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Strong and Extremely Strong Ditkin sets for the Banach Algebras $A_p^r(G) = A_p \cap L^r(G)$

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Abstract. Let $A_p(G)$ be the Figa-Talamanca, Herz Banach Algebra on G; thus $A_2(G)$ is the Fourier algebra. Strong Ditkin (SD) and Extremely Strong Ditkin (ESD) sets for the Banach algebras $A_p^r(G)$ are investigated for abelian and nonabelian locally compact groups G. It is shown that SD and ESD sets for $A_p(G)$ remain SD and ESD sets for $A_p^r(G)$, with strict inclusion for ESD sets. The case for the strict inclusion of SD sets is left open.

A result on the weak sequential completeness of $A_2(F)$ for ESD sets F is proved and used to show that Varopoulos, Helson, and Sidon sets are not ESD sets for $A_2(G)$, yet they are such for $A_2^r(G)$ for discrete groups G, for any $1 \leq r \leq 2$.

A result is given on the equivalence of the sequential and the net definitions of SD or ESD sets for σ-compact groups.

The above results are new even if G is abelian.

1 Introduction

Let X be a locally compact space and $C(X)$ $[C_0(X)]$ $\{C_c(X)\}$, denote the complex bounded continuous functions [which tend to 0 at infinity] {which have compact support} respectively. For $v \in C(X)$, let $spt v = cl\{x; v(x) \neq 0\}$, where cl denotes closure.

Let $\mathbb{A}(X)$ be a subalgebra of $C_0(X)$ with a norm that renders it into a Banach Algebra, see [HR, (39.1)]. If $F \subset X$ is closed, let $I_F = \{v \in \mathbb{A}; v = 0 \text{ on } F\}$ and $J_F^0 = \{v \in \mathbb{A} \cap C_c(X); F \cap spt v = \phi\}$. Denote, for $u \in \mathbb{A}$,

$$\|u\|_{M[F]} = \sup\{\|uv\|_{\mathbb{A}}; v \in I_F, \|v\|_{\mathbb{A}} \leq 1\},$$

the multiplier norm on I_F. If $F = \phi$, the void set, denote by $\|u\|_M = \|u\|_{M[\phi]}$, i.e. the multiplier norm on $\mathbb{A}$. Let $\mathbb{A}(F) = \mathbb{A}(X)/I_F$.

Definition 1.1 (i) The closed set F is a set of synthesis (**S**) if $cl J_F^0 = I_F$.
(ii) F is a Ditkin set (**D**) if for all $v \in I_F$ there exists a net $v_\alpha \in J_F^0$ such that

$$\|vv_\alpha - v\|_{\mathbb{A}} \to 0.$$

(iii) F is a Strong Ditkin (**SD**) [Extremely Strong Ditkin (**ESD**)] set if there exists a net $v_\alpha \in J_F^0$ such that $\|vv_\alpha - v\|_{\mathbb{A}} \to 0$ for all $v \in I_F$ and $\sup \|v_\alpha\|_{M[F]} < \infty$, [$\sup \|v_\alpha\|_M < \infty$], respectively. (This is consistent with [Gi, Ro, Sch].)

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Note that ϕ is a **SD** set (if and only if ϕ is a **ESD** set) if and only if there exists a net $v_\alpha \in \mathbb{A} \cap C_c(X)$, such that $\|vv_\alpha - v\|_{\mathbb{A}} \to 0$ for all $v \in \mathbb{A}$, and $\sup \|v_\alpha\|_M < \infty$, *i.e.*, there exists an approximate identity in $\mathbb{A} \cap C_c(X)$, bounded in multiplier norm on $\mathbb{A}$.

Definition 1.2 Let $F \subset X$ be a closed set.

(i) F is a Helson set for $\mathbb{A}(X)$ if $\mathbb{A}(F) = C_0(F)$.

(ii) F is a Varopoulos set for $\mathbb{A}(X)$ if there exist locally compact spaces $E_1, \dots, E_n$ such that $\mathbb{A}(F) \approx C_0(E_1)\hat{\otimes} \cdots \hat{\otimes} C_0(E_n)$ (projective tensor product), a Banach space isomorphism.

G will denote all through a locally compact group with a left Haar measure $\lambda = dx$.

If G is discrete and amenable, then Helson sets for the Fourier algebra $A(G)$, as in [Ey1], are also called Sidon sets; see [Pic].

The following theorems of Graham [Grh] and of Varopoulos [Va1, Va2], [GMc, 11.8.4] for the Fourier algebra $A(G)$ of the abelian group G are the motivation for the above definition.

Theorem 1.3 *If G is a metrisable abelian locally compact group and $X_1, \dots, X_n$, are perfect compact subsets of G then there are pairwise disjoint perfect subsets $Y_1 \subset X_1, \dots, Y_n \subset X_n$, such that Y_j is either a Kronecker set or a translate of a K_{p_j} set and $A(Y_1 + \cdots + Y_n) \approx C(Y_1)\hat{\otimes} \cdots \hat{\otimes} C(Y_n)$, an isomorphism (see* [Grh]*).*

Thus $Y = Y_1 + \cdots + Y_n$ is a Varopoulos set.

Theorem 1.4 *Let G be abelian discrete and infinite.*

(i) *Then G contains an infinite set F_0, which is a $1/3$-Kronecker set or a K_p-set for some prime $p > 1$.*

(ii) *Let X, Y be countably infinite disjoint subsets of G, such that $F_0 = X \cup Y$. Then $A(X + Y) = C_0(X)\hat{\otimes} C_0(Y)$, an isomorphism (see* [Va2], [GMc, 11.8.4]*).*

Thus $Z = X + Y$ is a Varopoulos set.

