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OSCILLATION AND COMPARISON THEOREMS FOR SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS WITH INTEGRABLE COEFFICIENTS

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1. The classical comparison and interlacing theorems of Sturm were originally proved for the equations

(1) $(p(t)u'(t))' + q(t)u(t) = 0,$

(2) $(P(t)U'(t))' + Q(t)U(t) = 0,$

under the assumption that all coefficients are real-valued, continuous, and $p > 0$, $P > 0$. Atkinson [**1**, Chapter 8] has carried out the standard theory for eigenvalue problems involving (1), under the more general hypothesis

(H) $(1/p)$, $(1/P)$, Q and q belong to $L^1(a, b)$,

with additional positivity hypotheses where needed (cf. p. 204). Under Assumption (H), equation (1) is to be thought of as a system

$$u' = (1/p)v, \quad v' = -qu,$$

and a solution is a pair of absolutely continuous functions (u, v) that satisfy the system almost everywhere. In particular, p may be undefined on a set of measure zero, it may vanish on a set of measure zero as long as $1/p$ remains integrable, and we can have $p = \pm\infty$ (i.e., $(1/p) = 0$) on sets of arbitrary measure. Note that if $p(t) = +\infty$ on $[a, c]$, then

$$u(t) = u(a), \quad v(t) = v(a) - u(a)\int_a^t q(s)ds \text{ on } [a, c].$$

We show that assumption (H) and the condition $(1/p) \geqq 0$ a.e. are enough to imply that the Sturm comparison theorem holds. We then prove a slight modification of a standard result (cf. Hartman [**4**, p. 351]) on the relation between disconjugacy and the existence of positive solutions. Finally, we investigate the situation when the zeros of $p(t)$ cluster in $[a, b]$. All coefficients are assumed real-valued.

Diaz and McLaughlin [**3**, Theorems 1-5] extend the Sturm theorems under different hypotheses. For example, they prove (Theorem 3): "Suppose y_1, y_2 are real-valued and continuous on $[a, b]$, that y_1' and y_2' exist on (a, b), and $y_1(a) = y_1(b) = 0$. Suppose further that y_1, y_2 satisfy the respective equations

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$(r_i y_i')' + p_i y_i = 0$, $i = 1, 2$, in the sense that $(r_1 y_1')'$ and $(r_2 y_2')'$ exist on (a, b) with $r_1 \geqq r_2 > 0$, $p_1 \leqq p_2$ on (a, b). Then y_2 must have a zero in $[a, b]$". Note that they do not place any smoothness or integrability conditions on r_i, p_i. They do assume that r_i, p_i are finite-valued on $[a, b]$. Now consider the equations treated by Diaz and McLaughlin under our assumption (H). We choose $r_1(t) \equiv r_2(t)$, $p_1(t) \equiv p_2(t)$ where

$$r_i(t) = \begin{cases} 1, \ 0 \leqq t \leqq \pi; \\ +\infty, \ \pi < t \leqq 2\pi; \\ 1, \ 2\pi < t \leqq 3\pi; \end{cases} \quad p_i(t) \equiv 1.$$

Then let

$$y_1(t) = \begin{cases} \sin t, \ 0 \leqq t \leqq \pi; \\ 0, \ \pi < t \leqq 2\pi; \\ -\sin t, \ 2\pi < t \leqq 3\pi; \end{cases} \quad y_2(t) = \begin{cases} \cos t, \ 0 \leqq t \leqq \pi; \\ -1, \ \pi < t \leqq 2\pi; \\ -\cos t, \ 2\pi < t \leqq 3\pi. \end{cases}$$

$y_1(t)$ has an infinite number of zeros in $[3\pi/4, 9\pi/4]$, yet $y_2(t)$ has no zeros in this interval.

