{V}_i$ is integrable, then*

(a) $\quad S(A_i, B_i, C_i, D_i) = -S(A_i, B_i, D_i, C_i)$

(b) $\quad S(A_i, B_i, C_i, D_i) = R'(A_i, B_i, C_i, D_i)$

where R' is the curvature tensor of the Levi-Civita connection of the leaves of $\mathcal{V}_i$.

If, further, there exists j, $i \neq j$, such that

(1) $\oplus_{h \neq j} \mathcal{V}_h$ is integrable, or

(2) for every h, $h \neq j$, $\mathcal{V}_i \oplus \mathcal{V}_h$ is integrable,

then,

(c) $\quad S(A_i, B_i, C_j, D_j) = 0.$ //

Henceforth, we will consider an additional condition on one of the distributions, namely,

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DEFINITION [3], [6]. *A distribution $\mathcal{V}$ will be said to be of type D_1 if $(\nabla_A v)A = 0$ for all A in $\mathcal{V}$; where v is the orthogonal projection onto $\mathcal{V}$.*

THEOREM 5. *A p-dimensional distribution $\mathcal{V}$ in an n-dimensional Riemannian manifold $(\mathcal{M}, g)$ is of type D_1 if and only if every geodesic of $\mathcal{M}$ with initial tangent vector in $\mathcal{V}$ remains tangent to $\mathcal{V}$.*

PROOF. Let σ be a curve in $\mathcal{M}$, and let $\alpha = \dot{\sigma}$ be its tangent vector field, which is a curve in $T\mathcal{M}$. Then, σ is a geodesic if and only if α is an integral curve of the spray ξ associated to ∇. Let $\{E_1, \ldots, E_n\}$ be a local frame defined in an open set U of $\mathcal{M}$ such that $E_1, \ldots, E_p \in \mathcal{V}$ and $E_{p+1}, \ldots, E_n \in \mathcal{V}$. This frame defines a trivialization $TU \cong U \times \mathbb{R}^n$. Under this identification the field ξ can be written as

$$\xi = \sum_i p^i E_i - \sum_{i,j,k} p^i p^j \theta^k(\nabla_{E_i} E_j) \frac{\partial}{\partial p^k},$$

where the p^i's $(1 \le i \le n)$ are the standard coordinates on $\mathbb{R}^n$, and $\{\theta^k\}$ is the dual frame of $E_1, \ldots, E_n$. On the other hand, the condition of $\mathcal{V}$ being of type D_1 is clearly equivalent to the fact that, for all $a, b = 1, \ldots, p$ and $u = p + 1, \ldots, n$,

$$(*) \qquad \theta^u(\nabla_{E_a} E_b) + \theta^u(\nabla_{E_b} E_a) = 0.$$

Further, the subbundle of $T\mathcal{M}$, determined by $\mathcal{V}$, which will be eqally denoted by $\mathcal{V}$, is a regular submanifold of $T\mathcal{M}$, and a tangent vector to $T\mathcal{M}$ is tangent to $\mathcal{V}$ if and only if it is a linear combination of $E_1, \ldots, E_p, \partial/\partial p^1, \ldots, \partial/\partial p^p$ (under the above identification). So, (*) is equivalent to the fact that the restriction of ξ to $\mathcal{V}$ be tangent to $\mathcal{V}$; in other words, the fact that the integral curves of ξ with initial values in $\mathcal{V}$ be contained in $\mathcal{V}$, whence the result follows. //

Notice that if $\mathcal{F}$ is a foliation in $(\mathcal{M}, g)$, then $\mathcal{F}$ is of type D_1 if and only if the metric is bundle-like [6], and thus, theorem 5 generalizes Reinhart's result about geodesics in a foliated manifold with bundle-like metric [8].

EXAMPLE. Consider the sphere S^{4n+3} and the Hopf fibration onto the quaternionic n-space $S^{4n+3} \to \mathbb{H}^n$. Let $\mathcal{V}$ be the distribution determined by the vector spaces tangent to the fibres, and let $\mathcal{H} = \mathcal{V}^\perp$. On the other hand, consider $S^r \times \mathbb{R}^t$, where $r + t = 4m + 3$ for some integer m. As an orientable hypersurface in $\mathbb{R}^{4m+4}$, $S^r \times \mathbb{R}^t$ has a global normal vector field N. If J_1, J_2, J_3 are the three canonical almost-complex structures on $\mathbb{R}^{4m+4} = \mathbb{H}^m$, then J_1N, J_2N, J_3N are vector fields tangent to $S^r \times \mathbb{R}^t$ which span a 3-dimensional distribution $\mathcal{V}'$. Then, on $S^{4n+3} \times S^r \times \mathbb{R}^t$, $\mathcal{H} \oplus \mathcal{V}'$ is a non-integrable distribution of type D_1, whose orthogonal complement is not integrable. For more details and further examples, see [5].