For $1 < p < \infty$, $1 \leq r \leq \infty$, denote by $A_p^r(G) = A_p \cap L^r(G)$, with $\|u\|_{A_p^r} = \|u\|_{A_p} + \|u\|_{L^r}$, a Figa-Talamanca, Herz, Lebesgue Banach Algebra on G, as defined and studied in our paper [Gr1]; see history loc. cit. Here $A_p(G)$ is a Figa-Talamanca, Herz algebra on G as in Eymard [Ey2], (thus is $A_{p'}(G)$, a la Herz [Hz], where $1/p + 1/p' = 1$). Hence $A_2(G) = A(G)$ is the Fourier algebra of G as defined and studied in [Ey1]. Let $A_p^\infty(G) = A_p(G)$ equipped with the A_p norm, (this being equivalent to the A_p^∞ norm). Chose and *fix* some p and let $1 \leq r < \infty$. Let **S**, **D**, **SD**, and **ESD** ($\mathbf{S}_r$, $\mathbf{D}_r$, $\mathbf{SD}_r$, and $\mathbf{ESD}_r$) denote the Synthesis, Ditkin, Strong Ditkin, and Extremely Strong Ditkin sets for A_p ($[A_p^r]$, respectively).

We have proved in [Gr1, p. 414] that $\mathbf{S} = \mathbf{S}_r$ under the mild assumption that

$$\text{the empty set } \phi \text{ is an } \mathbf{SD} \text{ set for } A_p. \tag{1.1}$$

We prove in Theorem 2.1 that $\mathbf{D} = \mathbf{D}_r$ under the same assumption (1.1). Thus,

$A_p(G)$ and $A_p^r(G)$ have the same sets of Synthesis and the same Ditkin sets, in this case. This is *not* the case if r is fixed but p varies, as shown, even for abelian G, in [Gr1, p. 414].

We are not able to prove for *arbitrary* G that $\phi \in \mathbf{S}_r$, even though $\phi \in \mathbf{S}$, as is well known.

We consider next the equality question for **SD** sets and **ESD** sets.

Theorem 1.5 (i) *For any group G, $\mathbf{SD} \subset \mathbf{SD}_r$ and $\mathbf{ESD} \subset \mathbf{ESD}_r$.*
(ii) *If G is infinite discrete and is abelian or solvable or type I or FC, and $p = 2$ then*

$$\mathbf{ESD} \neq \bigcap\{\mathbf{ESD}_r; 1 \leq r \leq 2\},$$

*thus there exist infinite subsets F (such as Sidon or Varopoulos sets, see sequel) that are **ESD** for $A_2^r(G)$, for all $1 \leq r \leq 2$, yet they are not **ESD** for $A_2(G)$.*

Haagerup has shown that $G = SL(2, R) \propto R^2$, does not satisfy (1.1) for A_2 (see [Do]), where R denotes the additive reals. We were unable to find a *discrete* group G for which (1.1) does not hold for A_2. Such G would satisfy for $p = 2$ that $\phi \notin \mathbf{SD}$ but $\phi \in \mathbf{SD}_r$ for all $1 \leq r \leq 2$, hence the first inclusion in (i) would also be proper. We conjecture that there exist some G, for which this inclusion is proper.

Note that if G is amenable, and the closed subset F is **ESD** for $A_2(G)$, then F is a closed set in the coset ring $R(G_d)$ of G with the discrete topology, as proved by Cohen, LeFranc, Host, and Forrest (see [Ho] and [Fo1, Prop. 3.5]). This holds since the norm and the multiplier norm on $A_2(G)$ are equivalent when G is amenable.

If G is a free group on 2 generators, it has to contain an infinite Leinert set L, thus $A_2(L) = \ell^2(L)$, $1_M A_2(G) = I_L$, if $M = G \sim L$ and M and L do not belong to $R(G)$, see [Le, FTP]. L is a **ESD** set that is not in $R(G)$. (By Haagerup's result, see [Do], $A_2(G)$ has an approximate identity $v_n \in A_2 \cap C_c$, thus $1_M v_n$ is an approximate identity of I_L bounded in multiplier norm on $A_2(G)$.)

But if G is any discrete group, every subset of G is **ESD** for $A_2^r(G)$ if $1 \leq r \leq 2$, see sequel, hence $R(G)$ plays no role at all in this case.

The main result of Section 3 is the following.

Theorem 1.6 *Let G be a locally compact group and H be an amenable closed subgroup. Let $F \subset H$ be closed. If F is **ESD** for $A_2(G)$, then $A_2(F)$ (hence any of its closed subspaces) is a weak sequentially complete w.s.c. Banach space.*

*Thus infinite closed subsets F of H that are Varopoulos sets for $A_2(G)$ are not **ESD** sets for $A_2(G)$.*

The last part substantially improves a result mentioned in [DuR, p. 59] for **ESD** sets in discrete abelian G.

Furthermore, using a result of De Michele and Soardi [MS] for the FC case and [Ru, Rob, Tho], we get a result that improves Theorem 2.3(ii).

Corollary 1.7 *Let G be a discrete group and H be a subgroup which is abelian or solvable or FC or type I. Then every infinite subset of H contains an infinite Sidon subset F and as such, is not **ESD** for $A_2(G)$, but is **ESD** for $A_2^r(G)$, for all $1 \leq r \leq 2$.*

Question Can one assume only that H is amenable?