W. T. Reid has developed Sturmian theory for both scalar and vector equations; his results as well as those of many others are well summarized in his recent book [**10**] (especially Chapter VII). He assumes (in our context) that $P(t) \geqq 0$, $p(t) \neq 0$ a.e., with P, p, $1/p$, q, Q measurable and essentially bounded. His approach is via the calculus of variations. No doubt one can use such an approach under (H). Our approach is rather different, being based on the geometrical point of view as developed (beginning with Prüfer and Ettlinger) by Atkinson [**1**], Jakubovic [**5**] and Krasnosel'skii *et. al.* [**6**, p. 145]. One could as well use a Picone type of identity or Riccati arguments; each method has its own advantages.

Definitions. Let $y(t)$ solve (1) on $[a, b]$ under (H). If $c \in (a, b)$, $y(c) = 0$ with $y(t) \not\equiv 0$ in some deleted neighbourhood of c, then c is an *isolated* zero of y. A zero c (not necessarily isolated) is a *proper zero* if $y(t)$ changes sign at c, that is $y(t) \geqq 0$ on one side of c, and $y(t) \leqq 0$ on the other side of c. If $y(t) \equiv 0$ on a maximal closed proper sub-interval $J \subset (a, b)$, then J is counted as a single zero of y when speaking of, or counting, distinct zeros of y. We say (1) is disconjugate on $[a, b]$ if no solution has more than one zero in $[a, b]$, counting according to the convention just described.

Remark. An example of a nonisolated proper zero is the function $f(x) = x^5|\sin 1/x|$, $f(0) = 0$, at $x = 0$.

Examples. (1) Let $p(t) = +\infty$ on $[a, b]$. Then $y(t) \equiv$ constant for any solution $y(t)$ regardless of $q(t)$. This equation is thus disconjugate on $[a, b]$, for any $q \in L^1(a, b)$.

(2) Let $u(t)$ be any absolutely continuous function on $[a, b]$ with $[1/u'(t)] \in L^1(a, b)$. Then $u(t)$ solves (1) with $q \equiv 0$, $p = (1/u')$. In particular, u and u' can vanish simultaneously on a set of measure zero, and u may have nonproper and/or nonisolated zeros.

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2. Consider the two-dimensional systems

$$(3) \quad x' = A_1(t)x,$$

$$(4) \quad y' = A_2(t)y, \quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

$t \in [a, b]$, with A_1, A_2 matrices of real-valued functions in $L^1(a, b)$. Solutions are understood to be absolutely continuous on $[a, b]$, satisfying the appropriate equation almost everywhere. Let

$$J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

As is usual we say a 2×2 matrix B is nonnegative definite ($B \geqq 0$) if the quadratic form $x^*Bx \geqq 0$ for all x.

If $x(t)$ solves (1), then we define a continuous argument function $(\arg x) = \arg(x_1 + ix_2)$, by requiring that $(\arg x)(a) \in [0, 2\pi)$.

THEOREM 1. *Let $A_1(t)$, $A_2(t)$ be as described above, and suppose*

$$J[A_2(t) - A_1(t)] \geqq 0 \text{ a.e. in } [a, b].$$

If $\arg x(a) \leqq \arg y(a)$, *and if* $x(t)$, $y(t)$ *solve* (3), (4) *respectively, then* $\arg x(b) \leqq \arg y(b)$.

Proof. If $\theta(t) = \arg x(t)$, $\phi(t) = \arg y(t)$, then

$$\theta'(t) = \frac{(Jx')^*x}{x^*x}, \quad \phi'(t) = \frac{(Jy')^*y}{y^*y}, \quad \text{(* indicates transpose)}$$

so

$$\theta'(t) = (x^*A_1{}^*J^*x)/x^*x = [\cos\theta, \sin\theta]A_1{}^*J^*\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix},$$

$$\phi'(t) = (y^*A_2J^*y)/y^*y = [\cos\phi, \sin\phi]A_2{}^*J^*\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}.$$

Thus $\theta(a) \leqq \phi(a)$, $\theta'(t) = f(t, \theta)$, $\phi'(t) = g(t, \phi)$ where f, g satisfy the Caratheodory conditions [**10**, p. 90], with $f(t, u) \leqq g(t, u)$ a.e. We can therefore apply Cafiero's extension of Peano's basic comparison theorem (Cafiero [**2**]; cf. Muldowney [**7**] for a more general result, with a full discussion) to conclude that $\theta(b) \leqq \phi(b)$.

