THE SCHWARZIAN DERIVATIVE AND DISCONJUGACY OF nth ORDER LINEAR DIFFERENTIAL EQUATIONS

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1. Introduction. In this paper we deal with the number of zeros of a solution of the nth order linear differential equation

$$(1.1)\qquad y^{(n)}(z) + p_{n-2}(z)y^{(n-2)}(z) + \ldots + p_0(z)y(z) = 0, \qquad n = 2, 3, \ldots,$$

where the functions $p_j(z)$ $(j = 0, 1, \ldots, n-2)$ are assumed to be regular in a given domain D of the complex plane. *The differential equation* (1.1) *is called disconjugate in* D, *if no* (*non-trivial*) *solution of* (1.1) *has more than* $(n-1)$ *zeros in* D. (*The zeros are counted by their multiplicity.*)

The ideas of this paper are related to those of Nehari (**7**; **9**) on second order differential equations. In (**7**), he pointed out the following basic relationship. *The function*

$$(1.2)\qquad f(z) = \frac{y_1(z)}{y_2(z)},$$

where $y_1(z)$ *and* $y_2(z)$ *are two linearly independent solutions of*

$$(1.3)\qquad y''(z) + p(z)y(z) = 0,$$

is univalent in D, *if and only if no solution of equation* (1.3) *has more than one zero in* D, *i.e., if and only if* (1.3) *is disconjugate in* D.

The coefficient $p(z)$ of (1.3) is expressed in terms of the function (1.2) by the identity

$$(1.4)\qquad 2p(z) = \{f, z\},$$

where $\{f, z\}$ denotes the Schwarzian derivative of $f(z)$ with respect to z, namely

$$(1.5)\qquad \{f, z\} = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$

It is well known that the Schwarzian derivative (1.5) is invariant under a linear transformation

$$(1.6)\qquad Tf = \frac{Af + B}{Cf + D}, \qquad AD - BC \neq 0.$$

Thus, (1.4) is independent of our choice of the solutions $y_1(z)$ and $y_2(z)$ in (1.2).

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By making use of the duality relationship between disconjugacy of (1.3) and univalence of (1.2) (for the necessary condition), and of an integral inequality (for the sufficient condition), Nehari proved the following theorem (**7**, Theorem 1), which we state here as a disconjugacy criteria. *In order that equation* (1.3) *be disconjugate in* $|z| < 1$, *it is necessary that*

$$|p(z)| \leqq \frac{3}{(1 - |z|^2)^2}, \qquad |z| < 1, \tag{1.7}$$

and sufficient that

$$|p(z)| \leqq \frac{1}{(1 - |z|^2)^2}, \qquad |z| < 1. \tag{1.8}$$

Both conditions are sharp as shown by the Koebe function and by an example due to Hille (**6**).

2. $(n - 2)$ **parameter family of univalent functions.** Our study of equation (1.1) starts with a problem suggested to us by Nehari. In view of (**7**; **9**), what are, if any, the function-theoretic aspects of disconjugacy of nth order linear differential equations? In the following, we shall prove that a disconjugate equation (1.1) is related to an $(n - 2)$ parameter family of univalent functions.

In analogy with (1.2), we consider the function

$$f(z, a_1, a_2, \ldots, a_{n-2}) = \frac{y_1(z)}{y_2(z)}, \tag{2.1}$$

where $y_1(z)$ and $y_2(z)$ are two linearly independent solutions of (1.1), which vanish on a given set S of $(n - 2)$ points $a_1, a_2, \ldots, a_{n-2}$ of D. (Some of these zeros may coincide, giving rise to zeros of higher order.) The existence of at least two such linear independent solutions is an immediate consequence of the existence of a fundamental set of n linearly independent solutions $\eta_1(z), \eta_2(z), \ldots, \eta_n(z)$ of equation (1.1). Indeed, setting now,

$$y(z) = \sum_{k=1}^{n} \alpha_k \eta_k(z),$$

and writing

$$y(a_j) = 0, \qquad j = 1, 2, \ldots, (n - 2), \tag{2.2}$$

one obtains a system of $(n - 2)$ homogeneous equations for the n unknown constants α_k, and there always exist at least two linearly independent solutions of (2.2). In case of a zero of higher order, e.g.,

$$a_1 = a_2 = \ldots = a_m, 1 < m \leqq n - 2,$$

(2.2) is replaced by

$$y(a_1) = 0, y'(a_1) = 0, \ldots, y^{m-1}(a_1) = 0, y(a_{m+1}) = 0, \ldots, y(a_{n-2}) = 0,$$

and the same conclusion follows. Moreover, if $m = n - 2$, then there exist exactly two linearly independent solutions which vanish $(n - 2)$ times at the

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point $a_1 \in D$; however, for $1 \leqq m < n - 2$ it does not follow from the general existence theorem that any three solutions of (1.1) which vanish on a set of $(n - 2)$ points are linearly dependent. In the following lemma we give two sufficient conditions which guarantee such a situation.

