Canad. Math. Bull. Vol. **26** (3), 1983

THE USUAL BEHAVIOUR OF RATIONAL APPROXIMATIONS

BY
PETER B. BORWEIN

ABSTRACT. Questions concerning the convergence of Padé and best rational approximations are considered from a categorical point of view in the complete metric space of entire functions. The set of functions for which a subsequence of the mth row of the Padé table converges uniformly on compact subsets of the complex plane is shown to be residual.

The speed of convergence of best uniform rational approximations and Padé approximations on the unit disc is compared. It is shown that, in a categorical sense, it is expected that subsequences of these approximants will converge at the same rate.

Likewise, it is expected that the poles of certain sequences of best uniform rational approximations wil be dense in the entire plane.

1. **Introduction.** We examine questions concerning rational approximations to entire functions from the point of view of how they usually behave. The notion of "usually" we adopt is the categorical notion in the complete metric space of entire functions.

We show, for example, that Baker's conjecture concerning uniform convergence of a subsequence of the mth row of the Padé table is "usually" correct. We then show that "usually" there is a subsequence of the mth row of the Padé table that is asymptotically as efficient as a best rational approximation of corresponding degree on the unit disc. This analysis also reveals that, in some sense, we expect approximations of degree (n, m) (either best or Padé) to behave better than approximations of degree $(n+k, m-k)$.

In [5] Gončar raises the question: does there exist an entire function f so that the poles of the best rational approximations to f are dense in the entire plane? Once again in a categorical sense we show, perhaps surprisingly, that the answer to the above question is "usually" yes.

Received by the editors March 5, 1982 and, in revised form, September 21, 1982.

AMS Subject Classification: 41 A20,A21,A25

This research was supported in part by the National Science and Engineering Research Council of Canada.

https://doi.org/10.4153/CMB-1983-051-1 Published online by Cambridge University Press

2. **Preliminaries.** Let $D_n=\{z \mid |z|<n\}$ and let d_n denote the supremum norm on $\bar{D}_n$ that is

$$d_n(f, g)=\sup_{|z|\leq n} |f(z)-g(z)|:=\|f-g\|_{\bar{D}_n}.$$

Let $d(f, g)$ be defined by

$$d(f, g)=\sum_{n=1}^{\infty}\frac{1}{2^n}\frac{d_n(f, g)}{1+d_n(f, g)}.$$

Let A denote the class of entire functions equipped with the above metric. Then A is a complete metric space where convergence is equivalent to uniform convergence on compact subsets. We adopt the familiar categorical vocabulary. A set B is "nowhere dense" if the interior of B closure is empty. A set B is "category 1" if B is a countable union of nowhere dense sets. A set B is "residual" if it is the complement of a category 1 set.

Let π_n denote the polynomials of degree at most n. A rational function is of type (n, m) if it has numerator of degree at most n and denominator of degree at most m. The (n, m) Padé approximant to f is the unique (n, m) rational function p_n/q_m that satisfies

$$q_m(z)f(z)-p_n(z)=su^z+\text{higher order terms}$$

where $s\neq 0$, u is chosen to be as large as possible and where $q_m\neq 0$. (See [3] or [4].)

We will call p_n/q_m regular if $u=n+m+1$, q_m and p_n have exact degree m and n respectively, and $q_m(0)\neq 0$. (This is not a standard usage.) If the (n, m) Padé approximant is regular for all n and m then we call the Padé table normal.

If $f(z)=\sum_{n=0}^{\infty} C_n z^n$ then f has normal Padé table if for all m and n

$$\det\begin{vmatrix} C_m & C_{m-1} & \cdots & C_{m-n+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ C_{m+n-1} & C_{m+n-2} & \cdots & C_m \end{vmatrix}\neq 0$$

where $C_i=0$ if $i<0$ (see [3] or [4]). The following lemma will be needed.

LEMMA 1. *Fix N. The set of $f\in A$ with regular (n, m) Padé approximants for all $n+m\leq N$ is open and dense in A.*

The Lemma is immediate from the above normality criterion and the fact that the regularity of the (n, m) Padé approximant to f depends only on the first $n+m+1$ coefficients of f. From this it is clear that the collection of f with normal Padé tables is residual in A.

https://doi.org/10.4153/CMB-1983-051-1 Published online by Cambridge University Press

We are allowing the functions in A and the corresponding approximants to have complex coefficients. It is worth pointing out that all the results go through verbatim if we restrict our attention to functions and approximants with real coefficients.

