Canad. Math. Bull. Vol. **47** (1), 2004 pp. 144–151

On the Uniqueness of Wave Operators Associated with Non-Trace Class Perturbations

Jingbo Xia

Abstract. Voiculescu has previously established the uniqueness of the wave operator for the problem of $\mathcal{C}^{(0)}$-perturbation of commuting tuples of self-adjoint operators in the case where the norm ideal $\mathcal{C}$ has the property $\lim_{n\to\infty} n^{-1/2}\|P_n\|_{\mathcal{C}} = 0$, where $\{P_n\}$ is any sequence of orthogonal projections with $\operatorname{rank}(P_n) = n$. We prove that the same uniqueness result holds true so long as $\mathcal{C}$ is not the trace class. (It is well known that there is no such uniqueness in the case of trace-class perturbation.)

1 Introduction

In this note, all Hilbert spaces are assumed to be separable and all operators bounded.

As the title indicates, this note concerns wave operators arising from the problem of perturbation by operators in norm ideals. But before discussing the details, it is necessary to recall the relevant definitions and known facts.

A *norm ideal* of compact operators is a linear space $\mathcal{C}$ of compact operators on a Hilbert space H which has the following properties:

(a) There is a norm $\|\cdot\|_{\mathcal{C}}$ on $\mathcal{C}$ with respect which $\mathcal{C}$ is a Banach space.

(b) If $T \in \mathcal{C}$ and $A, B \in \mathcal{B}(H)$, then $ATB \in \mathcal{C}$ and $\|ATB\|_{\mathcal{C}} \le \|A\|\,\|T\|_{\mathcal{C}}\|B\|$.

(c) If $T \in \mathcal{C}$, then $T^* \in \mathcal{C}$ and $\|T\|_{\mathcal{C}} = \|T^*\|_{\mathcal{C}}$.

(d) $\|T\| \le \|T\|_{\mathcal{C}}$ for every $T \in \mathcal{C}$, and the equality holds whenever $\operatorname{rank}(T) = 1$.

(e) $\mathcal{C} \neq \{0\}$.

Each norm ideal is intrinsically given by the associated symmetric gauge function. One can therefore speak of norm ideal of compact operators without reference to the underlying Hilbert space. We cite [4] as a standard reference for norm ideals.

Given a norm ideal $\mathcal{C}$ of compact operators, let $\mathcal{C}^{(0)}$ denote the $\|\cdot\|_{\mathcal{C}}$-closure of the collection of finite-rank operators in $\mathcal{C}$. With the norm $\|\cdot\|_{\mathcal{C}}$, $\mathcal{C}^{(0)}$ is itself a norm ideal. In general $\mathcal{C}$ and $\mathcal{C}^{(0)}$ need not coincide. If $\mathcal{C}$ is not the trace class, then the dual $\mathcal{C}'$ of $\mathcal{C}^{(0)}$ is also a norm ideal of compact operators [4, Theorem III.12.2]. In fact, for any bounded linear functional F on $\mathcal{C}^{(0)}$, there is an $X = X_F \in \mathcal{C}'$ such that

$$F(T) = \operatorname{tr}(TX), \qquad T \in \mathcal{C}^{(0)}.$$

Recall that an operator D is said to be *diagonal* if it is unitarily equivalent to an operator $\operatorname{diag}\{a_j\}_{j=1}^{\infty}$ on ℓ^2_+ defined by the formula

$$\operatorname{diag}\{a_j\}_{j=1}^{\infty}\{c_1, \dots, c_j, \dots\} = \{a_1c_1, \dots, a_jc_j, \dots\}.$$

Received by the editors June 7, 2002.
Research supported in part by National Science Foundation grant DMS-0100249.
AMS subject classification: 47A40, 47B10.

If $A = (A_1, \dots, A_N)$ is a commuting tuple of self-adjoint operators and $\mathcal{C}$ is a norm ideal of compact operators, we say that A is *simultaneously diagonalizable modulo* $\mathcal{C}$ if there is a commuting tuple $(D_1, \dots, D_N)$ of self-adjoint diagonal operators such that $A_j - D_j \in \mathcal{C}$, $j = 1, \dots, N$.

