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ON MAHLER'S p-ADIC U_m-NUMBERS

GÜLCAN KEKEÇ

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Abstract

The aim of this work is to adapt a construction of the so-called U_m-numbers ($m > 1$), which are extended Liouville numbers with respect to algebraic numbers of degree m but not with respect to algebraic numbers of degree less than m, to the p-adic frame.

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1. Introduction

Let p be a fixed prime number, and let $|\cdot|_p$ denote the p-adic absolute value function on the field $\mathbb{Q}$ of rational numbers. We shall denote the unique extension of $|\cdot|_p$ to the field $\mathbb{Q}_p$ of p-adic numbers, the completion of $\mathbb{Q}$ with respect to $|\cdot|_p$, by the same notation $|\cdot|_p$.

By analogy with his classification of complex numbers, Mahler [3], in 1934, proposed a classification of p-adic numbers. Let $P(x) = a_n x^n + \cdots + a_0$ be a polynomial with rational integral coefficients. The height $H(P)$ of P is defined by $H(P) = \max(|a_n|, \ldots, |a_0|)$, where $|\cdot|$ denotes the usual absolute value function on the field $\mathbb{R}$ of real numbers, and we shall denote the degree of P by $\deg(P)$. Given a p-adic number ξ and natural numbers n and H (recall that a natural number means a positive rational integer), define the quantity (see Bugeaud [2])

$$w_n(H, \xi) = \min\{|P(\xi)|_p : P(x) \in \mathbb{Z}[x], \deg(P) \le n, H(P) \le H \text{ and } P(\xi) \ne 0\}$$

and set

$$w_n(\xi) = \limsup_{H\to\infty} \frac{-\log(H w_n(H, \xi))}{\log H} \quad \text{and} \quad w(\xi) = \limsup_{n\to\infty} \frac{w_n(\xi)}{n}.$$

The inequalities $0 \le w_n(\xi) \le \infty$ and $0 \le w(\xi) \le \infty$ hold. If $w_n(\xi) = \infty$ for some integers n, then $\mu(\xi)$ is defined as the smallest such integer. If $w_n(\xi) < \infty$ for every n, put $\mu(\xi) = \infty$. Hence, $\mu(\xi)$ and $w(\xi)$ are uniquely determined and are never finite simultaneously. Therefore, there are the following four possibilities for the p-adic number ξ. We call ξ:

- a p-adic A-number if $w(\xi) = 0$ and $\mu(\xi) = \infty$;
- a p-adic S-number if $0 < w(\xi) < \infty$ and $\mu(\xi) = \infty$;
- a p-adic T-number if $w(\xi) = \infty$ and $\mu(\xi) = \infty$;
- a p-adic U-number if $w(\xi) = \infty$ and $\mu(\xi) < \infty$.

Every p-adic number ξ is of precisely one of these four types. The p-adic A-numbers are precisely the p-adic algebraic numbers. Let S, T, and U denote the set of p-adic S-numbers, the set of p-adic T-numbers, and the set of p-adic U-numbers, respectively. Then the p-adic transcendental numbers are distributed into the three disjoint classes S, T, and U. Let ξ be a p-adic U-number such that $\mu(\xi) = m$, and let U_m denote the set of all such numbers, that is, $U_m = \{\xi \in U : \mu(\xi) = m\}$. Obviously, the set U_m ($m = 1, 2, 3, \ldots$) is a subclass of U, and U is the union of all the disjoint sets U_m. An element of U_m is called a p-adic U_m-number. (See Bugeaud [2] for more information about Mahler's classification in $\mathbb{Q}_p$.)

Suppose that α is an algebraic number. Let $P(x)$ be the minimal defining polynomial of α such that its coefficients are rational integers and relatively prime, and its highest coefficient is positive. Then the height $H(\alpha)$ of α is defined by $H(\alpha) = H(P)$, and the degree $\deg(\alpha)$ of α is defined as the degree of P.

