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ON AN INTERESTING PROPERTY OF A COMBINATORIAL FUNCTION

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1. **Introduction.** For any two integers n and k we take, as usual,

$$\binom{n}{k} = \begin{cases} \dfrac{n!}{k!(n-k)!}, & 0 \le k \le n \\ 0, & \text{otherwise.} \end{cases}$$

Using the above symbol we define a new function $F(q, 2n)$ by a sum of two finite sums given by

$$F(q, 2n) = \sum_{i=0}^{2n-1} (-1)^{i+\frac{1}{2}q-1} \frac{\binom{2n-1}{i}\binom{n-1}{\frac{1}{2}q-1}}{i-q} + \sum_{r=0}^{n-1} (-1)^{r+q+1} \frac{\binom{2n-1}{q}\binom{n-1}{r}}{q-2r-2} \tag{1.1}$$

for $i \neq q$, $r \neq \frac{1}{2}q-1$ and $\binom{n-1}{\frac{1}{2}q-1}=0$ whenever q is an odd integer.

The function $F(q, n)$ is of frequent occurrence in some transformations of Gauss hypergeometric functions and in the evaluation of probability distribution functions (Anderson [1], Consul [2], Kemp [3]) of likelihood criteria.

We shall prove the following important property of the function $F(q, 2n)$.

2. **Property.** For all integral values of q and n such that $q < 2n$,

$$F(q, 2n) = (-1)^{n-1} F(2n-q, 2n) \tag{2.1}$$

and whenever n is an even integer

$$F(n, 2n) = 0. \tag{2.2}$$

To prove this property we shall first prove two simple lemmas.

LEMMA 1. *For any integer n, $q \neq 2r+2$ and $r \le n-2$*

$$\frac{2n-q}{q-2r-2}\binom{n-1}{r} = -\binom{n}{r+1} + \frac{q}{q-2r-2}\binom{n-1}{r+1}. \tag{2.3}$$

Proof.

$$\frac{2n-q}{q-2r-2}\binom{n-1}{r} = \left[2n\binom{n-1}{r}-q\left\{\binom{n-1}{r}+\binom{n-1}{r+1}\right\}+q\binom{n-1}{r+1}\right](q-2r-2)^{-1}$$

$$= \left[2(r+1)\binom{n}{r+1}-q\binom{n}{r+1}\right](q-2r-2)^{-1}+\frac{q}{q-2r-2}\binom{n-1}{r+1}$$

which gives the result of Lemma 1.

LEMMA 2. *For* $i \le m-1$ *and* $i \ne q$,

$$\frac{q}{i-q}\binom{m-1}{i} = -\binom{m}{i}+\frac{m-q}{i-q}\binom{m-1}{i-1}. \tag{2.4}$$

Proof.

$$\frac{q}{i-q}\binom{m-1}{i} = -\binom{m-1}{i}+\frac{i}{i-q}\binom{m-1}{i}$$

$$= -\left[\binom{m-1}{i}+\binom{m-1}{i-1}\right]+\binom{m-1}{i-1}\left[1+\frac{m-i}{i-q}\right]$$

which gives the result of Lemma 2.

Since the first sum in $F(q, 2n)$ is zero when q is an odd integer, the property in (2.1) will now be proved separately for odd and even integral values of q.

Case I. When q is an odd integer,

$$F(q, 2n) = \binom{2n-1}{q}(-1)^{q+1}\sum_{r=0}^{n-1}(-1)^r\binom{n-1}{r}/(q-2r-2)$$

$$= \binom{2n-1}{2n-q}(-1)^{q+1}\frac{2n-q}{q}\left[\sum_{r=0}^{n-2}(-1)^r\binom{n-1}{r}/(q-2r-2)+\frac{(-1)^{n-1}}{q-2n}\right]$$

which, by using Lemma 1 and some simplification, becomes

$$F(q, 2n) = (-1)^{q+1}\binom{2n-1}{2n-q}\left[\sum_{r=0}^{n-2}\left\{\frac{(-1)^{r+1}}{q}\binom{n}{r+1}+\frac{(-1)^r}{q-2r-2}\binom{n-1}{r+1}\right\}+\frac{(-1)^n}{q}\right].$$

By incorporating q^{-1} and the last term in first summation and rearrangement, the above expression gives

$$=(-1)^{q+1}\binom{2n-1}{2n-q}\left[q^{-1}\sum_{s=0}^{n}(-1)^s\binom{n}{s}+\sum_{r=-1}^{n-2}\frac{(-1)^r}{q-2r-2}\binom{n-1}{r+1}\right]. \tag{2.5}$$