Voiculescu showed in [10] that, given a commuting tuple $A = (A_1, \dots, A_N)$ of self-adjoint operators on a Hilbert space H and a norm ideal $\mathcal{C}$ of compact operators on H, there is a spatial decomposition

$$H = H_{\mathrm{d}}(A; \mathcal{C}) \oplus H_{\mathrm{nd}}(A; \mathcal{C}) \tag{1}$$

with the following properties:

(i) Both $H_{\mathrm{d}}(A; \mathcal{C})$ and $H_{\mathrm{nd}}(A; \mathcal{C})$ are invariant under A.

(ii) $A|H_{\mathrm{d}}(A; \mathcal{C})$ is simultaneously diagonalizable modulo $\mathcal{C}^{(0)}$.

(iii) $H_{\mathrm{nd}}(A; \mathcal{C})$ contains no nonzero invariant subspace for A on which A can be simultaneously diagonalized modulo $\mathcal{C}^{(0)}$.

The orthogonal projection from H onto $H_{\mathrm{nd}}(A; \mathcal{C})$ will be denoted by $P_{\mathrm{nd}}(A; \mathcal{C})$.

Given a commuting tuple $A = (A_1, \dots, A_N)$ of self-adjoint operators on a Hilbert space H, it is elementary that we have the decomposition $H = H_{\mathrm{p}}(A) \oplus H_{\mathrm{c}}(A)$ such that both $H_{\mathrm{p}}(A)$ and $H_{\mathrm{c}}(A)$ are invariant under A, $A|H_{\mathrm{p}}(A)$ is a tuple of diagonal operators, and $\mathcal{E}_c(\{x\}) = 0$ for every singleton set $\{x\}$ in $\mathbf{R}^N$, where $\mathcal{E}_c$ is the spectral resolution of $A|H_{\mathrm{c}}(A)$. Let $P_{\mathrm{c}}(A)\colon H \to H_{\mathrm{c}}(A)$ be the orthogonal projections. It is obvious that $P_{\mathrm{c}}(A) \ge P_{\mathrm{nd}}(A; \mathcal{C})$ for any norm ideal $\mathcal{C}$ of compact operators.

Definition 1 Let $\mathcal{C}$ be a norm ideal of compact operators and $A = (A_1, \dots, A_N)$ be a commuting tuple of self-adjoint operators. A sequence $\{\varphi_n\}$ of functions on $\mathbf{R}^N$ is said to be $\mathcal{C}$-*admissible* for A if the following hold true:

(i) For every n, $|\varphi_n| = 1$ on $\mathbf{R}^N$ and φ_n is continuous on $\mathbf{R}^N$.

(ii) $w\text{-}\lim_{n\to\infty} \varphi_n(A) P_{\mathrm{nd}}(A; \mathcal{C}) = 0$.

Given commuting N-tuples A and A' of self-adjoint operators and a norm ideal $\mathcal{C}$, there exists a sequence which is $\mathcal{C}$-admissible for both A and A'. Indeed, because $P_{\mathrm{c}}(A \oplus A') \ge P_{\mathrm{nd}}(A; \mathcal{C}) \oplus P_{\mathrm{nd}}(A'; \mathcal{C})$, an even stronger statement, [12, Proposition 2.6], holds true.