LEMMA 1. *If there exist more than two linearly independent solutions of equation* (1.1), *which vanish on the set S of $(n - 2)$ points of D, then equation* (1.1) *is conjugate in D and at least one of the functions of the type* (2.1) *is non-univalent in D.*

Proof. Assume that there exist three linearly independent solutions $y_1(z)$, $y_2(z)$, and $y_3(z)$, which vanish on S. Let $b \in D$, such that $y_2(b) \neq 0$, and set

$$y^*(z) = \alpha_1 y_1(z) + \alpha_2 y_2(z) + \alpha_3 y_3(z), \quad y^*(b) = 0, y^{*\prime}(b) = 0.$$

It follows that there always exists a non-trivial solution $y^*(z)$ which vanishes at least n times in D. Moreover, $y^*(z)/y_2(z)$, which is a function of the type (2.1) is non-univalent in D.

We are now ready to formulate the connection between the function (2.1) and the equation (1.1).

THEOREM 1. *Equation* (1.1) *is disconjugate in D if and only if all the functions of the type* (2.1) *are univalent in D; i.e., if and only if the ratio of any two linearly independent solutions, which vanish at $(n - 2)$ points of D, is univalent in D.*

Proof. (i) Disconjugacy implies univalence. If $f(b_1) = f(b_2) = -\beta\alpha^{-1}$, it follows from (2.1) that the solution $\alpha y_1(z) + \beta y_2(z)$ has n zeros in D at the points $a_1, a_2, \ldots, a_{n-2}, b_1, b_2$, and (1.1) is conjugate in D.

(ii) Univalence implies disconjugacy. Suppose that there exists a solution $y_1(z)$ which vanishes at $a_1, a_2, \ldots, a_n$. There always exists a solution $y_2(z)$ which vanishes at $a_1, a_2, \ldots, a_{n-2}$ and is linearly independent of $y_1(z)$. Now, if

$$y_2(a_{n-1}) \neq 0, \qquad y_2(a_n) \neq 0, \tag{2.3}$$

then the function (2.1) is non-univalent in D. Thus, suppose that (2.3) is false and denote by Σ the set of common zeros of $y_1(z)$ and $y_2(z)$. We may assume, without loss of generality, that $a_{n-1} \in \Sigma$. Let now $b \in D$, such that $b \notin \Sigma$. There exists a solution $y_3(z) = \alpha_1 y_1(z) + \alpha_2 y_2(z)$, which vanishes at b and at all the points of Σ. Moreover, there exists another solution $y_4(z)$ which vanishes at b and at $a_1, \ldots, a_{n-3}$ and is linearly independent of $y_3(z)$. Now, $y_4(a_{n-2}) \neq 0$, $y_4(a_{n-1}) \neq 0$ since, suppose that $y_4(a_{n-2}) = 0$, then by Lemma 1 it follows from the univalence of all the functions of the type (2.1), that $y_4(z)$ is a linear combination of $y_1(z)$ and $y_2(z)$; i.e., $y_4(z) = \beta_1 y_1(z) + \beta_2 y_2(z)$. However, since $y_3(z)$ and $y_4(z)$ are linearly independent and since $y_3(b) = 0$ and $y_4(b) = 0$, it follows that $y_1(b) = y_2(b) = 0$, which contradicts our assumption that $b \notin \Sigma$. Therefore, $y_4(z)$ does not vanish at a_{n-2} nor at a_{n-1}.

Considering now the function

$$f(z, a_1, \ldots, a_{n-3}, b) = \frac{y_3(z)}{y_4(z)}, \tag{2.4}$$

we obtain that (2.4) is non-univalent in D.