As far as **SD** sets for *discrete* groups G are concerned we show in Proposition 3.7 that every subset F of G is **SD** for $A_p^r(G)$ for all $1 \leq r \leq \infty$, provided ϕ is **SD** for $A_p(G)$.

Section 4 is concerned mostly with the abelian case. As an application of Theorem 2.3(i) we get an improvement of an important result of Saeki [Sa] namely

Theorem 1.8 *Let G be a locally compact abelian group and F be a nowhere dense closed subset. If F is **SD** set for $A_2(G)$, then F is a **ESD** set for $A_2^r(G)$ for all $1 \leq r \leq \infty$.*

Recall that the closed interval $[0, 1]$ *is* **SD**, but is *not* **ESD** for $A_2(R)$, see [Ru, Ro, GMc].

The main result in Section 4 is the following.

Theorem 1.9 *Let G be an abelian, metrisable, locally compact group. Let F be a compact scattered subset of G and $1 \leq r \leq \infty$. If $A_p^r(F)$ is w.s.c., then F is finite.*

Corollary 1.10 *Let G be a metrisable locally compact group and H be a closed abelian subgroup. Let $F \subset H$ be a compact scattered subset. If F is a **SD** for $A_2(G)$, then F is finite.*

In Section 5 we prove that the *sequential* and the *net* definitions of **SD** [or **ESD**] sets are equivalent, provided the group G is σ*-compact*.

2 Strong and Extremely Strong Ditkin Sets

Let $1 < p < \infty$ *be fixed*. If F is a closed subset of G, let $I_F^r = \{v \in A_p^r; v = 0 \text{ on } F\}$, for $1 \leq r \leq \infty$. If $r = \infty$, let $I_F^r = I_F$. Denote by, $J_F^0 = \{v \in A_p \cap C_c; F \cap sptv = \phi\}$, thus $J_F^0 \subset I_F^r$ if $1 \leq r \leq \infty$. For $u \in A_p^r$, denote by

$$\|u\|_{M^r[F]} = \sup\{\|uv\|_{A_p^r}; v \in I_F^r, \|v\|_{A_p^r} \leq 1\}$$

and if $F = \phi$, the void set, then denote by $\|u\|_{M^r[\phi]} = \|u\|_{M^r}$, the multiplier norm on A_p^r. Furthermore if $r = \infty$, omit r and denote for example $\|u\|_{M^r} = \|u\|_M$ (the multiplier norm on A_p, etc. ...)

In the sequel we will study **D**, **SD**, and **ESD** sets for the algebras $A_p^r(G)$, $1 \leq r \leq \infty$. To clarify these notions we note the following:

(1) The closed interval $[a, b]$ for $a < b$ is **SD** but not **ESD** for $A_2(R)$, see [Ro, p. 188]. However, $F = [a, b] \times \{0\} \subset R^2$ is **D** but not **SD** for $A_2(R^2)$, see [Ru, 7.5.2] and [Ro, p. 187] or [GMc, p. 73], for much more.

(2) If G is a *discrete abelian* group, then infinite Sidon sets are **SD** but not **ESD** for $A_2(G)$ (see [DuR, p. 59]), yet they are **ESD** for $A_2^r(G)$ for all $1 \leq r \leq 2$. We prove in the sequel that the above and substantially more holds true.

(3) In the context of arbitrary G, clearly

ϕ is a **SD** set for $A_p(G)$ if and only if there exists a net $v_\alpha \in A_p \cap C_c$ such that $\|vv_\alpha - v\|_{A_p} \to 0$ for all $v \in A_p(G)$ and $\sup \|v_\alpha\|_M < \infty$.

(a) This is the case if G is amenable. This is also the case for $p = 2$ if G is the free group on $n > 1$ generators and for many more nonamenable groups (see [dCH] and [Do, p. 709]).

(b) If $G = SL(n, R^n) \propto R^n$, and $n > 1$, then the void set ϕ is not **SD** for the Fourier algebra $A_2(G)$ by Haagerup's result, see [Do].

Theorem 2.1 *Let G be arbitrary, F a closed subset and $1 \le r < \infty$. Assume that ϕ is an **SD** set for $A_p(G)$. Then F is a **D** set for $A_p(G)$ if and only if F is a **D** set for $A_p^r(G)$.*

Proof Let F be a **D** set for A_p. Let $u_\alpha \in A_p \cap C_c$ be an approximate identity for A_p such that $\|u_\alpha\|_M \le B < \infty$. Let $v \in I_F \cap C_c$. There exist $v_n \in J_F^0$ such that $\|v_n v - v\|_{A_p} \to 0$. But then $\|v_n v - v\|_{L^r} \to 0$, since $v \in C_c$. Thus $\|v_n v - v\|_{A_p^r} \to 0$.

Assume now that $v \in I_F^r$. By [Gr1, Cor. 2], A_p^r has an approximate identity $e_\alpha \in A_p \cap C_c$.

Let $\in > 0$ and α_0 satisfy $\|ve_\alpha - v\|_{A_p^r} < \in /2$ if $\alpha \ge \alpha_0$. Since $ve_{\alpha_0} \in I_F \cap C_c$, let $v_n \in J_F^0$ satisfy $\|v_n ve_{\alpha_0} - ve_{\alpha_0}\|_{A_p^r} \to 0$, by the above. Chose m such that

$$\|v_m ve_{\alpha_0} - ve_{\alpha_0}\|_{A_p^r} < \ \in /2.$$

Thus $\|v - v_m e_{\alpha_0} v\|_{A_p^r} < \in$. Hence F is a **D** set for $A_p^r(G)$.