COROLLARY. *Let* (H) *hold, and assume that* $Q(t) \geqq q(t)$, $1/P(t) \geqq 1/p(t) \geqq 0$ *a.e. in* $[a, b]$. *Let* $u(t)$ *solve* (1), *with* $u(a) = u(b) = 0$, $u(t) \neq 0$ *for* $t \in (a, b)$. *Then each solution of* (2) *vanishes on* $[a, b]$.

Proof. We convert (1) and (2) to systems of the form (3) in the standard way, and define $\theta(t) = \arg(pu' + iu)$, $\phi(t) = \arg(PU' + iU)$ in the complex plane, with $0 = \theta(a) \leqq \phi(a) < 2\pi$. The hypotheses of the corollary imply that the assumptions of Theorem 1 hold; therefore we may conclude $\theta(b) \leqq$

$\phi(b)$. Since $U(t)$ is zero if and only if $\phi(t) = j\pi$ for some integer j, we need only show $\theta(b) = \pi$. This follows easily from a result of Atkinson [**1**, Theorem 8.4.3], which states that $\theta(t)$ cannot tend to any multiple of π from above as t increases. Since $u(b) = 0$, $\theta(b)$ must be a positive multiple of π, and since $u(t) \not\equiv 0$ in (a, b), we conclude $\theta(b) = \pi$.

The above theorem can be applied to certain three-term recurrence relations of the form

$$C_n(y_{n+1} - y_n) - C_{n-1}(y_n - y_{n-1}) + b_n y_n = 0, \quad n = 0, 1, 2, \ldots$$

in the spirit of Atkinson [**1**, pp. 202-203] and Reid [**9**].

3. *Definition.* $\{t_1, t_2\}$ in $[a, b]$ is called a *p-infinite pair* if $p(t) = +\infty$ a.e. on $[t_1, t_2]$ (equivalently, $[t_1, t_2]$ is called a p-infinite interval). If $\{t_1, t_2\}$ does not have this property, it is called a *p-finite pair.*

LEMMA (Compare Hartman [**4**, p. 357]). *Assume* (H) *holds. Then* (1) *is disconjugate on* $[a, b]$ *if and only if the two-point boundary value problem* (1), $u(t_1) = \alpha_1$, $u(t_2) = \alpha_2$, *has a unique solution for all p-finite pairs* $\{t_1, t_2\}$ *in* $[a, b]$ *and all real numbers* α_1, α_2.

Proof. As a preliminary, let (v, pv') and (w, pw') be an independent pair of solutions of (1), and let $[t_1, t_2] \subset [a, b]$ be fixed. As is well-known, there exists a unique solution to the above two-point boundary value problem for all α_1, α_2 if and only if

$$\det \begin{bmatrix} v(t_1) & w(t_1) \\ v(t_2) & w(t_2) \end{bmatrix} \neq 0.$$

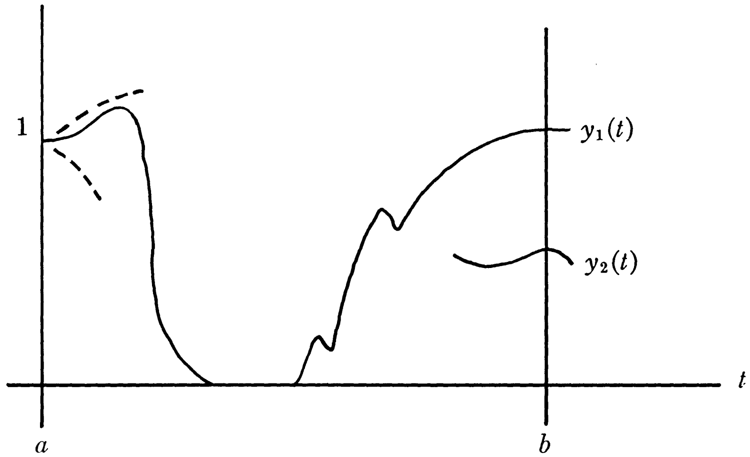
We may for definiteness assume $v(t_1) \neq 0$.