3. Quantities invariant under linear transformations. Our next goal is to express the coefficients of (1.1) in terms of the function (2.1). Replacement of the solutions $y_1(z)$ and $y_2(z)$ in (2.1) by the linearly independent solutions $y_3(z) = Ay_1(z) + By_2(z)$ and $y_4(z) = Cy_1(z) + Dy_2(z)$, respectively, would have resulted in a function Tf, where T is of the type (1.6). Hence, any identity connecting the coefficients of equation (1.1) with the function (2.1) should be expressed by quantities invariant under the transformation (1.6). The simplest quantity of this type is the Schwarzian derivative

$$s(z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{3.1}$$

Other invariant quantities may be obtained by differentiating (3.1) and by producing various combinations of $s(z)$ and its derivatives. But, basically, all these invariant quantities are derived from $s(z)$. A different way of obtaining invariant quantities is by expanding the function

$$g(z) = \frac{f'(\zeta)}{f(z+\zeta) - f(\zeta)}$$

into a power series of the form

$$g(z) = \frac{1}{z} + \sum_{j=0}^{\infty} I_j[f(\zeta)]z^j. \tag{3.2}$$

It is easily checked that the coefficients $I_j[f(\zeta)]$ $(j = 1, 2, \ldots)$ are invariant under a linear transformation (1.6). Examination of the first coefficients shows that

$$I_1[f(\zeta)] = -\frac{1}{3!}\left[\frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2\right] = -\frac{s(\zeta)}{6} \tag{3.3}$$

and

$$I_2[f(\zeta)] = -\frac{1}{4!}\left[\frac{f^{(4)}}{f'} - \frac{4f'''f''}{(f')^2} + 3\left(\frac{f''}{f'}\right)^3\right] = -\frac{s'(\zeta)}{24}. \tag{3.4}$$

While for $I_j[f(\zeta)]$ $(j = 2, 3, \ldots)$ we have, by a recent result due to Ahronov (**1**), that the following recursion formula holds:

$$(j+3)I_{j+1}[f(\zeta)] = I_j'[f(\zeta)] - \sum_{k=1}^{j-1} I_k[f(\zeta)]I_{j-k}[f(\zeta)], \quad j = 1, 2, \ldots. \tag{3.5}$$

In view of (3.3), it now follows from (3.5) that all the invariants $I_j[f]$ $(j = 1, 2, \ldots)$ are also derived from (3.1). This information raises the question of whether there exists (at all) an invariant quantity which is not derived from the Schwarzian derivative $s(z)$. In the following theorem we answer this question in the negative, provided the function $f(z)$ is meromorphic in D, with at most simple poles, and such that $f'(z) \neq 0$. We shall say that such functions belong to the *restricted class* in D (see **10**), and denote the class by $\mathrm{RC}(D)$. Evidently, for $f(z) \in \mathrm{RC}(D)$, (3.1) is a regular function.

THEOREM 2. *Let $f(z) \in \mathrm{RC}(D)$ and let*

$$E[f(z)] = E[f(z), f'(z), \ldots, f^{(n)}(z)] = I(z) \tag{3.6}$$

be a differential operator of order n, operating on $f(z)$. If (3.6) *is invariant when f is subject to a linear transformation* (1.6), *namely, if*

$$E[Tf(z)] = E[f(z)] = I(z), \qquad z \in D, \tag{3.7}$$

then $I(z)$ is derived from $s(z)$, and $E[f(z)]$ is identical to a differential operator of order $(n-3)$, operating on $s(z)$, i.e.,

$$I(z) = E[f(z)] = E^*[s(z)] = E^*[s(z), s'(z), \ldots, s^{(n-3)}(z)]. \tag{3.8}$$

Proof. Let $z_0 \in D$. We may assume without loss of generality that

$$f(z_0) = 0, \quad f'(z_0) = 1, \quad f''(z_0) = 0. \tag{3.9}$$

Since, if (3.9) is not true and $f(z_0) = \alpha, f'(z_0) = \beta \neq 0$, and $f''(z_0) = 2\gamma$, then the function

$$F(z) = \frac{\beta[f(z) - \alpha]}{\beta^2 + \gamma[f(z) - \alpha]}, \qquad \alpha \neq \infty,$$

satisfies (3.9), and by (3.7) we have that $E[F(z)] = E[f(z)]$. If $\alpha = \infty$, then apply first a transformation $f \to f^{-1}$ and then proceed as before. Setting $z = z_0$ in (3.1) and (3.4), it follows by (3.9) that

$$s(z_0) = f'''(z_0) \tag{3.10}$$

and

$$s'(z_0) = f^{(4)}(z_0). \tag{3.11}$$

By differentiation of (3.4) and by induction we obtain

$$s^{(m)}(z) = \frac{f^{(m+3)}(z)}{f'(z)} + \frac{P_{m+2}[f^{(m+2)}(z), \ldots, f'(z)]}{[f'(z)]^{m+2}}, \quad m = 0, 1, 2, \ldots, \tag{3.12}$$

where P_{m+2} is a homogeneous polynomial of order $(m+2)$, in which the highest degree of $f'(z)$ is m. Using (3.9), it follows from (3.10), (3.11), and (3.12) that

$$\begin{aligned} s^{(m)}(z_0) &= f^{(m+3)}(z_0) \\ &\quad + P_{m+2}[f^{(m+2)}(z_0), \ldots, f^{(5)}(z_0), s'(z_0), s(z_0), 0, 1], \quad m = 2, 3, \ldots. \end{aligned} \tag{3.13}$$