Assume now that F is a **D** set for A_p^r. If $v \in I_F \cap C_c \subset A_p^r$, there are $v_n \in J_F^0$ such that $\|v_n v - v\|_{A_p} \le \|v_n v - v\|_{A_p^r} \to 0$. If now $v \in I_F$ and $\in > 0$, let α_0 satisfy $\|vu_{\alpha_0} - v\|_{A_p} < \in /2$.

Let $v_0 \in J_F^0$ satisfy $\|vu_{\alpha_0} v_0 - vu_{\alpha_0}\|_{A_p} < \in /2$. Then $\|v - vu_{\alpha_0} v_0\|_{A_p} < \in$ and $u_{\alpha_0} v_0 \in J_F^0$. (We only used that ϕ is a **D** set for A_p in this part.) ■

Remark (i) This theorem has been proved in the case where F is a single point in [Bu1].

(ii) It does not seem to be known whether for $A_2(R^N)$, a set F is an **S** set if and only if it is a **D** set (see [RS, (2.5.5)]).

Lemma 2.2 *For any closed subset F of G and $u \in I_F$, $\|u\|_{M^r[F]} \le \|u\|_{M[F]}$, for all $1 \le r < \infty$. In particular $\|u\|_{M^r} \le \|u\|_M$.*

Proof If $u \in I_F$, then $\|u\|_\infty \le \|u\|_{M[F]}$. Since $x \notin F$, let V be a neighborhood of x such that its closure $\bar{V}$ is compact and disjoint from F. There is some $v \in A_p$ such that $sptv \subset V$ and $1 = v(x) = \|v\|_{A_p}$. Then

$$|u(x)| = |u(x)v(x)| \le \|uv\|_{A_p} \le \|u\|_{M[F]} \|v\|_{A_p} = \|u\|_{M[F]}.$$

Hence if $u \in I_F$, $v \in I_F^r$, then

$$\begin{aligned} \|uv\|_{A_p^r} &= \|uv\|_r + \|uv\|_{A_p} \le \|u\|_\infty \|v\|_r + \|u\|_{M[F]} \|v\|_{A_p} \\ &\le \|u\|_{M[F]} \left(\|v\|_r + \|v\|_{A_p} \right) = \|u\|_{M[F]} \|v\|_{A_p^r}. \end{aligned}$$

Hence $\|u\|_{M^r[F]} \le \|u\|_{M[F]}$. ■

Let G be discrete. Then G is FC if and only if it has finite conjugacy classes. If G is countable, then G is Type I if and only if G is an extension of a finite group by an abelian one, see [Dix, 13.11.12]. All of these are amenable groups, see [MS].

Theorem 2.3 *Let G be arbitrary, F be a closed subset ,and $1 \le r < \infty$. Then,*

(i) *if F is **SD** [F is **ESD**] for $A_p(G)$ then F is **SD** [F is **ESD**] for $A_p^r(G)$, respectively;*

(ii) *let G be discrete and contain an abelian or solvable or type I or a FC group H. Let $p = 2$. Then every infinite subset E of H contains an infinite subset F, such that $A_2(F) = C_0(F)$. The set F, is **ESD** for $A_2^r(G)$, for all $1 \le r \le 2$, but is not **ESD** for $A_2(G)$.*

Proof (i) Let, $v_\alpha \in J_F^0$ satisfy $\|v_\alpha v - v\|_{A_p} \to 0$ for all $v \in I_F$ and $\sup \|v_\alpha\|_{M[F]} = B$, [$\sup \|v_\alpha\|_M = B$]. Then $\|v_\alpha\|_\infty \le B$. Hence $\|v_\alpha - 1\|_\infty \le 1 + B$. If $v \in I_F^r$ and $\in > 0$, let $K \subset G$ be compact and satisfy $(1+B)^r \int_{G\sim K} |v|^r \, d\lambda < \in$. Chose α_0 such that $\left(\|(v_\alpha - 1)v\|_{A_p}\right)^r < \in / \lambda(K)$ if $\alpha \ge \alpha_0$. Hence

$$\int |(v_\alpha - 1)v|^r \le \int_K |(v_\alpha - 1)v|^r + (1+B)^r \int_{G\sim K} |v|^r < 2\in \text{ if } \alpha \ge \alpha_0.$$

Thus $\|v_\alpha v - v\|_{A_p^r} \to 0$ for all $v \in I_F^r$.

But by the above lemma $\|u\|_{M^r[F]} \le \|u\|_{M[F]}$, $\left[\|u\|_{M^r} \le \|u\|_M\right]$, respectively. Hence F is a **SD** set [**ESD** set] for A_p^r, respectively.

(ii) In such discrete amenable groups H, E has to contain an infinite Sidon subset F (*i.e.*, such that $A_2(F) = C_0(F)$) by [Ru, Rob, Tho], and especially [MS]. And such infinite sets F satisfy the above; see Corollary 3.6 for much more. ■

Question Do there exist groups G and subsets F that are **SD** for $A_p^r(G)$, for some $1 \le r < \infty$, but are not **SD** for $A_p(G)$?

Conjecture There exists a discrete group G such that ϕ is not a **SD** set for $A_2(G)$ (a candidate for such G could be $SL(2,Z) \propto Z^2$, see [HaKr], and compare with Miao [Mi]). Since ϕ is a **SD** set for $A_2^r(G)$ for $1 \le r \le 2$, a proof of this conjecture would show that the above question has an affirmative answer.

3 Subsets of G that are not ESD

At times we denote $A_2(G)$ by $A(G)$. The main tool in this section is the following lemma.