The connection between (1) and wave operators was first established by Voiculescu in [10]. Among the many results in [10], we in particular recall

Theorem 2 **[10, Theorem 1.5]** *Let $\mathcal{C}$ be a norm ideal of compact operators such that*

$$\lim_{n\to\infty} n^{-1/2} \|\xi_1 \otimes \xi_1 + \cdots + \xi_n \otimes \xi_n\|_{\mathcal{C}} = 0, \tag{2}$$

where $\{\xi_1, \dots, \xi_n, \dots\}$ is any orthonormal set. Let $A = (A_1, \dots, A_N)$ and $A' = (A'_1, \dots, A'_N)$ be tuples of self-adjoint operators on a Hilbert space H such that $[A_i, A_j] = 0 = [A'_i, A'_j]$ for all $i, j \in \{1, \dots, N\}$ and such that

$$A_j - A'_j \in \mathcal{C}^{(0)}, \quad j = 1, \dots, N.$$

Then there exists a partial isometry W such that

$$W = s\text{-}\lim_{n\to\infty} \varphi_n^*(A') \varphi_n(A) P_{\mathrm{nd}}(A; \mathcal{C})$$

for any sequence $\{\varphi_n\}$ which is $\mathcal{C}$-admissible for A.

https://doi.org/10.4153/CMB-2004-015-4 Published online by Cambridge University Press

The most striking feature of this result is the *uniqueness* of W. In other words, under condition (2), the wave operator W is independent of the choice of the sequence $\{\varphi_n\}$, so long as it is $\mathcal{C}$-admissible for A. The reason we say this is striking is that, in the original setting of wave operators, *i.e.*, in the case of trace-class perturbation of a single self-adjoint operator, not only is such uniqueness statement decidedly false, but the non-uniqueness of wave operators is a fundamental fact of nature. Since this uniqueness is the main interest of the present note, a little more elaboration is called for.

Let $\mathcal{C}_1$ denote the trace class. For a single self-adjoint operator T on a Hilbert space H, the decomposition $H = H_{\mathrm{d}}(T;\mathcal{C}_1) \oplus H_{\mathrm{nd}}(T;\mathcal{C}_1)$ given by (1) coincides with the more familiar decomposition $H = H_{\mathrm{s}}(T) \oplus H_{\mathrm{ac}}(T)$. Indeed, it follows from a theorem of Carey and Pincus [1] that $H_{\mathrm{s}}(T) \subset H_{\mathrm{d}}(T;\mathcal{C}_1)$, whereas the inclusion $H_{\mathrm{ac}}(T) \subset H_{\mathrm{nd}}(T;\mathcal{C}_1)$ is a consequence of the Kato-Rosenblum theorem [5,7]. Therefore $H_{\mathrm{s}}(T) = H_{\mathrm{d}}(T;\mathcal{C}_1)$ and $H_{\mathrm{ac}}(T) = H_{\mathrm{nd}}(T;\mathcal{C}_1)$, *i.e.*, $P_{\mathrm{ac}}(T) = P_{\mathrm{nd}}(T;\mathcal{C}_1)$. Let $e_n^+(x) = e^{inx}$ and $e_n^-(x) = e^{-inx}$, $n \in \mathbf{N}$. According to Definition 1, both sequences $\{e_n^+\}$ and $\{e_n^-\}$ are $\mathcal{C}_1$-admissible for any self-adjoint operator T. It is a well-known fact that there exist self-adjoint operators T and T' such that $T - T'$ is of trace class and such that the wave operators

$$W_+(T', T) = s\text{-}\lim_{n\to\infty} \{e_n^+(T')\}^* e_n^+(T) P_{\mathrm{ac}}(T),$$

$$W_-(T', T) = s\text{-}\lim_{n\to\infty} \{e_n^-(T')\}^* e_n^-(T) P_{\mathrm{ac}}(T)$$

do not coincide — see the Appendix. In other words, if we set $f_{2n} = e_n^+$ and $f_{2n-1} = e_n^-$ for $n = 1, 2, \dots$, then the sequence $\{f_n^*(T') f_n(T) P_{\mathrm{nd}}(T;\mathcal{C}_1)\}$ can fail to converge in the strong operator topology, even though the sequence $\{f_n\}$ is still $\mathcal{C}_1$-admissible for T.

The non-uniqueness of the wave operators for the perturbation problem $T \to T'$ has actual physical significance, for the operator $S = W_-^*(T', T) W_+(T', T)$ contains scattering information [8]. Also, the fact that $W_+(T', T)$ and $W_-(T', T)$ may differ reflects a dichotomy [13] of the spectral flow of the canonical commutation relation.