3. **On Baker's conjecture.** Baker has conjectured that if f is entire then there exists a subsequence of the mth row of the Padé table that converges to f uniformly on compact subsets. He proved this for $m=2$ [1]. The $m=0$ case is the Taylor series case. The $m=1$ case has been proved by Beardon. If $f=\sum_{n=0}^{\infty} C_n Z^n$ then the denominator of the $(n, 1)$ Padé approximant is given by $q_{n,1}=C_n/C_{n+1}-Z$. In particular, if $|C_{n+1}|>|C_n|$ then $q_{n,1}$ has a pole in D_1. It is apparent that given any sequence of distinct integers $\{n_i\}$ tending to ∞ it is possible to construct an entire f so that the $(n_i, 1)$ Padé approximants tend to f uniformly on D_1 but no subsequence of the first row that doesn't eventually become a subsequence of the above sequence converges in D_1. Thus, not only is passing to a subsequence essential in Baker's conjecture but this subsequence may be arbitrarily sparse.

Our first result shows that Baker's conjecture is "usually" true.

THEOREM 1. *Let (n_i, m_i) be a sequence of positive integer pairs where n_i and n_i-m_i tend to infinity. The entire functions f for which there exists a subsequence (n'_i, m'_i) of (n_i, m_i) so that the (n'_i, m'_i) Padé approximants converge to f (in the d metric on A) are residual in A.*

Proof. Let B be the set of entire functions which satisfy the conclusion of the above theorem.

For each k and γ let

$B_{k,\gamma}=\{f\in A \mid$ the (n_k, m_k) Padé approximant to f has no poles in $\overline{D_\gamma}$ and for all $n\le n_k$ and $m\le m_k$ the (n, m) Padé approximant to f *is regular*$\}$.

A "small" perturbation of the coefficients of an f possessing a regular (n, m) Padé approximant with no poles in $\overline{D_\gamma}$ results in a g whose (n, m) Padé approximant has the same properties. This observation and Lemma 1 guarantee that $B_{k,\gamma}$ is open.

A sequence of Padé approximant to an $f\in A$ will converge uniformly on D_λ if none of the approximant have any poles in some larger disc of radius γ_λ. (See [2]). We note that

$$B\supset \bigcap_{\gamma=1}^{\infty}\bigcap_{h=1}^{\infty}\bigcup_{k=h}^{\infty} B_{k,\gamma}=B^*$$

and that, for each m, $\bigcup_{k=h}^{\infty} B_{k,\gamma}=B^*_{h,\gamma}$ is open. We can finish the result by Baire's Theorem if we show that $B^*_{h,\gamma}$ is dense in A. Note that elements of B^* have a subsequence of Padé approximants with poles tending to infinity.

https://doi.org/10.4153/CMB-1983-051-1 Published online by Cambridge University Press

Padé showed (see [6]) that as $n+m\to\infty$ the poles of the (n, m) Padé approximants to e^z tend to infinity. From this it follows that if $n+m\to\infty$ and $n-m\to\infty$ then the poles of the (n, m) Padé approximants to $e^z+p(z)$, p a polynomial, also tend to infinity. A perturbation argument on the coefficients of $ae^z+p(z)$ allows us to construct a dense collection of functions in each of the sets $B^*_{h,\gamma}$ (the Padé table of e^z is normal). $\square$

4. **Rates of convergence.** We wish to show that (n, m) Padé approximants and (n, m) best approximants converge similarly on the unit disc. Let

$$\delta_{n,m}(f)=\|f-p_n/q_m\|_{\bar{D}_1}$$

where p_n/q_m is the (n, m) Padé approximant to f. Let

$$r_{n,m}(f)=\inf_{p_n\in\pi_n,\ q_m\in\pi_m}\|f-p_n/q_m\|_{\bar{D}_1}$$

THEOREM 2. *Let m be fixed. Let S be the subset of A such that, for each $f\in S$ there exists an increasing sequence of integers $\{n_i\}$ with*

(a) $\displaystyle\lim_{i\to\infty}\frac{\delta_{n_i,m}(f)}{r_{n_i,m}(f)}=1$

(b) $\displaystyle\lim_{i\to\infty}\frac{\delta_{n_i+k,m-k}(f)}{n_i^k\delta_{n_i,m}(f)}=\frac{(m-k)!}{m!}\qquad k=0,\ldots,m,$

(c) $\displaystyle\lim_{i\to\infty}\frac{r_{n_i+k,m-k}(f)}{n_i^k\delta_{n_i,m}(f)}=\frac{(m-k)!}{m!}\qquad k=0,\ldots,m.$

Then S is residual in A.