Lemma 3.1 *Let G be any locally compact group and $F \subset G$ be a closed subset. If $1_F \in B(G_d)$, then $A(F)$ (hence any of its closed subspaces) is a weak sequentially complete (w.s.c) Banach space.*

Proof If $v_i \in I_F$ and $u \in A(G)$, then $1_F(u + v_1) = 1_F(u + v_2)$. If $w \in A(F)$, there is some $w' \in A(G)$ such that $w'(x) = w(x)$ for all $x \in F$. Define by $e_F(w) \in B(G_d)$, the function $e_F(w) = 1_F w'$. Clearly $e_F \colon A(F) \to A(G)1_F \subset B(G_d)$ is linear, one-to-one, and onto. But

$$\|e_F(w)\|_{B(G_d)} = \|1_F w'\|_{B(G_d)} \le \|1_F\|_{B(G_d)} \|w\|_{A(F)} \quad \forall w \in A(F). \tag{3.1}$$

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Since for all $v \in I_F$, $1_F(w' + v) = 1_F w'$ thus

$$\|1_F w'\|_{B(G_d)} = \|1_F(w' + v)\|_{B(G_d)} \leq \|1_F\|_{B(G_d)} \|w' + v\|_{A(G)}.$$

By taking inf. over all $v \in I_F$, we get (3.1). We have used above that for all $u \in A(G)$, $\|u\|_{A(G)} = \|u\|_{B(G)} = \|u\|_{B(G_d)}$, by [Ey1, (2.24), pp. 202, 208]. Hence $e_F \colon A(F) \to A(G)1_F \subset B(G_d)$ is $1-1$ continuous and onto.

We will show that $A(G)1_F$ is a norm closed subspace of the w.s.c. Banach space $B(G_d)$ (as any predual of any W^* algebra by [Tak, p. 148, Cor. 5.2]. But closed subspaces of w.s.c. Banach spaces are also w.s.c. , see below. It will hence follow that $A(F)$ is w.s.c..

Let $\{u_n 1_F\}$ be a norm Cauchy sequence in $A(G)1_F$ hence in $B(G_d)$, with $u_n \in A(G)$. Clearly $I_F = \{v \in A(G), v = 0 \text{ on } F\}$ is a norm closed subspace of $A(G)$ hence of $B(G_d)$ (it may not be an ideal in $B(G_d)$). Let $t \colon B(G_d) \to B(G_d)/I_F$ denote the quotient map into the quotient Banach space $B(G_d)/I_F$. Now $t(u_n 1_F) \in A(G)/I_F \subset B(G_d)/I_F$ thus, for some $u \in A(G)/I_F = A(F)$, $t(u_n 1_F) \to u$, in the norm of $A(F)$. Let $u' \in A(G)$ satisfy $t(u') = u$. Then $u'(x) = u(x)$ for all x in F and $e_F(u) = 1_F u' \in A(G)1_F$. But $\{u_n 1_F\}$ is norm Cauchy in $B(G_d)$, hence $u_n 1_F \to w$ in the norm of $B(G_d)$, for some $w \in B(G_d)$. But $\|v\|_\infty \leq \|v\|_{B(G_d)}$. Hence $u_n 1_F(x) \to w(x)$ for all x in G. But if $x \in F$, $t(u_n 1_F)(x) = u_n(x) \to u(x)$, since $t(u_n) \in A(G)/I_F$ and $\|v\|_\infty \leq \|v\|_{A(F)}$, for $v \in A(F)$. Hence $e_F(u) = u' 1_F = w$, and $w \in A(G)1_F$.

Any closed subspace Y of a w.s.c. Banach space X is w.s.c. Since a weak Cauchy sequence in Y is such in X, it hence converges weakly to some element of X, which has to belong to Y. ∎

We recall the following results of Herz [Hz1, p. 92], which are needed for the next lemma.

Let H be a closed subgroup of the locally compact group G. Then :

(i) Restriction of functions from G to H is a contraction from $A_p(G)$ onto $A_p(H)$.

(ii) For all $h \in A_p(H)$ and $\in > 0$ $\exists g \in A_p(G)$, such that $g = h$ on H and $\|g\|_{A_p(G)} \leq \|h\|_{A_p(H)} + \in$.

(If $p = 2$, one may take $\|g\|_{A_p(G)} = \|h\|_{A_p(H)}$).

Lemma 3.2 *Let H be a closed subgroup of G and $F \subset H$ be a closed subset. If F is* ***SD [ESD]*** *for $A_p(G)$, then it is* ***SD [ESD]*** *for $A_p(H)$, respectively.*

Proof Denote by

$$I_F^G = \{f \in A_p(G); f = 0 \text{ on } F\}, \quad I_F^H = \{f \in A_p(H); f = 0 \text{ on } F\}, \quad \text{and}$$

$$J_F^{0G} = \{f \in A_p \cap C_c(G); F \cap spt\, f = \phi\}, \; J_F^{0H} = \{f \in A_p \cap C_c(H); F \cap spt\, f = \phi\}.$$

Let $u_\alpha \in J_F^{0G}$ satisfy $\|u_\alpha w - w\|_{A_p(G)} \to 0$ for all $w \in I_F^G$. Let $v_\alpha = u_\alpha$ on H. Then $v_\alpha \in J_F^{0H}$, since $F \cap spt v_\alpha \subset F \cap spt u_\alpha = \phi$. If $v \in I_F^H$, let $u \in A_p(G)$ extend v. Then $u \in I_F^G$ and $\|v_\alpha v - v\|_{A_p(H)} \leq \|u_\alpha u - u\|_{A_p(G)} \to 0$.