To summarize, the wave operator for any $\mathcal{C}^{(0)}$-perturbation problem is unique if $\mathcal{C}$ satisfies (2), and, for good reason, there is no such uniqueness when $\mathcal{C}$ is the trace class. This naturally and obviously leads to the following question: what happens if $\mathcal{C}$ is neither the trace class nor does it satisfy (2)? The purpose of this note is to report that the uniqueness of W in Theorem 2 in fact holds whenever $\mathcal{C}$ is not the trace class.

Theorem 3 *Let $\mathcal{C}$ be a norm ideal of compact operators which is not the trace class. Let $A = (A_1, \dots, A_N)$ and $A' = (A_1', \dots, A_N')$ be tuples of self-adjoint operators on a Hilbert space H such that $[A_i, A_j] = 0 = [A_i', A_j']$ for all $i, j \in \{1, \dots, N\}$ and such that*

$$A_j - A_j' \in \mathcal{C}^{(0)}, \quad j = 1, \dots, N.$$

Then there exists a partial isometry W such that

$$W = s\text{-}\lim_{n\to\infty} \varphi_n^*(A') \varphi_n(A) P_{\mathrm{nd}}(A;\mathcal{C})$$

for any sequence $\{\varphi_n\}$ which is $\mathcal{C}$-admissible for A.

Our proof will be given in the next section. As it turns out, to replace (2) by the more general condition that $\mathcal{C} \neq \mathcal{C}_1$, we only need one new technical step (Lemma 4 below) in addition to Voiculescu's ideas.

2 Proof of Theorem 3

Let A, A', $\mathcal{C}$, $\{\varphi_n\}$, *etc.*, be the same as in Theorem 3. To simplify notation, we write

$$W_n = \varphi_n^*(A')\varphi_n(A) \quad \text{and} \quad P = P_{\mathrm{nd}}(A;\mathcal{C}).$$

Lemma 4 *Let $\{k(n)\}$ be a sequence of integers such that $k(n) > n$ for every $n \in \mathbf{N}$. Let $T_n = W_{k(n)}^* W_n - 1$ and $Y_n = PT_n^* T_n P$, $n \in \mathbf{N}$. Then $[Y_n, A_j] \in \mathcal{C}^{(0)}$ for all $n \in \mathbf{N}$ and $j = 1, \dots, N$. Furthermore, there is a sequence $\{Z_n\}$ in the convex hull of $\{Y_1, Y_2, \dots, Y_\ell, \dots\}$ such that*

$$\lim_{n\to\infty} \|[Z_n, A_j]\|_{\mathcal{C}} = 0, \quad j = 1, \dots, N. \tag{3}$$

Proof Let $S_n = PT_nP$, $n \in \mathbf{N}$. Then $[S_n, A_j] = P[T_n, A_j]P$. We first prove that

$$[S_n, A_j] \in \mathcal{C}^{(0)}, \quad n \in \mathbf{N} \text{ and } 1 \le j \le N, \tag{4}$$

and that, for every $j \in \{1, \dots, N\}$,

$$\lim_{n\to\infty} \mathrm{tr}([S_n, A_j]X) = 0 \quad \text{if } X \in \mathcal{C}'. \tag{5}$$

Let $K_j = A_j - A_j'$, $1 \le j \le N$. Because $K_j \in \mathcal{C}^{(0)}$, (4) follows from the identity

$$[S_n, A_j] = P\{W_{k(n)}^* \varphi_n^*(A') K_j \varphi_n(A) - \varphi_{k(n)}^*(A) K_j \varphi_{k(n)}(A') W_n\} P. \tag{6}$$