PROPOSITION 6. *If $\mathcal{V}_i$ is of type D_1, then*

(a) $S(M, N, A_i, B_i) = -S(M, N, B_i, A_i)$

(b) The curvature tensors S and R of $\tilde{\nabla}$ and ∇, respectively, agree when acting upon arguments of $\mathcal{V}_i$ if and only if $\mathcal{V}_i$ is integrable with totally geodesic leaves.

The proof follows by computation from properties (i), (ii), (iii) and the following.

LEMMA 7. *If* $\mathcal{V}_i$ *is of type* D_1 and $M \in \oplus_{h \neq i} \mathcal{V}_h$, *then*

$$M(g(A_i, B_i)) = g([M, A_i], B_i) + g(A_i, [M, B_i]),$$

or, equivalently,

$$(\mathcal{L}_M g)(A_i, B_i) = 0 \ //$$

where $\mathcal{L}$ *denotes the Lie derivative.* //

COROLLARY 8. *If* $\mathcal{V}_i$ *is of type* D_1 *and if further one of conditions* (a), (b), (c) *of corollary 3 is satisfied, then* $S(A_j, B_i, C_i, D_i) = 0$. //

3. **The obstruction theorem**. Let E be a vector bundle on $\mathcal{M}$, which we can assume as a subbundle of $T\mathcal{M}$, with fibre $\mathbb{R}^r$ and D a connection in E. If we consider the associated principal bundle we have the Weyl homomorphism [4]

$$w: I(\mathrm{Gl}(r, \mathbb{R})) \rightarrow H^*(\mathcal{M}, \mathbb{R})$$

where $I(\mathrm{Gl}(r, \mathbb{R}))$ $(= \sum I^k(\mathrm{Gl}(r, \mathbb{R})))$ is the graded algebra of ad-$\mathrm{Gl}(r, \mathbb{R})$-invariant polynomials on $\mathfrak{gl}(r, \mathbb{R})$, the Lie algebra of $\mathrm{Gl}(r, \mathbb{R})$. The image of this homomorphism is the algebra of characteristic classes of E and will be denoted by Pont (E). For each k, $w(I^k(\mathrm{Gl}(r, \mathbb{R})))$ is the space of characteristic classes of degree $2k$, and will be denoted by $\mathrm{Pont}^{2k}(E)$.

The algebra Pont (E) is spanned by the classes $w(f_0)$, $w(f_1)$, $\ldots$, $w(f_{2m})$ with $r = 2m$ or $2m + 1$, where $f_0, f_1, \ldots, f_r$ are ad-$\mathrm{Gl}(r, \mathbb{R})$-invariant polynomials on $\mathfrak{gl}(r, \mathbb{R})$, defined by

$$\det\left(\lambda I_r - \frac{1}{2\pi} X\right) = \sum_{k=0}^{r} f_k(X) \lambda^{r-k}$$

for all $X \in \mathfrak{gl}(r, \mathbb{R})$, where I_r is the identity matrix of dimension $r \times r$. For each $k \in \{0, \ldots, m\}$ the class $w(f_{2k}) \in H^{4k}(\mathcal{M}, \mathbb{R})$ is the k-th Pontrjagin class $p_k(E)$ of the bundle E.

From the relations between the curvature tensor S and the curvature form Ω of D, it follows that if f is an invariant homogeneous polynomial of degree k, then there exists a representative α in $w(f)$ such that, if $\{Y_j\}_{j=1,\ldots,r}$ is a local orthonormal frame of E and $X_1, \ldots, X_{2k}$ are linearly independent vector fields on $\mathcal{M}$, then $\alpha(X_1, \ldots, X_{2k})$ is a homogeneous polynomial of degree k in the variables $\{S(X_1, X_m, Y_i, Y_j)\}$, $i, j \in \{1, \ldots, r\}$, m, $1 \in \{1, \ldots, 2k\}$, so that, within each monomial, the first two arguments of each factor perform a permutation of $X_1, \ldots, X_{2k}$.