Proof. Let $T_{N,\varepsilon}$ be the set of $f\in A$ for which there exists $i\geq N$ so that

$$1-\varepsilon<\frac{\delta_{n_i,m}(f)}{r_{n_i,m}(f)}<1+\varepsilon$$

$$1-\varepsilon<\frac{\delta_{n_i+k,m-k}(f)}{\delta_{n_i,m}(f)}\frac{m!}{n_i^k(m-k)!}<1+\varepsilon\qquad k=0,\ldots,m$$

and

$$1-\varepsilon<\frac{r_{n_i+k,m-k}(f)}{r_{n_i,m}(f)}\frac{m!}{n_i^k(m-k)!}<1+\varepsilon\qquad k=0,\ldots,m$$

with the additional requirement that the (i, j) Padé approximants involved in satisfying the inequalities be regular for $i=0, 1,\ldots, n_i+m$ and $j=0, 1,\ldots, m$. This regularity ensures that the mapping

$$f\to\delta_{i,j}(f)$$

is continuous at each f in $T_{N,\varepsilon}$ for the appropriate values of i and j. The

https://doi.org/10.4153/CMB-1983-051-1 Published online by Cambridge University Press

mapping

$$f \to r_{i,j}(f)$$

is also continuous. Thus, we can verify that $T_{N,\varepsilon}$ is open.

We can finish, more or less as in Theorem 1, if we can show that $T_{N,\varepsilon}$ is dense. That is, we take a sequence ε_j tending to zero and note that

$$\bigcap_{j=1}^{\infty} T_{j,\varepsilon_j} \subset S.$$

To show that $T_{j,\varepsilon}$ is dense we observe that if we can show that e^z satisfies (a), (b), and (c) of the statement of the Theorem then for any polynomial p we can find an arbitrarily small perturbation of the coefficients of $ae^z + p(z)$ so that this perturbed function is in T_{j,ε_j}.

That e^z satisfies the desired conditions follows from the facts that

$$\delta_{n,m}(e^z) = \frac{n!\, m!}{(m+n)!\,(m+n+1)!}(1+o(1)) \qquad n \to \infty.$$

and

$$r_{n,m}(e^z) = \frac{n!\, m!}{(m+n)!\,(m+n+1)!}(1+o(1)) \qquad n \to \infty.$$

Both of these results may be found in Saff [7]. □

The preceding theorem seems to say that for most entire functions there exists a subsequence of indices for which Padé and best approximants behave the same and for which the closer to the main diagonal the indices become the better the approximation becomes.

However, any "canonical behaviour" will generate a theorem like Theorem 2. That is, if we can find a dense set in A so that conditions like the initial conditions of the proof of Theorem 2 are satisfied then we will get a similar type of result. With this in mind consider

$$f(z) = \sum C_k z^{\gamma_k}$$

where the C_n are chosen so that

$$\sum_{k=n}^{\infty} |C_{k+1}| < \frac{1}{n} |C_n|.$$

Let p be a polynomial of fixed degree h and suppose that

$$h + 2v < \gamma_n \quad \text{and} \quad v > \gamma_{n-1} > h.$$

Suppose that p_v/q_v is a best (v, v) approximant to $f+p$ on $\bar{D}_1$. By comparing p_v/q_v to $f - \sum_0^{n-1} C_k z^{\gamma_k}$ we can deduce (via Rouché's Theorem) that

$$\left(1 - \frac{1}{n}\right) C_n \le \left\| f - \frac{p_v}{q_v} \right\|_{\bar{d}_1} \le \left(1 + \frac{1}{n}\right) C_n.$$

https://doi.org/10.4153/CMB-1983-051-1 Published online by Cambridge University Press

Also,

$$\left(1-\frac{1}{n}\right)C_n \le \|f-s_v\|_{\bar{D}_1} \le \left(1+\frac{1}{n}\right)C_n.$$

where s_v is a best polynomial approximation of degree v. (See [2].)