Given an $\epsilon > 0$, write $K_j = F + G$ in (6), where $\mathrm{rank}(F) < \infty$ and $\|G\|_{\mathcal{C}} \le \epsilon$. Then

$$\begin{aligned}
&\mathrm{tr}([S_n, A_j]X) \\
&\quad = \big(\mathrm{tr}(PW_{k(n)}^*\varphi_n^*(A')F\varphi_n(A)PX) - \mathrm{tr}(XP\varphi_{k(n)}^*(A)F\varphi_{k(n)}(A')W_nP)\big) \\
&\qquad + \big\{\mathrm{tr}\big(PW_{k(n)}^*\varphi_n^*(A')G\varphi_n(A)PX\big) - \mathrm{tr}\big(P\varphi_{k(n)}^*(A)G\varphi_{k(n)}(A')W_nPX\big)\big\} \\
&\quad = \big(f_1(n) - f_2(n)\big) + \{g_1(n) - g_2(n)\}.
\end{aligned}$$

We have the obvious estimates $|g_1(n)| + |g_2(n)| \le 2\|G\|_{\mathcal{C}}\|X\|_{\mathcal{C}'} \le 2\epsilon\|X\|_{\mathcal{C}'}$ and

$$|f_1(n)| \le \mathrm{rank}(F)\|PW_{k(n)}^*\varphi_n^*(A')F\varphi_n(A)PX\| \le \mathrm{rank}(F)\|F\varphi_n(A)PX\|,$$
$$|f_2(n)| \le \mathrm{rank}(F)\|XP\varphi_{k(n)}^*(A)F\varphi_{k(n)}(A')W_nP\| \le \mathrm{rank}(F)\|F^*\varphi_{k(n)}(A)PX^*\|.$$

We now use the condition that $\mathcal{C}$ is not the trace class. It simply means that $\mathcal{C}'$ consists of compact operators [4, Theorem III.12.2]. In particular, X is compact. Since F

https://doi.org/10.4153/CMB-2004-015-4 Published online by Cambridge University Press

is also compact, the weak convergence $w\text{-}\lim_{n\to\infty} \varphi_n(A)P = 0$ (see Definition 1) implies

$$\lim_{n\to\infty} \|F\varphi_n(A)PX\| = 0 = \lim_{n\to\infty} \|F^*\varphi_{k(n)}(A)PX^*\|.$$

Therefore $\lim_{n\to\infty}\big(|f_1(n)| + |f_2(n)|\big) = 0$. This proves (5).

Now $T_n^*T_n + T_n^* + T_n = (T_n+1)^*(T_n+1) - 1 = 0$ because $T_n + 1 = W_{k(n)}^* W_n$ is a unitary operator. Recall that $Y_n = PT_n^*T_nP$. Thus $[Y_n, A_j] = [S_n, A_j]^* - [S_n, A_j]$, and, by (4) and (5), we have $[Y_n, A_j] \in \mathcal{C}^{(0)}$ and

$$\lim_{n\to\infty} \operatorname{tr}([Y_n, A_j]X) = 0 \quad \text{for all } X \in \mathcal{C}' \text{ and } j = 1, \dots, N.$$

Since $\mathcal{C}'$ is the dual of $\mathcal{C}^{(0)}$, this means that $0 \oplus \cdots \oplus 0$ cannot be separated from

$$\Omega = \text{the convex hull of } \{[Y_\ell, A_1] \oplus \cdots \oplus [Y_\ell, A_N] : \ell \in \mathbf{N}\}$$

by any bounded functional on $\mathcal{C}^{(0)}$. By the Hahn-Banach separation theorem, $0 \oplus \cdots \oplus 0$ lies in the $\|\cdot\|_{\mathcal{C}}$-closure of Ω. By the identity $[B, A_1] \oplus \cdots \oplus [B, A_N] = [B \oplus \cdots \oplus B, A_1 \oplus \cdots \oplus A_N]$, there is a sequence $\{Z_n\}$ in the convex hull of $\{Y_1, Y_2, \dots, Y_\ell, \dots\}$ for which (3) holds. ■

Except for the use of Lemma 4, our proof of Theorem 3 follows the general strategy of [10, Section 1]. In particular it requires the following two well-known lemmas.