Given a p-adic number ξ and natural numbers n and H, by analogy with Koksma's classification of complex numbers, define the quantity (see Bugeaud [2] and Schlickewei [5])

$$w_n^*(H, \xi) = \min\{|\xi - \alpha|_p : \alpha \text{ is a } p\text{-adic algebraic number}, \\ \deg(\alpha) \le n, H(\alpha) \le H \text{ and } \alpha \ne \xi\}$$

and set

$$w_n^*(\xi) = \limsup_{H\to\infty} \frac{-\log(H w_n^*(H, \xi))}{\log H} \quad \text{and} \quad w^*(\xi) = \limsup_{n\to\infty} \frac{w_n^*(\xi)}{n}.$$

The inequalities $0 \le w_n^*(\xi) \le \infty$ and $0 \le w^*(\xi) \le \infty$ hold. If $w_n^*(\xi) = \infty$ for some integers n, then $\mu^*(\xi)$ is defined as the smallest such integer. If $w_n^*(\xi) < \infty$ for every n, put $\mu^*(\xi) = \infty$. Hence, $\mu^*(\xi)$ and $w^*(\xi)$ are uniquely determined and are never finite simultaneously. Therefore, there are the following four possibilities for the p-adic number ξ. We call ξ:

- a p-adic A^*-number if $w^*(\xi) = 0$ and $\mu^*(\xi) = \infty$;
- a p-adic S^*-number if $0 < w^*(\xi) < \infty$ and $\mu^*(\xi) = \infty$;
- a p-adic T^*-number if $w^*(\xi) = \infty$ and $\mu^*(\xi) = \infty$;
- a p-adic U^*-number if $w^*(\xi) = \infty$ and $\mu^*(\xi) < \infty$.

Every p-adic number ξ is of precisely one of these four types. Let A^*, S^*, T^*, and U^* denote the set of p-adic A^*-numbers, the set of p-adic S^*-numbers, the set of p-adic T^*-numbers, and the set of p-adic U^*-numbers, respectively. Then the p-adic numbers are distributed into the four disjoint classes A^*, S^*, T^*, and U^*. Let ξ be a p-adic U^*-number such that $\mu^*(\xi) = m$, and let U_m^* denote the set of all such numbers, that is, $U_m^* = \{\xi \in U^* : \mu^*(\xi) = m\}$. Obviously, the set U_m^* $(m = 1, 2, 3, \ldots)$ is a subclass of U^*, and U^* is the union of all the disjoint sets U_m^*. An element of U_m^* is called a p-adic U_m^*-number.

Both classifications are equivalent, that is, the classes A, S, T, and U are the same as the classes A^*, S^*, T^*, and U^*, respectively. Moreover, $U_m = U_m^*$ $(m = 1, 2, 3, \ldots)$ holds. (See Bugeaud [2] for all references and Schlickewei [5].)

The main purpose of this work is to give a new method for constructing p-adic U_m-numbers ξ with upper bounds for $w_n^*(\xi)$, where $n = 1, \ldots, m-1$. In the present work, the method given in [1] is extended to the p-adic case. Our new results are stated in Section 2 and proved in Section 4, and the lemmas we need to prove the main result of this work are given in Section 3.

2. The main result

The following theorem can be regarded as a p-adic version of the theorem in [1], and in the proof of the following theorem, the method given in [1] is extended to the p-adic case.

THEOREM 2.1. *Let $m \geq 2$ be an integer, and let $\epsilon > 0$ be any real number. Then there are uncountably many p-adic U_m-numbers ξ with*

$$w_n^*(\xi) \leq n + m - 1 + \epsilon \quad \text{for } n = 1, \ldots, m-1. \tag{2.1}$$

To prove Theorem 2.1, we construct by induction a rapidly converging sequence of p-adic algebraic numbers of degree m, whose limit is a p-adic U_m-number with (2.1).