Now we can prove the lemma as follows. First suppose that (1) is disconjugate on $[a, b]$. Consider the solution $u(t) = w(t_1)v(t) - v(t_1)w(t)$, which vanishes at t_1. Since (1) is disconjugate, it follows that $u(t_2) \neq 0$ whenever $\{t_1, t_2\}$ is a p-finite pair. But $u(t_2)$ is just the above determinant.

For the converse, suppose $u(t)$ is a solution of (1) with $u(t_1) = u(t_2) = 0$, $u(t) \not\equiv 0$ on $[t_1, t_2]$. Then $\{t_1, t_2\}$ is p-finite, yet the two-point boundary value problem with $\alpha_1 = \alpha_2 = 0$ does not have a unique solution, since $y(t_1) \equiv 0$ is another.

THEOREM 2. *Assume* (H) *holds. If* (1) *is disconjugate on an interval* $[a, b]$, *then there is a solution* $u(t) > 0$ *on* $[a, b]$. *The converse is true if* $p(t) \geqq 0$ *almost everywhere on* $[a, b]$.

Proof. If $[a, b]$ is p-infinite, then any solution of (1) is constant on $[a, b]$ and the conclusion is immediate. Now suppose that $[a, b]$ is p-finite, and that (1) is disconjugate on $[a, b]$. By the lemma, there exist independent solutions y_1, y_2 of (1) such that $y_1(a) = y_1(b) = y_2(a) = 1, y_2(b) = 1/2$. If $y_1(t) \neq 0$ on $[a, b]$, then it is the desired solution. Suppose that in fact $y_1(t)$ vanishes on $[a, b]$. Since (1) is disconjugate and $y_1(a) > 0$, $y_1(b) > 0$, y_1 has only this one zero, and does not change sign (see the sketch below). But it is easy to see that y_2 must then intersect y_1 in the open interval (a, b), so the solution $y_1 - y_2$ has two zeros in $[a, b]$, a contradiction.



Note that the possibility that $y_1 \equiv y_2$ on some (maximal) interval $[a, c]$ is of no consequence, since this would imply $y_1 - y_2 \equiv 0$, which in turn implies $[1/p(t)] = 0$ a.e. on $[a, c]$, so that $y_1(t) \equiv 1$ on $[a, c]$, and we could just begin the argument at c.

Conversely, suppose $u_0(t)$ solves (1) with $u_0(t) > 0$ on $[a, b]$. The Sturm interlacing theorem (the Corollary to Theorem 1 with $p = P$, $q = Q$) implies that no solution of (1) may vanish more than once on $[a, b]$.

Remark. We have stated the above result in the form of the standard result. In fact we have shown that, under (H), if $u(t)$ is a solution of a disconjugate equation on $[a, b]$ with sgn $u(a) =$ sgn $u(b)$, then $u(t)$ cannot vanish on $[a, b]$. Thus every zero must be proper.

In the case of continuous coefficients there is a result analogous to the above theorem for disconjugacy on $[a, \infty)$: (1) is disconjugate on $[a, \infty)$ if and only if there exists a solution $u(t) > 0$ on (a, ∞) (the *open* interval). The "if" part follows easily in our case from the Sturm Theorem; the "only if" is not, in general, true. In the continuous case it is proved by taking $u(a) = 0, u'(a) = 1$. However, under (H) this solution may not have the desired property of positivity. If $p(t) = +\infty$ a.e. on an interval $[a, c)$, then $u(t) \equiv 0$ on this interval.