***Lemma 5* [10, Theorem 1.2]** *Let $A = (A_1, \dots, A_N)$ be a commuting tuple of self-adjoint operators and let $\mathcal{C}$ be a norm ideal of compact operators. If $\{B_n\}$ is a sequence of finite-rank positive contractions such that $\lim_{n\to\infty} \|[B_n, A_j]\|_{\mathcal{C}} = 0$, $j = 1, \dots, N$, then*

$$s\text{-}\lim_{n\to\infty} B_n P_{\mathrm{nd}}(A; \mathcal{C}) = 0.$$

Lemma 6 *For each $\eta \in C_c^\infty(\mathbf{R})$, there is a constant $0 < C(\eta) < \infty$ such that the following holds true: Let $\mathcal{C}$ be a norm ideal of compact operators and let $X, Y \in \mathcal{B}(H)$ be such that $[X, Y] \in \mathcal{C}$. If X is self-adjoint, then $[\eta(X), Y] \in \mathcal{C}$ and $\|[\eta(X), Y]\|_{\mathcal{C}} \le C(\eta)\|[X, Y]\|_{\mathcal{C}}$.*

Lemma 6 goes back to the 1970s, and there are numerous references for it. See, *e.g.*, [11, Lemma 1.4]. It can also be found in [2].

Proof of Theorem 3 We only need to prove the strong convergence of the sequence $\{W_nP\}$, where, as we recall, $W_n = \varphi_n^*(A')\varphi_n(A)$ and $P = P_{\mathrm{nd}}(A; \mathcal{C})$. Since $\{\varphi_n\}$ is an arbitrary $\mathcal{C}$-admissible sequence for A, the fact that the limit operator W is independent of the choice of $\{\varphi_n\}$ follows from this convergence and the following observation borrowed from Voiculescu [10, page 91]: If one mixes two such sequences, then one obtains a new sequence of the same kind.

To prove the strong convergence of $\{W_nP\}$, we suppose the contrary. Then there would be a unit vector $\xi \in PH$ and a $0 < d \le 1$ such that $\sup_{n \le k < k'} \|W_k\xi - W_{k'}\xi\| \ge 3d$ for all $n \in \mathbf{N}$. Thus, for each $n \in \mathbf{N}$, there would be a $k(n) > n$ such that

$$\|W_n\xi - W_{k(n)}\xi\| \ge d. \tag{7}$$

https://doi.org/10.4153/CMB-2004-015-4 Published online by Cambridge University Press

We will complete the proof by showing that this leads to a contradiction.

Let $T_n = W^*_{k(n)}W_n - 1$ and $Y_n = PT^*_nT_nP$, $n \in \mathbf{N}$. By Lemma 4, there is a sequence $\{Z_n\}$ in the convex hull of $\{Y_1, Y_2, \dots, Y_\ell, \dots\}$ such that (3) holds. Now pick a function $\eta \in C^\infty_c(\mathbf{R})$ such that $0 \le \eta \le 1$ on $\mathbf{R}$, $\eta = 0$ on $\mathbf{R} \setminus [d^2/4, 5]$, and $\eta = 1$ on $[d^2/3, 4]$. Define $B_n = \eta(Z_n)$, $n \in \mathbf{N}$. It follows from Lemma 6 and (3) that

$$\lim_{n\to\infty} \|[B_n, A_j]\|_{\mathcal{C}} \le C(\eta) \lim_{n\to\infty} \|[Z_n, A_j]\|_{\mathcal{C}} = 0, \quad j = 1, \dots, N. \tag{8}$$