3. Auxiliary results

The following lemma is a consequence of Bugeaud [2, Theorem 9.4], and the proof of the main result of this work is based on it.

LEMMA 3.1. *Let $m \geq 2$ be an integer and ξ be a p-adic algebraic number of degree m, and let $\varepsilon > 0$ be any real number. Then there exist a real constant $\kappa(\xi, \varepsilon, m) > 1$, depending only on ξ, ε, and m, and infinitely many p-adic algebraic numbers α of degree m such that*

$$0 < |\xi - \alpha|_p < \kappa(\xi, \varepsilon, m) H(\alpha)^{-m+\varepsilon}.$$

The following lemma is a form of the so-called Liouville inequality the proof of which can be found in many references (see, for example, Pejkovic [4, Lemma 2.5]).

LEMMA 3.2. *Let α_1 and α_2 be two distinct p-adic algebraic numbers of degree n_1 and n_2, respectively. Then*

$$|\alpha_1 - \alpha_2|_p \geq (n_1 + 1)^{-n_2} (n_2 + 1)^{-n_1} H(\alpha_1)^{-n_2} H(\alpha_2)^{-n_1}.$$

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4. Proof of Theorem 2.1

Let $m \geq 2$ be an integer, β be any p-adic algebraic number of degree less than or equal to $m-1$, and ξ_1 be a p-adic algebraic number of degree m. By Lemma 3.2,

$$|\xi_1 - \beta|_p \geq (m+1)^{-2m} H(\xi_1)^{-\deg(\beta)} H(\beta)^{-m}. \tag{4.1}$$

Let ε be any real number satisfying $0 < \varepsilon \leq 1$. Then, by Lemma 3.1, there exist a real constant $\kappa(\xi_1, \varepsilon, m) > 1$, depending only on ξ_1, ε, and m, and infinitely many p-adic algebraic numbers ξ_2 of degree m such that

$$0 < |\xi_1 - \xi_2|_p < \kappa(\xi_1, \varepsilon, m) H(\xi_2)^{-m+\varepsilon}. \tag{4.2}$$

If

$$H(\beta) \leq (m+1)^{-2} H(\xi_1)^{-1} \kappa(\xi_1, \varepsilon, m)^{-1/m} H(\xi_2)^{1-\varepsilon/m} =: A$$

holds, then it follows from (4.1) and (4.2) that

$$|\xi_2 - \beta|_p \geq c_1 H(\beta)^{-m}, \tag{4.3}$$

where

$$c_1 := (m+1)^{-2m} H(\xi_1)^{-m} < 1.$$

In the case $H(\beta) > A$, we use Lemma 3.2 and obtain

$$|\xi_2 - \beta|_p \geq (m+1)^{-2m} H(\xi_2)^{-\deg(\beta)} H(\beta)^{-m}. \tag{4.4}$$

We deduce from (4.4) and $H(\beta) > A$ that

$$|\xi_2 - \beta|_p > c_2 H(\beta)^{-m-\deg(\beta)/(1-\varepsilon/m)} > H(\beta)^{-m-\deg(\beta)/(1-\varepsilon/m)} (\log(3H(\beta)))^{-1}, \tag{4.5}$$

where

$$c_2 := (m+1)^{-2m} ((m+1)^2 H(\xi_1) \kappa(\xi_1, \varepsilon, m)^{1/m})^{-\deg(\beta)/(1-\varepsilon/m)},$$

holds true if

$$H(\beta) > c_3 := \tfrac{1}{3} \exp((m+1)^{2m} ((m+1)^2 H(\xi_1) \kappa(\xi_1, \varepsilon, m)^{1/m})^{m/(1-\varepsilon/m)}).$$

But we choose ξ_2 such that $A > c_3$. Then we infer from (4.3) and (4.5) that

$$|\xi_2 - \beta|_p > c_1 H(\beta)^{-m-\deg(\beta)/(1-\varepsilon/m)} (\log(3H(\beta)))^{-1} \tag{4.6}$$

holds for any p-adic algebraic number β of degree less than or equal to $m-1$. We now prove the following claim by induction.