Now, let $(\mathcal{M}, g)$ be a Riemannian manifold and let $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_3$ be three orthogonal distributions on $\mathcal{M}$ of dimensions p_1, p_2, p_3, respectively, such that $\mathcal{V}_1 \oplus \mathcal{V}_2 \oplus \mathcal{V}_3 = T\mathcal{M}$, $\mathcal{V}_1$ and $\mathcal{V}_1 \oplus \mathcal{V}_2$ are integrable, and, v_i being the projector of $T\mathcal{M}$ onto $\mathcal{V}_i$, $(i = 1, 2, 3)$

$$(\nabla_{A_2} v_2) A_2 = 0$$

for all $A_2 \in \mathcal{V}_2$, where ∇ is the Levi-Civita connection.

Then, we have

LEMMA 9.

$$\begin{aligned} S(A_1, B_1, C_2, D_2) &= S(A_1, B_1, C_3, D_3) \\ &= S(A_1, B_2, C_2, D_2) \\ &= S(A_1, B_2, C_3, D_3) = 0. \end{aligned}$$

PROOF. It follows from corollary 4 (c), that $S(A_1, B_1, C_3, D_3) = 0$. From proposition 2($b$), and the integrability of $\mathcal{V}_1 \oplus \mathcal{V}_2$, we get $S(A_1, B_1, C_2, D_2) = 0$. $S(A_1, B_2, C_2, D_2) = 0$ follows from proposition 2(c), and 6(a). Finally,

$$\begin{aligned} S(A_1, B_2, C_3, D_3) &= g([v_1[A_1, B_2], C_3], D_3) + g([v_2[A_1, B_2], C_3], D_3) \\ &\quad + g([B_2, v_3[A_1, C_3]], D_3) - g([A_1, v_3[B_2, C_3]], D_3) \\ &\quad + g([[A_1, B_2], C_3], D_3) - g([v_3[A_1, B_2], C_3], D_3) = 0 \end{aligned}$$

by the integrability of $\mathcal{V}_1 \oplus \mathcal{V}_2$ and the Jacobi identity. //

From this lemma and the preceding remarks, we have

THEOREM 10.

$$\mathrm{Pont}^i\,(\mathcal{V}_2)\,\mathrm{Pont}^j\,(\mathcal{V}_3) = 0 \text{ for } j > 2p_3 \text{ or } i + j > p_2 + 2p_3.$$

PROOF. If w_2 and w_3 are forms representing elements of $\mathrm{Pont}^i\,(\mathcal{V}_2)$ and $\mathrm{Pont}^j\,(\mathcal{V}_3)$, respectively, with $i + j > p_2 + 2p_3$, then $w_2 \wedge w_3$ decomposes as a sum of monomials, each of which vanishes due to the saturation of the exterior products of curvature forms that, according to lemma 9, are not zero. //

Observe that if $p_2 = 0$ (resp. $p_3 = 0$), Bott (resp. Pasternack) theorem is obtained [1] (resp. [7]).

REFERENCES

1. R. Bott, *Lectures on characteristic classes and foliations*, Lecture Notes in Mathematics, Springer-Verlag, **279** (1972), pp. 1–80.

2. L. A. Cordero and X. Masa, *Characteristic classes of subfoliations*, Ann. Inst. Fourier, Grenoble **31**, 2 (1981), pp. 61–86.

3. O. Gil-Medrano, *On the geometric properties of some classes of almost-product structures*, Rend. Circ. Mat. Palermo **32** (1983), 315–329.

4. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I and II, Interscience, 1963, 1969.

5. V. Miquel, *Some examples of Riemannian almost-product manifolds*, Pac. J. Math. **111** (1984), 163–178.

6. A. M. Naveira, *A classification of Riemannian almost-product manifolds*, Rendiconti di Matematica, **3** (1983), 577–592.

7. J. S. Pasternack, *Foliations and compact Lie group actions*, Comm. Math. Helv. **46** (1971), pp. 467–477.

8. B. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. of Math. **69** (1959), pp. 119–131.

9. I. Vaisman, *Almost-multifoliate Riemannian manifolds*, Ann. St. Univ. "Al. I. Cuza", XVI (1970), pp. 97–104.

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