Note that (7) implies $\langle T^*_nT_n\xi, \xi\rangle \ge d^2$, $n \in \mathbf{N}$. By definition, $\langle Y_n\xi, \xi\rangle = \langle PT^*_nT_nP\xi, \xi\rangle = \langle T^*_nT_n\xi, \xi\rangle$. Because $\{Z_n\}$ is in the convex hull of $\{Y_n\}$, this gives us

$$\langle Z_n\xi, \xi\rangle \ge d^2, \quad n \in \mathbf{N}. \tag{9}$$

Since $\|T_\ell\| \le 2$, we have $0 \le Z_n \le 4$. The conditions $\eta = 1$ on $[d^2/3, 4]$ and $\eta \ge 0$ on $\mathbf{R}$ ensure that $B_n \ge (1/4)\{Z_n - (d^2/2)\chi_{[0,d^2/2]}(Z_n)\}$. Thus, by (9) and the condition $\|\xi\| = 1$,

$$\langle B_n\xi, \xi\rangle \ge (1/4)\{d^2 - (d^2/2)\} = d^2/8, \quad n \in \mathbf{N}. \tag{10}$$

Since $A_j - A'_j \in \mathcal{C}^{(0)}$, $j = 1, \dots, N$, and φ_k is continuous on $\mathbf{R}^N$, every $W_k - 1$ is a compact operator. Consequently, $\{(W_{k(n)} - 1)^* + 1\}\{(W_n - 1) + 1\} - 1 = T_n$ is compact. Thus each Z_n is also compact. Since $\eta = 0$ in a neighborhood of 0, the rank of each positive contraction $B_n = \eta(Z_n)$ is finite. Because $\xi \in PH = P_{\mathrm{nd}}(A; \mathcal{C})H$, (8) and (10) together contradict Lemma 5. This completes the proof. ∎

Appendix

The fact that the two wave operators

$$W_+(T', T) = s\text{-}\lim_{t\to+\infty} e^{-itT'}e^{itT}P_{\mathrm{ac}}(T),$$
$$W_-(T', T) = s\text{-}\lim_{t\to-\infty} e^{-itT'}e^{itT}P_{\mathrm{ac}}(T)$$

can differ for self-adjoint operators T, T' satisfying the condition $T - T' \in \mathcal{C}_1$ is well known. But since the main point of this note is the uniqueness of wave operators for non-trace-class perturbation problems, it is only appropriate to include a contrasting example in the case of trace-class perturbation. We will therefore present a pair of self-adjoint operators T, T' such that $T - T'$ belongs to the trace class and such that $W_+(T', T) \ne W_-(T', T)$. We reiterate that the material presented below contains nothing new and is only meant to save the non-expert reader an unnecessary trip to the library.

Let D denote the differential operator $(1/i)d/dx$ on its usual domain in $L^2(\mathbf{R})$. Let $V \in C^\infty_c(\mathbf{R})$ be a function such that $0 \le V \le 1$ on $\mathbf{R}$, $V = 0$ on $\mathbf{R} \setminus [-1, 2]$, and $V = 1$ on $[0, 1]$. Define $\eta(x) = \int_{-\infty}^x V(y)\,dy$, $x \in \mathbf{R}$. Let $D' = D+V$ and let U be the unitary operator of multiplication by $e^{i\eta}$ on $L^2(\mathbf{R})$. Then $U^*DU = D'$. Obviously,

$P_{ac}(D) = 1$. Furthermore, $e^{-itD'}e^{itD} = U^*e^{-itD}Ue^{itD}$ and $(e^{itD}f)(x) = f(x+t)$, $f \in L^2(\mathbf{R})$. Thus the two wave operators $W_\pm(D', D) = s\text{-}\lim_{t\to\pm\infty} e^{-itD'}e^{itD}$ are easily computed: We have

$$W_+(D, D') = U^* \text{ and } W_-(D, D') = e^{i\theta}U^*, \quad \text{where } \theta = \int_{-\infty}^{\infty} V(y)\,dy \in [1, 3].$$

It is well known (see, *e.g.*, [8, Theorem XI.20]) that $\chi_{[-1,2]}(D+i)^{-1}$ belongs to the Hilbert-Schmidt class. (This also follows from the fact that $\chi_{[-1,2]}(D^2+1)^{-1}\chi_{[-1,2]} \in \mathcal{C}_1$, which is easy to establish because the kernel function of $(D^2+1)^{-1}$ can be computed using Fourier transform.) Therefore

$$(D+i)^{-1} - (D'+i)^{-1} = (D'+i)^{-1}V(D+i)^{-1} \in \mathcal{C}_1. \tag{A.1}$$