Let $\in> 0$ and $v \in I_F^H$, $\|v\|_{A_p(H)} \leq 1$. Let $u \in A_p(G)$, $\|u\|_{A_p(G)} < 1+\in$, extend v to G. Then

$$\|v_\alpha v\|_{A_p(H)} \leq \|u_\alpha u\|_{A_p(G)} \leq \|u_\alpha\|_{M^G[F]}(1+\in).$$

Thus $\|v_\alpha\|_{M^H[F]} \leq \|u_\alpha\|_{M^G[F]}$, for arbitrary closed F, in particular for $F = \phi$. ∎

The main result of this section is the following theorem.

Theorem 3.3 *Let $H \subset G$ be a closed amenable subgroup of the locally compact group G. Let $F \subset H$ be closed. If F is **ESD** for $A_2(G)$ then $A_2(F)$ (hence any of its closed subspaces) is a w.s.c. Banach space.*

Proof By Lemma 3.2, F is **ESD** for $A_2(H)$ and H is amenable, hence the multiplier norm and the actual norm coincide on $A_2(H)$. Thus by [Fo1, Lemma 3.3], $1_F \in B(H_d)$. By our Lemma 3.1, $A_2^H(F) = A_2(H)/I_F^H$ is w.s.c. But $A_2(G)/I_F^G = A_2^H(F)$, as a set of functions. Let $v \in A_2(H)$, $w \in I_F^H$, and let $u, w_1 \in A_2(G)$ extend v, w to G. Then $\|v + w\|_{A_2(H)} \leq \|u + w_1\|_{A_2(G)}$. Hence $\|v\|_{A_2^H(F)} \leq \|u\|_{A_2(F)}$. By the open mapping theorem, $A_2^H(F)$ and $A_2(F)$ are isomorphic. Thus $A_2(F)$ is w.s.c. ∎

Corollary 3.4 *Let $H \subset G$ be a closed subgroup that is amenable and $F \subset H$ be closed. If F is an infinite Varopoulos set for $A_2(G)$, then F is not an **ESD** set for $A_2(G)$.*

Proof By definition $A_2(F) \approx C_0(E_1)\hat{\otimes} \cdots \hat{\otimes} C_0(E_n)$, an isomorphism. If the right hand side is w.s.c. so is every subspace and in particular so is each $C_0(E_i)$. But then the identity $I\colon C_0(E_i) \to C_0(E_i)$ is a weakly compact operator, by [DS, Theorem VI.7.6] Hence I^2 is a compact operator, by [DS, Theorem VI.7.5. p. 494], and $C_0(E_i)$ is finite dimensional and F is finite. ∎

Remark The above is a vast improvement of [DuR, (5.5.5), p. 59] who mentioned it for Sidon subsets of discrete abelian groups.

Example Let $G = Z$ denote the additive integers. Let $F \subset Z$ be an infinite $1/3$-Kronecker set, see [Va1, Va2], or [GMc, 11.8.4]. For *any* decomposition of F into disjoint infinite subsets as $F = X \cup Y$, the set $X + Y$ is a Varopoulos set, and as such, is not an **ESD** set for $A_2(Z)$, yet it is an **ESD** set for $A_2^r(Z)$, for all $1 \leq r \leq 2$. Z can be replaced by any discrete abelian group G.

Proposition 3.5 *Let G be discrete and $1 \leq r \leq \max(p, p')$. Then every subset F of G is **ESD** for $A_p^r(G)$.*

Proof If $G \sim F = E$ is finite, let $e_\alpha = 1_E$. Then for all $v \in I_F^r$, $e_\alpha v = v$, and $\|e_\alpha\|_{M^r} \leq \|1_E\|_{A_p^r} < \infty$. Thus F is **ESD** for A_p^r. If E is infinite, it is proved in [Gr1, Theorem 7] that $A_p^r(G) = \ell^r(G)$, with norm equivalence, for the above r. Let $E_\alpha \subset E$ be a net of finite sets such that $E_\alpha \uparrow E$ and let $e_\alpha = 1_{E_\alpha}$.

If $v \in I_F^r$, then

$$\|e_\alpha v - v\|_{A_p^r} \leq d\|e_\alpha v - v\|_{\ell^r(G)} = \|1_{E-E_\alpha} v\|_{\ell^r(G)} \to 0.$$

And, $\|e_\alpha v\|_{\ell^r(G)} \leq \|v\|_{\ell^r(G)}$ thus $\|e_\alpha\|_{M^r} \leq d'$, where d, d' are constants. ∎

Corollary 3.6 *Let G be a discrete group. Assume that G contains an infinite subgroup H that is abelian (or solvable, or type I or FC). Then every infinite subset E of H contains an infinite subset F that is* **ESD** *for $A_2^r(G)$, for all $1 \leq r \leq 2$ but is not* **ESD** *for $A_2(G)$.*

Proof As mentioned before E contains an infinite Sidon set F for $A_2(H)$, by [Ru, Rob, Tho, MS]. Hence $A_2(G)$ restricted to F coincides with $C_0(F)$, which is not w.s.c. F is a **ESD** set for $A_2^r(G)$, for all $1 \leq r \leq 2$, by Proposition 3.5. ∎

Question Does there exist an infinite discrete group G for which all subsets are **ESD** for $A_2(G)$?