4. We now turn to questions involving sign changes of solutions at zeros, under assumption (H). We are interested only in zeros in the interior (a, b).

THEOREM 3. *Assume* (H) *holds, and let* $u(t)$ *solve* (1) *on* $[a, b]$.

(a) *If* $p(t) \geqq 0$ *on* $[a, b]$, *then every zero of* $u(t)$ *is proper.*

(b) *If* $p(t) \geqq 0$ *on* $[a, b]$, *and if* $c \in [a, b]$ *is a non-isolated zero of* $u(t)$, *then* c *belongs to a* p-*infinite interval.*

(c) *If there is a* (relative) *open cover of* $[a, b]$ *by intervals of disconjugacy for* (1), *then each solution has only proper zeros. The converse is true if* $p(t)$ *changes sign at most a finite number of times in* $[a, b]$, *in the sense that*

$$[a, b] = \bigcup_{j=1}^{N} I_j$$

with either $p(t) \geqq 0$, *or* $p(t) \leqq 0$, *a.e. on each* I_j.

Proof. (a) Atkinson [**1**, Theorem 8.4.3] has shown that for $p(t) \geqq 0$, the function $\theta(t) = \arg\,(pu' + iu)$ cannot tend to a multiple of π from above as t increases, nor can it tend to a multiple of π from below as t decreases. This immediately translates into (a) above for an isolated zero. Now suppose c is a nonisolated zero, and let $t_n \to c$ from the left, say, with $u(t_n) = 0$. Then $\theta(t_n)$ must be successively larger multiples of π, so $\lim_{n\to\infty}\theta(t_n) = \infty$ or else $\theta(t)$ is a constant multiple of π in some neighbourhood of c. But $\theta(t)$ solves $\theta' = (1/p)\cos^2\theta + q\sin^2\theta = 0$, all of whose solutions are extendable. This observation proves (b).

(c) If there is an open cover by intervals of disconjugacy, then there is a finite cover by *closed* intervals of disconjugacy. The Remark following the proof of Theorem 2 then implies that each zero of a solution is proper. For the converse, assume that $c \in (a, b)$, and that each interval containing c is not an interval of disconjugacy for (1). Then for $n = 1, 2, \ldots$ there exist solutions $u_1(t), u_2(t), \ldots$ of (1) such that $u_n(t)$ vanishes at c_n', c_n'', where $c - (1/n) < c_n' < c_n'' < c + (1/n)$. Since zeros of solutions are assumed proper, c_n' and c_n'' are proper zeros of $u_n(t)$. Also, we can normalize the solutions $u_n(t)$ so that $u_n^2(a) + [pu_n'(a)]^2 = 1$. By standard arguments (see [**4**, p. 14]), there is a subsequence, which we also call u_n, that converges to a solution $u_0(t)$ of (1), uniformly on $[a, b]$. Clearly $u_0(c) = 0$; we assume $pu_0'(c) > 0$. Since zeros of solutions are proper we may assume $u_0(t) \geqq 0$ in a right neighbourhood of c and $u_0(t) \leqq 0$ in a left neighbourhood of c; we can also assume $pu_0'(t) > 0$ on the union of these two neighbourhoods (the opposite case is similar).

Now the sign behaviour of $u_0(t)$ and $pu_0'(t)$ in a sufficiently small neighbourhood of c implies that there are sequences $\{t_i\}$, $\{s_i\}$ converging to c from above and below, respectively, for which $pu_0'(t_i)$, $pu_0'(s_i)$, are positive, and $u_0'(t_i)$, $u_0'(s_i)$ are nonnegative. Since p has only finitely many sign changes, it follows that $p > 0$ a.e. in some neighbourhood of c. For sufficiently large n, $u_n(t)$ has two zeros and $pu_n'(t) > 0$ in this neighbourhood of c. Thus $u_n'(t) \geqq 0$ a.e. in this neighbourhood, a contradiction to the existence of two distinct zeros. This completes the proof of Theorem 3.