Claim. *There exist p-adic algebraic numbers $\xi_2, \xi_3, \ldots$ of degree m such that*

$$|\xi_{i+1} - \beta|_p > c_1 H(\beta)^{-m-\deg(\beta)/(1-\varepsilon/m)} (\log(3H(\beta)))^{-1} \quad (i = 1, 2, \ldots) \tag{4.7}$$

for any p-adic algebraic number β of degree less than or equal to $m-1$,

$$0 < |\xi_i - \xi_{i+1}|_p < \kappa(\xi_i, \varepsilon, m) H(\xi_{i+1})^{-m+\varepsilon} \quad \text{and} \quad H(\xi_{i+1}) > \kappa(\xi_i, \varepsilon, m) H(\xi_i)^i$$
$$(i = 1, 2, \ldots), \tag{4.8}$$

where $\kappa(\xi_i, \varepsilon, m) > 1$ is a real constant depending only on ξ_i, ε, m.

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Proof of the claim. By (4.6) and (4.2), the claim is true for $i = 1$. Let us assume that the claim is true for $i = k - 1$ ($k > 1, k \in \mathbb{N}$), that is, there exist p-adic algebraic numbers $\xi_2, \ldots, \xi_k$ of degree m such that (4.7) and (4.8) hold true for $i = 1, \ldots, k - 1$. By Lemma 3.1, there exists a p-adic algebraic number ξ_{k+1} of degree m such that

$$0 < |\xi_k - \xi_{k+1}|_p < \kappa(\xi_k, \varepsilon, m)H(\xi_{k+1})^{-m+\varepsilon} \quad \text{and} \quad H(\xi_{k+1}) > \kappa(\xi_k, \varepsilon, m)H(\xi_k)^k, \tag{4.9}$$

where $\kappa(\xi_k, \varepsilon, m) > 1$ is a real constant depending only on ξ_k, ε, m. By assumption,

$$|\xi_k - \beta|_p > c_1 H(\beta)^{-m-\deg(\beta)/(1-\varepsilon/m)}(\log(3H(\beta)))^{-1} \tag{4.10}$$

for any p-adic algebraic number β of degree less than or equal to $m - 1$. If

$$H(\beta)^{m+\deg(\beta)/(1-\varepsilon/m)} \log(3H(\beta)) \le c_1\kappa(\xi_k, \varepsilon, m)^{-1}H(\xi_{k+1})^{m-\varepsilon} \tag{4.11}$$

holds, then it follows from (4.9) and (4.10) that

$$|\xi_{k+1} - \beta|_p > c_1 H(\beta)^{-m-\deg(\beta)/(1-\varepsilon/m)}(\log(3H(\beta)))^{-1}. \tag{4.12}$$

By Lemma 3.2,

$$|\xi_k - \beta|_p \ge (m + 1)^{-2m}H(\xi_k)^{-\deg(\beta)}H(\beta)^{-m}. \tag{4.13}$$

If

$$H(\beta)^m \le (m + 1)^{-2m}H(\xi_k)^{-\deg(\beta)}\kappa(\xi_k, \varepsilon, m)^{-1}H(\xi_{k+1})^{m-\varepsilon} =: B$$

holds, then it follows from (4.9) and (4.13) that

$$|\xi_{k+1} - \beta|_p \ge (m + 1)^{-2m}H(\xi_k)^{-\deg(\beta)}H(\beta)^{-m}.$$

If β satisfies

$$((m + 1)^{2m}H(\xi_k)^{\deg(\beta)})^{(1-\varepsilon/m)/\deg(\beta)} \le H(\beta) \le B^{1/m}, \tag{4.14}$$

then

$$|\xi_{k+1} - \beta|_p \ge H(\beta)^{-m-\deg(\beta)/(1-\varepsilon/m)}. \tag{4.15}$$