Proposition 3.7 *Let G be discrete and satisfy that ϕ is* **SD** *for $A_p(G)$ (amenable groups or, free groups for $p = 2$, are such). Then any subset F of G is* **SD** *for $A_p^r(G)$ for all $1 \leq r \leq \infty$.*

Proof By [Gr1, Cor. 2] there exists a net $e_\alpha \in A_p$, with finite support, such that $\|e_\alpha v - v\|_{A_p^r} \to 0$ for all $v \in A_p^r$ and $\sup \|e_\alpha\|_{M^r} = B < \infty$. Let $E = G \sim F$ and, $u_\alpha = e_\alpha 1_E$. Then $u_\alpha \in A_p^r(G)$, since it has finite support. Clearly $u_\alpha \in I_F^r$ and $u_\alpha v = e_\alpha v$ for all $v \in I_F^r$. Hence, for $v \in I_F^r$,

$$\|u_\alpha v - v\|_{A_p^r} = \|e_\alpha v - v\|_{A_p^r} \to 0 \quad \text{and} \quad \|u_\alpha v\|_{A_p^r} = \|e_\alpha v\|_{A_p^r} \leq B\|v\|_{A_p^r}.$$

Thus $\|u_\alpha\|_{M^r[F]} \leq B < \infty$. ∎

4 Some Results for the Abelian Case

Theorem 2.3 allows one to improve a powerful result of Saeki [Sa], which definitively improves [Ro, Thm. 1.3]. Saeki's result, namely the $r = \infty$ case, is assumed in the proof.

Theorem 4.1 *Let G be a locally compact abelian group and F be a closed nowhere dense subset. If F is a* **SD** *set for $A_2(G)$, then F is an* **ESD** *set for $A_2^r(G)$, for all $1 \leq r \leq \infty$.*

Proof By Saeki [Sa] this holds for $A_2(G)$ *i.e.*, for $r = \infty$. Now apply Theorem 2.3(i). ∎

Remark Recall that $[0, 1]$ is **SD** but not **ESD** for $A_2(R)$. Meyer and Rosenthal have proved that a polygon P in R^2 is never **SD**. And the closure of its exterior is **SD** if and only if P is convex, and analogously in R^n, see [GMc, p. 73].

Theorem 4.2 *Let G be an amenable SIN group, see* [HR], *and H be a closed abelian subgroup. Let $F \subset H$ be a closed, nowhere dense subset of H. If F is* **SD** *for $A_2(G)$, then F is* **ESD** *for $A_2^r(G)$, for all $1 \leq r \leq \infty$.*

Proof By Lemma 3.2, F is **SD** for $A_2(H)$. Hence by Theorem 4.1, with $r = \infty$ (*i.e.*, Saeki's result) F is **ESD** for $A_2(H)$. Thus F is a closed set in $R(H_d)$, the ring generated by cosets of all subgroups of H_d, see [Sch]. Thus $F \in R(G_d)$. But then by Forrest

[Fo2, Theorem 3.11], I_F has an an approximate identity, bounded even in the norm of $A_2(G)$. Since F is a set of synthesis, (loc. cit.), the approximate identity can be chosen in J_F^0. Thus F is **ESD** for $A_2(G)$.Now apply Theorem 2.3(i). ■

In a sense, the next result complements and improves Theorem 3.3 in the case where G is a locally compact abelian group.

Let G be an *abelian metrisable* locally compact group. If F is a compact and scattered (*i.e.,* countable) subset of G, then F might be a Helson set for $A_2(G)$ in which case $A_2(F) = C(F)$ is not w.s.c. unless F is finite.

However there exist countable and compact subsets F of R, which are not Helson sets, see [Ru, (5.6.7), (5.6.8)]. Can $A_2(F)$ be w.s.c.? The next result shows that the answer is vastly NO.

Theorem 4.3 *Let G be an abelian, metrisable, locally compact group. Let F be a compact scattered subset of G and $1 \le r \le \infty$.*

If $A_p^r(F)$ is w.s.c., then F is finite.

Proof By [Gr1, Lemma 2], $A_p^r(F)$ and $A_p(F)$ are isomorphic Banach algebras. We will prove the result for $A_p(F)$. Now $A_p = A_{p'}$, since G is abelian ([Hz2, p. 72]), so let $1 < p \le 2$.

F is scattered and hence is a set of synthesis by [Ey2, p. 10]. Hence in the notation of [Lu2], $CV_p(F) = A_p(F)^*$. By [Lu2, Theorem 2.8], $A_p(F)^*$ has the RNP afortiori has the WRNP, if $1 < p \le 2$, see [Saa2, p. 422]. Hence by [Saa1, Theorem 1], by taking there the set K as the closed unit ball in $A_p(F)^*$, we get that every bounded sequence in $A_p(F)$ has a weak Cauchy subsequence.

If F is infinite, let x_0 be a nonisolated point of F. Let V_n be a neighborhood base at x_0 such that $\bar{V}_n$ is compact. Let $u_n \in A_p(G)$ be a tenting sequence at x_0, *i.e.,* $\|u_n\|_{A_p} = 1 = u_n(x_0)$, and $u_n = 0$ off $\bar{V}_n$. Let $v_n \in A_p(F)$, satisfy $v_n(x) = u_n(x)$ for $x \in F$.