By elimination and induction, (3.13) implies that

$$f^{(k)}(z_0) = s^{(k-3)}(z_0) + Q_{k-1}[s^{(k-4)}(z_0), \ldots, s(z_0)], \quad k = 3, 4, \ldots, \tag{3.14}$$

where Q_{k-1} $(k = 5, 6, \ldots)$ is a polynomial of order $(k - 1)$, free of terms of orders 0 and 1, and $Q_2 = Q_3 = 0$. Insertion of (3.9) and (3.14) in (3.6) yields

$$(3.15) \qquad I(z_0) = E[0, 1, 0, s(z_0), s'(z_0), f^{(5)}(z_0), \ldots, f^{(n)}(z_0)] = E^*[s(z_0), s'(z_0), \ldots, s^{(n-3)}(z_0)].$$

As (3.15) holds for every $z_0 \in D$, it implies the identity (3.8).

4. Relations between the coefficients of (1.1) and the Schwarzian derivative. If equation (1.1) is disconjugate in D, then, by Theorems 1 and 2, any connection between the coefficients of (1.1) and the function (2.1) has to be expressed in terms of the Schwarzian derivative of (2.1). However, when (1.1) is conjugate in D, (2.1) may or may not belong to $\mathrm{RC}(D)$ and Theorem 2 may not be applied. To take care of this problem, we replace the $(n - 2)$-parameter family of functions (2.1) by a one-parameter subfamily of functions $f(z, a)$ defined by

$$(4.1) \qquad f(z, a) = \frac{y_1(z)}{y_2(z)},$$

where $y_1(z)$ and $y_2(z)$ are linearly independent solutions of (1.1), which vanish $(n - 2)$ times at the point $a \in D$. Now, even if (1.1) is conjugate in D, we have that

$$\frac{df(z, a)}{dz}\Big|_{z=a} = f'(a, a) \neq 0,$$

and $f(z, a) \in \mathrm{RC}(N(a))$, where $N(a)$ is some neighbourhood in D of the point a. If

$$(4.2) \qquad s(z, a) = \{f(z, a), z\}$$

and

$$s^{(r)}(z, a) = \frac{d^r s(z, a)}{dz^r}, \qquad r = 1, 2, \ldots,$$

it follows that $s(z, a)$ and $s^{(r)}(z, a)$ are regular functions in $N(a)$. We are now ready to establish a relation between some of the coefficients of (1.1) and the derivatives of $s(z, a)$.

THEOREM 3. *Assume that*

$$(4.3) \qquad p_{n-2}(z) \equiv 0, p_{n-3}(z) \equiv 0, \ldots, p_{n-k+1}(z) \equiv 0, p_{n-k}(z) \not\equiv 0, \qquad 2 \leqq k \leqq n,$$

where $p_j(z)$ $(j = 0, 1, \ldots, n - 2)$ *are the coefficients of equation* (1.1). *Then*

$$(4.4) \qquad s(a, a) = 0, s'(a, a) = 0, \ldots, s^{(k-3)}(a, a) = 0, \qquad 3 \leqq k \leqq n,$$

and

$$(4.5) \qquad p_{n-k}(a) = \frac{(n + k - 1)!}{k(k + 1)!\,(n - 2)!} s^{(k-2)}(a, a), \qquad 2 \leqq k \leqq n.$$

Proof. Let $y_1(z)$ and $y_2(z)$ be two solutions of (1.1) which satisfy the following initial conditions.

$$(4.6)\qquad y_1(a) = 0, y_1'(a) = 0, \ldots, y_1^{(n-2)}(a) = 0, y_1^{(n-1)}(a) = (n-1)!,$$

$$(4.7)\qquad y_2(a) = 0, y_2'(a) = 0, \ldots, y_2^{(n-3)}(a) = 0,$$
$$y_2^{(n-2)}(a) = (n-2)!, y_2^{(n-1)}(a) = 0.$$