On the other hand, if

$$H(\beta) < ((m + 1)^{2m}H(\xi_k)^{\deg(\beta)})^{(1-\varepsilon/m)/\deg(\beta)},$$

then we choose ξ_{k+1} with sufficiently large $H(\xi_{k+1})$ such that (4.11), and so (4.12), is satisfied. Thus we infer from (4.14), (4.15) and the previous sentence that (4.12) is satisfied for $H(\beta) \le B^{1/m}$. Let us assume that $H(\beta) > B^{1/m}$. By Lemma 3.2,

$$|\xi_{k+1} - \beta|_p \ge (m + 1)^{-2m}H(\xi_{k+1})^{-\deg(\beta)}H(\beta)^{-m}. \tag{4.16}$$

We deduce from (4.16) and $H(\beta) > B^{1/m}$ that

$$|\xi_{k+1} - \beta|_p > c_4H(\beta)^{-m-\deg(\beta)/(1-\varepsilon/m)} > H(\beta)^{-m-\deg(\beta)/(1-\varepsilon/m)}(\log(3H(\beta)))^{-1}, \tag{4.17}$$

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where

$$c_4 := (m+1)^{-2m-2\deg(\beta)/(1-\varepsilon/m)} H(\xi_k)^{-(\deg(\beta))^2/(m-\varepsilon)} \kappa(\xi_k, \varepsilon, m)^{-\deg(\beta)/(m-\varepsilon)},$$

holds when $H(\xi_{k+1})$ is large enough. By (4.17), we see that (4.12) is satisfied for $H(\beta) > B^{1/m}$. Then (4.12) is satisfied for any p-adic algebraic number β of degree less than or equal to $m-1$. We infer from (4.9) and the previous sentence that the claim is true for $i = k$, and so the proof of the claim is complete. □

We deduce from (4.8) that

$$0 < |\xi_i - \xi_{i+1}|_p < H(\xi_i)^{-i} \quad \text{for } i = 2, 3, \ldots. \tag{4.18}$$

By (4.18) and using the non-Archimedean property of the p-adic absolute value,

$$|\xi_i - \xi_j|_p < H(\xi_i)^{-i} \tag{4.19}$$

for any integers i, j with $2 \le i < j$. Thus, $\{\xi_j\}_{j=2}^{\infty}$ is a Cauchy sequence in $\mathbb{Q}_p$ which is complete with respect to $|\cdot|_p$; let us denote its limit by $\xi \in \mathbb{Q}_p$. By letting j tend to infinity in (4.19),

$$|\xi - \xi_i|_p \le H(\xi_i)^{-i} \quad \text{for } i = 2, 3, \ldots. \tag{4.20}$$

We infer from (4.20) that $w_m^*(\xi) = \infty$, and so we obtain $\xi \in U^*$ with $\mu^*(\xi) \le m$. On the other hand, it follows from (4.7) that

$$w_n^*(\xi) \le \frac{n}{1-\varepsilon/m} + m - 1 \quad \text{for } n = 1, \ldots, m-1. \tag{4.21}$$

By (4.21) and the fact that $\xi \in U^*$ with $\mu^*(\xi) \le m$, we get $\mu^*(\xi) = m$, and hence it follows that $\xi \in U_m^*$. Since $U_i^* = U_i$ for any natural number i, we see that ξ is a p-adic U_m-number with $w_n^*(\xi) \le (n/(1-\varepsilon/m)) + m - 1$, for $n = 1, \ldots, m-1$. Finally, we observe that, at each step, we have infinitely many choices for the p-adic algebraic number ξ_i $(i \ge 2)$. Thus, we can construct uncountably many p-adic U_m-numbers satisfying the required properties. This completes the proof of Theorem 2.1.

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GÜLCAN KEKEÇ, Faculty of Science, Department of Mathematics,
Istanbul University, 34134 Vezneciler-Istanbul, Turkey
e-mail: gulkekec@istanbul.edu.tr

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