There exists a subsequence of v_n, again denoted by v_n, which is weak Cauchy. If $A_p(F)$ is w.s.c., there is some $v \in A_p(F)$ such that $(\phi, v_n) \to (\phi, v)$, for all $\phi \in A_p(F)^*$. But $M(F) \subset A_p(F)^*$. Hence $v_n(x) \to v(x)$, for all $x \in F$. Thus $v(x_0) = 1$ and $v(x) = 0$ if $x \ne x_0$. But $v \in C(F)$, which cannot be. Thus $A_p(F)$ is not w.s.c. ■

Corollary 4.4 *Let G be a metrisable locally compact group and H be a closed abelian subgroup. Let $F \subset H$ be a compact scattered subset. If F is a* ***SD*** *set for $A_2(G)$, then F is finite.*

Proof By Lemma 3.2, F is **SD** for $A_2(H)$. By Saeki's part of Theorem 4.1, F is **ESD** for $A_2(H)$. By Theorem 3.3, $A_2(H)/I_F^H$ is w.s.c., where $I_F^H = \{u \in A_2(H); u = 0 \text{ on } F\}$. Hence by Theorem 4.3, F is finite. ■

Remark One cannot replace the condition that "F is scattered" by "F is nowhere dense". Since if $G = T^2 = \{(e^{ix}, e^{iy}); 0 \le x, y < 2\pi\}$, then $(T, 1) = \{(e^{ix}, 1); 0 \le x < 2\pi\}$, is a closed infinite subgroup, hence is an **ESD** set, which is nowhere dense in G. See Rosenthal [Ro, pp. 187–190] for more.

5 When Sequences are Enough

The definition of **SD** and **ESD** sets in Gilbert [Gi], Rosenthal [Ro], and Schreiber [Sch] is given in terms of *sequences* of approximate identities (for the Fourier algebra of a locally compact *abelian* group). It is stated in [Ro, p. 191] and in [Sch, p. 811] that their results hold true if **SD** and **ESD** are defined in terms of nets, as done in this paper.

For which groups are these definitions equivalent? The next result shows that this equivalence, and more, holds for the case of σ-*compact* groups and for all the algebras $A_p^r(G)$.

Theorem 5.1 *Let G be a σ-compact locally compact group, F be a closed subset, and $1 \le r \le \infty$. If F is **SD** [**ESD**] for $A_p^r(G)$, then there exists a sequence $v_n \in J_F^0$ such that*

$$\|v_n v - v\|_{A_p^r} \to 0 \quad \forall v \in I_F^r \quad \textit{and} \quad \sup \|v_n\|_{M^r[F]} < \infty, \quad \left[\sup \|v_\alpha\|_{M^r} < \infty\right],$$

respectively.

*If a sequence $v_n \in J_F^0$, which is an approximate identity for I_F^r, exists, then F is a **SD** set but is not necessarily a **ESD** set for A_p^r.*

Proof Let F be a **SD** [**ESD**] set and let $v_\alpha \in J_F^0$ satisfy $\|v_\alpha v - v\|_{A_p^r} \to 0$ for all $v \in I_F^r$ and $\sup \|v_\alpha\|_{M^r[F]} < \infty$, [$\sup \|v_\alpha\|_{M^r} = B < \infty$], respectively.

Since G is σ-*compact*, so is $G \sim F$. Hence let K_n be compact, and let $U_n = U_n^{-1}$ be neighborhoods of e satisfying that $\bar{U}_n$ is compact and $K_n U_n^2 \subset K_{n+1}$ and $\bigcup K_n = G \sim F$. Let $h_n = \lambda(U_n)^{-1} 1_{K_n U_n} * 1_{U_n}$. Then $h_n = 1$ on K_n and $h_n = 0$ off $K_n U_n^2$. Let v_{α_n} satisfy $\|v_{\alpha_n} h_n - h_n\|_{A_p^r} \le 1$ and let $w_n = v_{\alpha_n} + h_n - v_{\alpha_n} h_n$. Then

$$\|w_n\|_{M^r[F]} \le 1 + \|v_{\alpha_n}\|_{M^r[F]} \le 1 + B,$$

[$\|w_{\alpha_n}\|_{M^r} \le 1 + B$], respectively. Thus $w_n = 1$ on K_n and $w_n = v_{\alpha_n}$ off $K_n U_n^2$. Hence $w_n \in J_F^0$.

Assume now that $v \in J_F^0$. Then there is some n_0 such that $sptv \subset K_n$ if $n \ge n_0$. Thus $w_n v - v = 0$ if $n \ge n_0$.

Let $v \in I_F^r$ be arbitrary and $\in > 0$. F is a synthesis set, hence there is some $u \in J_F^0$ such that $\|v - u\|_{A_p^r} < \in$. Hence

$$\begin{aligned}\|w_n v - v\|_{A_p^r} &\le \|w_n(v-u)\|_{A_p^r} + \|w_n u - u\|_{A_p^r} + \|u - v\|_{A_p^r} \\ &< (1+B) \in + \in + \|w_n u - u\|_{A_p^r}.\end{aligned}$$

But there is some n_0 such that $\|w_n u - u\|_{A_p^r} = 0$ if $n > n_0$. Thus $\|w_n v - v\|_{A_p^r} \to 0$ for all $v \in I_F^r$. This proves the first part.

Let G be abelian, discrete, and countable and let F be an infinite Sidon set for $A = A_2(G)$. Then F is a **SD** set but not an **ESD** set for A. Hence there exists a sequence $v_n \in J_F^0$ such that $\|v_n v - v\|_A \to 0$ for all $v \in I_F$. Thus $\sup \|v_n\|_{M[F]} < \infty$. But there is no net $v_\alpha \in J_F^0$ such that $\|v_\alpha v - v\|_A \to 0$ for all $v \in I_F$ and $\sup \|v_\alpha\|_M < \infty$ (note that $\|u\|_M = \|u\|_A$), since F is not an **ESD** set for A. ∎

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Remark Only the σ-*compactness* of $G \sim F$ has been used in the proof. This theorem improves [Gr1, p. 408(iii)].

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