In particular, if we chose the γ_n sufficiently gapped we can get the following result by considering $f+p$ as p varies:

PROPOSITION 1. *Let T be the subset of A such that, for each $f \in T$ there exists a subsequence $\{n_j\}$ with*

$$\lim_{i\to\infty} \frac{r_{n_i,n_i}(f)}{r_{n_i,0}(f)} = 1.$$

Then T is residual in A.

5. **The location of poles.** We begin by constructing an entire function whose best approximants of degree (n, n) have infinitely many prescribed poles. Consider

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} C_n\left(T_{27^{2n+1}}(x)+q_{2n+1}(x)+\frac{\delta_n}{x^2+a_nx+b_n}\right) \\ &\quad + C_n\left(T_{27^{2n}}(x)+q_{2n}(x)-\frac{\delta_n}{x^2+a_nx+b_n}\right) \\ &= \sum_{n=0}^{\infty} (C_n r_{2n+1}(x)+C_n r_{2n}(x)) \end{aligned}$$

where T_m is the mth Cebycev polynomial and where the q_n and $\delta_n \neq 0$ are chosen so that

$$\deg(q_n) \le 4.27^n$$

where $\|r_n\|_{[-1,1]}=1$ and $r_n = T_{27^n}$ at all points in $[-1, 1]$ where $|T_{27^n}|=1$. (This amounts to a Hermite interpolation problem where we must, of course, assume that $x^2+a_nx+b_n$ has no root in $[-1, 1]$.) We choose the $C_n>0$ to ensure that f is analytic. It is easily verified that

$$f(x)-\left(C_n r_{2n}+\sum_{h=0}^{n-1} C_h(r_{2h+1}+r_{2h})\right)$$

oscillates between its maximum and minimum modulus at least 27^{2n+1} times on $[-1, 1]$ and that

$$\left(C_n r_{2n}+\sum_{h=0}^{n-1} C_h(r_{2h+1}+r_{2h})\right) \text{ is of the form } \frac{p(x)}{x^2+a_nx+b_n}$$

where $p(x)$ is a polynomial of degree less than $5\cdot 27^{2n}$. It follows that this is the best approximation (either real or complex coefficiented) to f on $[-1, 1]$ of type

https://doi.org/10.4153/CMB-1983-051-1 Published online by Cambridge University Press

$(5\cdot 27^{2n}, j)$ where $j=2,\ldots, 5\cdot 27^{2n}$. Using functions of the form $f(x)+p(x)$ where f is of the above type and where p is a polynomial allows us to establish the following:

THEOREM 3. *Fix $m \geq 2$. Let $T(f)$ denote the set of all poles of a sequence of best (n, m) rational approximants to f on $[-1, 1]$. Then the collection of f for which $T(f)$ is dense in the complex plane is residual in A.*

The proof is very similar to that of Theorems 1 and 2 and is therefore omitted.

REFERENCES

1. G. A. Baker, Jr., and P. R. Graves-Morris, *Convergence of the Padé Table*, J. Math. Anal. App. **57** (1977), 323–339.

2. P. Borwein, *On Padé and Best Rational Approximations*, Canad. Math. Bull. **26** (1) (1983) 50–57.

3. C. K. Chui, *Recent results on Padé Approximants and related problems*, in Lorentz, G. G., Schumaker, L. L. (eds.), Proceedings, Conference on Approximation Theory, Academic Press, (1976).

4. W. B. Gragg, *The Padé table and its applications to certain algorithms of numerical analysis*, SIAM Review, **14** (1972), 1–62.

5. A A. Gončar, *Properties of functions related to their rate of approximability by rational functions*, Amer. Math. Soc. Transl. (2) **91** (1970), 99–128.

6. E. B. Saff, and R. S. Varga, *On the zeros and poles of Padé approximants to e^z*, Numer. Math., **25** (1975), 1–14.

7. E. B. Saff, *On the degree of best rational approximation to the exponential function*, J. Approx. Theory, **9** (1973), 97–101.

DALHOUSIE UNIVERSITY
HALIFAX, NOVA SCOTIA
CANADA B3H 4H8