By (1.1), (4.3), and (4.6), it follows that

$$(4.8)\qquad y_1(z) = (z-a)^{n-1}[1 + \alpha(z-a)^k + \ldots], \qquad 2 \leqq k \leqq n,$$

with

$$(4.9)\qquad \alpha = \frac{y_1^{(n+k-1)}(a)}{(n+k-1)!} = -\frac{p_{n-k}(a)y_1^{(n-1)}(a)}{(n+k-1)!} = -\frac{p_{n-k}(a)(n-1)!}{(n+k-1)!},$$

and in a similar way,

$$(4.10)\qquad y_2(z) = (z-a)^{n-2}[1 + \beta(z-a)^k + \ldots], \qquad 2 \leqq k \leqq n,$$

with

$$(4.11)\qquad \beta = -\frac{p_{n-k}(a)(n-2)!}{(n+k-2)!}.$$

By inserting (4.8) and (4.10) in (4.1) we obtain

$$(4.12)\qquad f(z, a) = (z-a)[1 + (\alpha - \beta)(z-a)^k + \ldots], \qquad 2 \leqq k \leqq n.$$

Hence,

$$(4.13)\qquad f(a, a) = 0, f'(a, a) = 1, f''(a, a) = 0, \ldots, f^{(k)}(a, a) = 0,$$
$$f^{(k+1)}(a, a) = (k+1)!(\alpha - \beta), \qquad 2 \leqq k \leqq n.$$

By (4.13) it follows from (3.12) that

$$s^{(m)}(a, a) = f^{(m+3)}(a, a), \qquad m = 0, 1, 2, \ldots, (k-2), \qquad 2 \leqq k \leqq n,$$

which implies (4.4) and (4.5).

Since any solution of (1.1), which has a zero of order $(n-2)$ at the point a, is a linear combination of the two particular solutions (4.8) and (4.10), a different choice of the two solutions in (4.1) would replace f by Tf, where T is of the form (1.6). However, $s(z, a)$ and $s^{(r)}(z, a)$ are invariant under the transformation (1.6); hence, (4.4) and (4.5) hold for any choice of the solutions $y_1(z)$ and $y_2(z)$ regardless of the initial conditions (4.6) and (4.7).

Remark. If for $n = 2$, (4.1) is interpreted as (1.2), then (4.5) implies the known relation (1.4).

5. Linear transformations of equation (1.1). We shall now use (4.4) and (4.5) in order to study the result of a linear transformation

$$z = \frac{A\zeta + B}{C\zeta + D}, \qquad AD - BC \neq 0, \tag{5.1}$$

upon the differential equation (1.1). We start by considering the effect of (5.1) on $s(z, a)$.

Suppose that $z = z(\zeta)$ is a one-to-one analytic transformation which maps the domain Δ onto D and $\alpha \in \Delta$ to $a \in D$, then

$$f(z, a) = f[z(\zeta), a] = \phi(\zeta, \alpha). \tag{5.2}$$

The Schwarzian derivative when subject to a transformation $z = z(\zeta)$ obeys the following rule, namely,

$$\sigma(\zeta, \alpha) = s(z, a)\left(\frac{dz}{d\zeta}\right)^2 + \{z(\zeta), \zeta\}, \tag{5.3}$$

where

$$\sigma(\zeta, \alpha) = \{\phi(\zeta, \alpha), \zeta\}, \qquad s(z, a) = \{f(z, a), z\}.$$

If $z(\zeta)$ is of the form (5.1), then $\{z(\zeta), \zeta\} \equiv 0$, and

$$\sigma(\zeta, \alpha) = s(z, a)\left(\frac{dz}{d\zeta}\right)^2. \tag{5.3$'$}$$

By (5.3)$'$, we have that $s(a, a) = 0$ if and only if $\sigma(\alpha, \alpha) = 0$. Differentiation of (5.3)$'$ with respect to ζ yields

$$\sigma'(\zeta, \alpha) = s'(z, a)\left(\frac{dz}{d\zeta}\right)^3 + 2s(z, a)\,\frac{dz}{d\zeta}\,\frac{d^2z}{d\zeta^2}. \tag{5.4}$$