Since the first summation is zero, the expression (2.5) can be put in the form

$$F(q, 2n) = (-1)^{2n-q+1}\binom{2n-1}{2n-q}\sum_{i=0}^{n-1}(-1)^{i-1}\binom{n-1}{i}/(q-2i).$$

Letting $i=n-1-r$ in the above expression, we find that

$$F(q, 2n) = (-1)^{n-1}F(2n-q, 2n).$$

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Case II. When q is an even integer, say, $q=2m$.

$$F(2m, 2n) = (-1)^{m-1}\binom{n-1}{m-1}\sum_{\substack{i=0\\ i\neq 2m}}^{2n-1}\frac{(-1)^i}{i-2m}\binom{2n-1}{i}$$

$$+\binom{2n-1}{2m}\sum_{\substack{r=0\\ r\neq m-1}}^{n-1}\frac{(-1)^{2m+r+1}}{2(m-r-1)}\binom{n-1}{r}$$

(2.6)
$$=\frac{(-1)^{m-1}}{2(n-m)}\binom{n-1}{n-m-1}\sum_{\substack{i=0\\ i\neq 2m}}^{2n-1}\frac{(-1)^i\cdot 2m}{i-2m}\binom{2n-1}{i}$$

$$+\frac{(-1)^{2m+1}}{2m}\binom{2n-1}{2n-2m}\sum_{\substack{r=0\\ r\neq m-1}}^{n-1}\frac{(-1)^r(n-m)}{m-r-1}\binom{n-1}{r}.$$

Now, by using Lemmas 1 and 2 and by some readjustment of terms, expression (2.6) becomes

$$F(2m, 2n) = (-1)^{m-1}\binom{n-1}{n-m-1}\left[(2n-2m)^{-1}\sum_{\substack{i=0\\ i\neq 2m}}^{2n-1}(-1)^{i+1}\binom{2n}{i}\right.$$

$$\left.+\sum_{\substack{i=1\\ i\neq 2m}}^{2n-1}\frac{(-1)^i}{i-2m}\binom{2n-1}{i-1}\right]$$

$$+(-1)^{2m+1}\binom{2n-1}{2n-2m}\left[(2m)^{-1}\sum_{\substack{r=0\\ r\neq m-1}}^{n-1}(-1)^{r+1}\binom{n}{r+1}\right.$$

$$\left.+\sum_{\substack{r=0\\ r\neq m-1}}^{n-2}(-1)^r\frac{\binom{n-1}{r+1}}{2(m-r-1)}\right].$$

By adding particular terms, two of the above summations become zero. Thus the above expression reduces to

$$F(2m, 2n) = (-1)^{m-1}\binom{n-1}{m-n-1}\left[\left\{1+(-1)^{2m}\binom{2n}{2m}\right\}(2n-2m)^{-1}\right.$$

$$\left.+\sum_{\substack{i=0\\ i\neq 2m-1}}^{2n-2}\frac{(-1)^{i+1}}{i+1-2m}\binom{2n-1}{i}\right]$$

(2.7)

$$+(-1)^{2m+1}\binom{2n-1}{2n-2m}\left[\frac{1}{2m}\left\{(-1)^{m-1}\binom{n}{m}-1\right\}\right.$$

$$\left.+\sum_{\substack{r=1\\ r\neq m}}^{n-1}\frac{(-1)^{r-1}}{2m-2r}\binom{n-1}{r}\right]$$

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which, on rearrangement and cancellation of two terms, gives

$$F(2m, 2n) = (-1)^{m-1}\binom{n-1}{n-m-1}\sum_{\substack{i=0\\ i\neq 2m-1}}^{2n-1}\frac{(-1)^{i+1}}{i+1-2m}\binom{2n-1}{i}$$

(2.8)

$$+(-1)^{2m+1}\binom{2n-1}{2n-2m}\sum_{\substack{r=0\\ r\neq m}}^{n-1}\frac{(-1)^{r-1}}{2m-2r}\binom{n-1}{r}.$$

By letting $i=2n-1-j$ and $r=n-1-s$ and with some trivial adjustment the above expression gets transformed into $(-1)^{n-1}F(2n-2m, 2n)$ which establishes the property of the function $F(q, 2n)$.

The proof of (2.2) follows trivially from (2.1).

References

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