Remarks. The existence of a cover by intervals of disconjugacy is important for the introduction of Morse quadratic forms (cf. Reid [**10**, pp. 253-261]).

If p has finitely many sign changes, then any solution of (1) has only finitely many zeros in $[a, b]$. For let u be a solution of (1) with zeros clustering at $\tau \in [a, b]$. Then $u(\tau) = 0$, $pu'(\tau) \neq 0$ and since u' has infinitely many sign changes in a neighbourhood of τ, so does p.

Let $\tau \in (a, b)$ be fixed. We define

$$\pi(t) = \int_\tau^t (1/p(s))ds.$$

If $\pi(t)$ has zeros accumulating at τ, then the equation $(p(t)y'(t))' = 0$ has a solution, namely $\pi(t)$, with zeros accumulating at τ. Thus for the given function $p(t)$, there exists a $q(t)$ such that (1) has a solution with zeros accumulating at τ. We can prove a partial result in the other direction, in the sense that for a given $p(t)$ with $\pi(t)$ non-oscillatory at τ, we can define a class of $q(t)$'s for which solutions of (1) cannot oscillate at τ.

THEOREM 4. *Let* (H) *hold, with $\tau \in (a, b)$ fixed. Suppose that $\pi(t)$ is different from zero in some deleted neighbourhood of τ. Then solutions of* (1) *will not have zeros accumulating at τ if $q(t)$ satisfies*

$$\liminf_{t \to \tau} \frac{\int_\tau^t (1/p(s))ds}{\int_\tau^t \frac{1}{|p(s)|} \int_\tau^s |q(r)|dr\, ds} > 0.$$

Proof. Suppose the contrary, that is all of the hypotheses of the theorem hold, yet $y(t)$ solves (1) with $y(t_n) = 0$ for some sequence $t_n \downarrow \tau$ (the opposite case is similar). Clearly $y(\tau) = 0$, and we may assume $py'(\tau) = 1$. Integrating (1) twice yields

$$y(t) = \int_\tau^t \frac{1}{p(s)} \left[1 - \int_\tau^s y(r)q(r)dr\right] ds$$

Let

$$\epsilon_n = \max_{\tau \leqslant t \leqslant t_n} |y(t)|.$$

Then

$$0 = y(t_n) \geqslant \int_\tau^{t_n} \frac{ds}{p(s)} - \epsilon_n \int_\tau^{t_n} \frac{1}{|p(s)|} \int_\tau^s |q(r)|dr\, ds$$

i.e.,

$$\epsilon_n \geqslant \frac{\int_\tau^{t_n} \frac{ds}{p(s)}}{\int_\tau^{t_n} \frac{1}{|p(s)|} \int_\tau^s |q(r)|dr\, ds},$$

which leads to a contradiction.

The condition that $\pi(t)$ is different from zero in some neighbourhood of τ is not, by itself, sufficient to guarantee that solutions of (1) will not have zeros accumulating at τ, as the following construction will indicate.

Let $f(t)$ be a nonnegative function on $[0, 1]$ with an absolutely continuous derivative. Let $g(t)$, $h(t)$ be in $C^0[0, 1] \cap C^1(0, 1)$ such that $g(t)$ and $h(t)$ are positive except at $t = 0$, and $g'(t)/t$ and $h'(t)/t$ are bounded on $(0, 1)$,
Set

$$G(t) = \int_0^t g(f(s))h(|f'(s)|)ds,$$

and define

$$p(t) = \frac{1}{f'(t) + G'(t)}, \quad q(t) = \frac{(p(t)G'(t))'}{f(t)}$$

(allowing $p(t)$ to be $+\infty$ when $f'(t) + G'(t)$ vanishes). Then $p(t)$, $q(t)$ are defined for all but a countable set of values of t in $[0, 1]$ and it is straightforward to verify that $1/p$ and q are in $L^1(0, 1)$ and that $y = f(t)$ is a solution of (1). However, $\pi(t) = f(t) + G(t) > 0$ for $t > 0$.

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