Suppose that $s(a, a) = 0$, then

$$\sigma'(\alpha, \alpha) = s'(a, a)\left(\frac{dz}{d\zeta}\right)^3\bigg|_{\zeta=\alpha}$$

and $s'(a, a) = 0$ if and only if $\sigma'(\alpha, \alpha) = 0$. By successive differentiation of (5.3)$'$ and by assuming (4.4), we obtain

$$\begin{aligned} &\sigma(\alpha, \alpha) = 0, \sigma'(\alpha, \alpha) = 0, \dots, \sigma^{(k-3)}(\alpha, \alpha) = 0, \qquad 3 \leqq k \leqq n, \\ &\sigma^{(k-2)}(\alpha, \alpha) = s^{(k-2)}(a, a)\left(\frac{dz}{d\zeta}\right)^k\bigg|_{\zeta=\alpha}, \qquad 2 \leqq k \leqq n, \end{aligned} \tag{5.5}$$

which can be rewritten as

$$\sigma^{(r)}(\alpha, \alpha) = s^{(r)}(a, a)\left(\frac{dz}{d\zeta}\right)^{r+2}\bigg|_{\zeta=\alpha}, \qquad r = 0, 1, 2, \dots, k-2. \tag{5.6}$$

Formula (5.6) provides us now with a deeper understanding of the mechanism which determines the form into which equation (1.1) is transformed when subject to a linear transformation (5.1). We now give a new proof to two theorems stated by Hadass for the case $k = n$; see (**5**, Theorems 1 and 2).

THEOREM 4. *Equation* (1.1) *with the additional assumption* (4.3) *is transformed by a one-to-one transformation*

$$z = z(\zeta), \quad w(\zeta) = y[z(\zeta)]\tau(\zeta), \qquad \tau(\zeta) \neq 0, \tag{5.7}$$

into an equation of the same form, namely,

$$w^{(n)}(\zeta) + q_{n-2}(\zeta)w^{(n-2)}(\zeta) + \ldots + q_0(\zeta)w(\zeta) = 0, \tag{5.8}$$

with

$$\begin{aligned} &q_{n-2}(\zeta) \equiv 0, \ldots, q_{n-k+1}(\zeta) \equiv 0, \\ &q_{n-k}(\zeta) = p_{n-k}[z(\zeta)]\left(\frac{dz}{d\zeta}\right)^k, \qquad 2 \leqq k \leqq n, \end{aligned} \tag{5.9}$$

if and only if $z(\zeta)$ *is of the form* (5.1).

Proof. Substitution of $z = z(\zeta)$ and $\omega(\zeta) = y[z(\zeta)]$ in (1.1) leads us to

$$\omega^{(n)}(\zeta) + R_{n-1}(\zeta)\omega^{(n-1)}(\zeta) + \ldots + R_0(\zeta)\omega(\zeta) = 0.$$

It is well known that the coefficient of $\omega^{(n-1)}(\zeta)$ can be removed by a suitable choice of the function $\tau(\zeta)$ in (5.7). Indeed, by setting

$$\tau(\zeta) = \exp\left[-\int \frac{R_{n-1}(\zeta)}{n}\,d\zeta\right],$$

one obtains equation (5.8). Thus, what we really have to prove is that (4.3) implies (5.9), if and only if $z(\zeta)$ is linear. Let $y_1(z)$ and $y_2(z)$ be linearly independent solutions of (1.1) possessing zeros of order $(n-2)$ at $a \in D$; then $w_1(\zeta) = y_1[z(\zeta)]\tau(\zeta)$ and $w_2(\zeta) = y_2[z(\zeta)]\tau(\zeta)$ are independent solutions of (5.8) with zeros of order $(n-2)$ at the point α ($z(\alpha) = a$), and

$$\frac{w_1(\zeta)}{w_2(\zeta)} = \frac{y_1[z(\zeta)]}{y_2[z(\zeta)]} = f[z(\zeta), a] = \phi(\zeta, \alpha) \tag{5.10}$$

holds.

Suppose now that $z(\zeta)$ is of the type (5.1). By Theorem 3, (4.3) implies (4.4) and (4.5) which imply (5.6). In view of (5.10), we may apply (4.5) to the coefficients $q_{n-2}(\zeta), \ldots, q_{n-k}(\zeta)$ of (5.8), and by (5.6), we obtain (5.9). Conversely, assume that $z(\zeta)$ is not linear; then $\{z(\zeta), \zeta\} \not\equiv 0$, i.e., there exists a point $\alpha \in \Delta$, such that $\{z(\zeta), \zeta\}|_{\zeta=\alpha} \neq 0$. For $3 \leqq k \leqq n$, it now follows from (5.3) and (4.4) that $\sigma(\alpha, \alpha) \neq 0$ which, by (4.5), implies that $q_{n-2}(\alpha) \neq 0$. For $k = 2$, it is trivial that (5.3) implies (5.9) if and only if $z(\zeta)$ is a linear transformation.