Define the function $\varphi(t) = (t^2+1)^{-1}$ on $\mathbf{R}$. Let $T = \varphi(D)$ and $T' = \varphi(D')$. Then T and T' are bounded self-adjoint operators. It follows from (A.1) that

$$T - T' = (D+i)^{-1}(D-i)^{-1} - (D'+i)^{-1}(D'-i)^{-1} \in \mathcal{C}_1.$$

It is easy to see that $T = \varphi(D)$ is purely absolutely continuous on $L^2(\mathbf{R})$, *i.e.*, $P_{ac}(T) = 1$. We have $\varphi' > 0$ on $(-\infty, 0)$ and $\varphi' < 0$ on $(0, \infty)$. For this reason we write $E_+ = E(-\infty, 0)$ and $E_- = E(0, \infty)$, where $E(\cdot)$ is the spectral resolution for D. (A.1) guarantees that the invariance principle for wave operators applies to the pair of operators D, D' and the function φ. More precisely, by [8, Theorem XI.11], we have

$$\begin{aligned} W_+(T', T) &= W_+\big(\varphi(D'), \varphi(D)\big) = W_+(D', D)E_+ + W_-(D', D)E_-, \\ W_-(T', T) &= W_-\big(\varphi(D'), \varphi(D)\big) = W_-(D', D)E_+ + W_+(D', D)E_-. \end{aligned}$$

Thus $W_+(T', T)E_+ = U^*E_+$ and $W_-(T', T)E_+ = e^{i\theta}U^*E_+$. Obviously, $E_+ \neq 0$. Since $e^{i\theta} \neq 1$, we have $W_+(T', T) \neq W_-(T', T)$ as promised.

References

[1] R. Carey and J. Pincus, *Unitary equivalence modulo the trace class for self-adjoint operators*. Amer. J. Math. **98**(1976), 481–514.

[2] ———, *Mosiacs, principal functions, and mean motion in von Neumann algebra*. Acta Math. **138**(1977), 153–218.

[3] G. David and D. Voiculescu, *s-Numbers of singular integrals for the invariance of absolutely continuous spectra in fractional dimensions*. J. Funct. Anal. **94**(1990), 14–26.

[4] I. Gohberg and M. Krein, *Introduction to the theory of linear nonselfadjoint operators*. Amer. Math. Soc., Transl. Math. Monogr. **18**, Providence, 1969.

[5] T. Kato, *Perturbation of continuous spectra by trace class operators*. Proc. Japan Acad. **33**(1957), 260–264.

[6] ———, *Perturbation theory for linear operators*. Springer-Verlag, New York, 1976.

[7] M. Rosenblum, *Perturbations of continuous spectrum and unitary equivalence*. Pacific J. Math. **7**(1957), 997–1010.

[8] M. Reed and B. Simon, *Methods of modern mathematical physics, III, Scattering theory*. Academic Press, New York, 1979.

[9] D. Voiculescu, *Some results on norm-ideal perturbations of Hilbert space operators*. J. Operator Theory **2**(1979), 3–37.

[10] ———, *Some results on norm-ideal perturbations of Hilbert space operators. II*. J. Operator Theory **5**(1981), 77–100.

[11] ———, *On the existence of quasicentral approximate units relative to normed ideals. Part I*. J. Funct. Anal. **91**(1990), 1–36.

[12] J. Xia, *An analogue of the Kato-Rosenblum theorem for commuting tuples of self-adjoint operators*. Comm. Math. Phys. **198**(1998), 187–197.

[13] ———, *Trace-class perturbation and strong convergence: wave operators revisited*. Proc. Amer. Math. Soc. **128**(2000), 3519–3522.

Department of Mathematics
State University of New York at Buffalo
Buffalo, New York 14260
USA
e-mail: jxia@acsu.buffalo.edu