Remark. As noted by Hadass, the necessary condition goes back to a theorem by Wilczynski (**11**).

6. Necessary condition for disconjugacy in the unit disk. We shall now use the results of Theorems 1, 3, and 4 to obtain a necessary condition for disconjugacy of equation (1.1) in the unit disk.

THEOREM 5. *Let the coefficients of* (1.1) *be regular for* $|z| < 1$ *and satisfy* (4.3) *there, and let equation* (1.1) *be disconjugate in the unit disk; then*

$$(6.1) \qquad |p_{n-k}(z)| \leqq \frac{2(n+k-1)!}{k^2(n-2)!\,(1-|z|^2)^k}, \qquad |z| < 1,\ 2 \leqq k \leqq n.$$

Proof. By Theorem 1, disconjugacy of (1.1) implies univalence of the function (2.1). In particular, the function (4.1) is univalent in the unit disk for any $|a| < 1$. Setting $a = 0$ in (4.1) and choosing $y_1(z)$ and $y_2(z)$ as in (4.6) and (4.7), we obtain by (4.12)

$$(4.12)' \qquad f(z, 0) = z + (\alpha - \beta)z^{k+1} + \ldots, \qquad 2 \leqq k \leqq n,$$

where α and β are given by (4.9) and (4.11). However, for the univalent function (4.12)′ we may apply the known inequality (see **3**; **4**),

$$(6.2) \qquad |\alpha - \beta| \leqq 2/k,$$

and equality holds in (6.2) if and only if

$$(6.3) \qquad f(z, 0) = \frac{z}{(1 - e^{i\theta}z^k)^{2/k}}, \qquad 0 \leqq \theta < 2\pi.$$

By (4.9) and (4.11) it follows from (6.2) that

$$(6.4) \qquad |p_{n-k}(0)| \leqq \frac{2(n+k-1)!}{k^2(n-2)!}, \qquad 2 \leqq k \leqq n,$$

which establishes (6.1) for $z = 0$. In order to prove (6.1) for any $|z| < 1$, we apply a linear transformation

$$(6.5) \qquad z = \frac{\zeta + a}{1 + \bar{a}\zeta}, \qquad |a| < 1,$$

which maps $|\zeta| < 1$ onto $|z| < 1$. By Theorem 4, equation (1.1) is transformed into equation (5.8), and by (5.9)

$$(6.6) \qquad q_{n-k}(0) = p_{n-k}(a)\left(\frac{dz}{d\zeta}\right)^k \bigg|_{\zeta=0}.$$

Since disconjugacy is preserved by the transformation (5.7), which in our case means that (1.1) is disconjugate in the unit circle if and only if (5.8) is, we may apply (6.4) to $q_{n-k}(0)$. Using the fact that for transformations of the unit circle on itself,

$$(6.7) \qquad \left|\frac{dz}{d\zeta}\right| = \frac{1 - |z|^2}{1 - |\zeta|^2}$$

holds, we obtain (6.1).

In view of (3.14) and (5.6), it is possible to state Theorem 5 also as a necessary condition for univalence of $f(z)$ in $|z| < 1$.

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THEOREM 5′. *Assume that* $f(z)$ *is univalent for* $|z| < 1$ *and let* $s(z) = \{f(z), z\}$. *Suppose that*

$$s(a) = s'(a) = \ldots = s^{(m-1)}(a) = 0, \qquad |a| < 1;$$

then

$$|s^{(m)}(a)| \leqq \frac{2(m+3)!}{(m+2)(1-|a|^2)^{m+2}}, \qquad |a| < 1, \quad m = 0, 1, 2, \ldots.$$

7. The equation $y^{(2m)} + py = 0$**.** For $k = n = 2$, (6.1) reduces to (1.7) which is the necessary condition given by Nehari for the disconjugacy of equation (1.3). The natural question to be asked next is whether it is possible to establish a sufficient condition for disconjugacy, which will generalize the sufficient condition (1.8). (Sufficient conditions of different type were given by Nehari in (**8**).) It is obvious that the easiest case to handle is that of the equation

$$y^{(n)}(z) + p(z)y(z) = 0, \tag{7.1}$$

where we have only one coefficient. For (7.1) we have the following conjecture.

CONJECTURE. *Assume that* $p(z)$ *is regular in* $|z| < 1$. *In order that* (7.1) *be disconjugate in* $|z| < 1$, *it is sufficient that*

$$|p(z)| \leqq \frac{A(n)}{(1-|z|^2)^n}, \qquad |z| < 1, \tag{7.2}$$

with a suitable constant $0 < A(n) \leqq 2(2n-1)!/n^2(n-2)!$.

Unfortunately, we have not succeeded in proving this conjecture nor in disproving it. Yet, weaker results partially supporting (7.2) were obtained for equations of even order

$$y^{(2m)}(z) + p(z)y(z) = 0, \qquad m = 1, 2, \ldots. \tag{7.3}$$

In the following theorem we prove that a condition of the type (7.2) guarantees the non-existence of a solution of equation (7.3) possessing two zeros of order m.

THEOREM 6. *Assume that* $p(z)$ *is regular in* $|z| < 1$ *and satisfies*

$$|p(z)| \leqq \frac{B(2m)}{(1-|z|^2)^{2m}}, \qquad |z| < 1, \tag{7.4}$$

where

$$B(2) = 1, \; B(4) = 9, \; B(2m) = 9\prod_{k=3}^{m}(4k-3), \qquad m = 3, 4, \ldots; \tag{7.5}$$

then no solution of (7.3) *has two zeros, of order* m *in* $|z| < 1$.

To prove Theorem 6 we need an integral inequality, which will be established in the following lemma.

LEMMA 2. *Let $U(x)$ be a real function with m continuous derivatives in the interval $[-1, 1]$, possessing zeros of order m at the points $x = \pm 1$; then*

$$(7.6)\qquad \int_{-1}^{+1} [U^{(m)}(x)]^2\,dx > B(2m) \int_{-1}^{+1} \frac{[U(x)]^2\,dx}{(1-x^2)^{2m}}, \qquad m = 1, 2, \ldots,$$

where $B(2m)$ are constants defined in (7.5).

Proof. (7.6) was proved for $m = 1$ by Nehari (**7**). By a slight change in Nehari's proof we first establish the following inequality

$$(7.7)\qquad \int_{-1}^{+1} \frac{[V'(x)]^2}{(1-x^2)^{2k-2}} \geqq (4k-3) \int_{-1}^{+1} \frac{[V(x)]^2}{(1-x^2)^{2k}}, \qquad k = 1, 2, \ldots,$$

for the real continuous function $V(x)$ with zeros of order k at ± 1. Expansion and integration by parts of the trivial inequality

$$\int_{-1}^{+1} \left[\frac{V'(x)}{(1-x^2)^{k-1}} + \frac{\gamma x V(x)}{(1-x^2)^k}\right]^2 dx \geqq 0$$

leads us to

$$\int_{-1}^{+1} \frac{[V'(x)]^2\,dx}{(1-x^2)^{2k-2}} \geqq \gamma \int_{-1}^{+1} \frac{1 + (4k-3-\gamma)x^2}{(1-x^2)^{2k}} [V(x)]^2\,dx, \qquad k = 1, 2, \ldots .$$

Setting now $\gamma = 4k - 3$, (7.7) follows. Equality may hold in (7.7) if and only if

$$(7.8)\qquad V(x) = C(1-x^2)^{2k-3/2}, \qquad k = 1, 2, \ldots .$$

For $k = 1$, (7.8) does not satisfy our hypotheses; thus, equality in (7.7) is excluded; however, for $k = 2, 3, \ldots,$ equality may hold in (7.7). Applying (7.7) successively to the functions $V(x) = U^{(m-1)}(x)$ $(k = 1)$, $V(x) = U^{(m-2)}(x)$ $(k = 2), \ldots, V(x) = U'(x)$ $(k = m - 1)$, we obtain

$$\int_{-1}^{+1} [U^{(m)}(x)]^2\,dx > \int_{-1}^{+1} \frac{[U^{(m-1)}(x)]^2}{(1-x^2)^2}\,dx \geqq 1 \cdot 5 \int_{-1}^{+1} \frac{[U^{(m-2)}(x)]^2}{(1-x^2)^4}\,dx \geqq \ldots \geqq$$
$$1 \cdot 5 \cdot 9 \cdot \ldots (4m-3) \int_{-1}^{+1} \frac{[U(x)]^2}{(1-x^2)^{2m}}\,dx.$$

Hence

$$(7.9)\qquad \int_{-1}^{+1} [U^{(m)}(x)]^2\,dx > \prod_{k=1}^{m} (4k-3) \int_{-1}^{+1} \frac{[U(x)]^2}{(1-x^2)^{2m}}\,dx.$$

Now, (7.9) differs from (7.6) only by a constant. To prove (7.6) one has to use Beesack's inequality (**2**, p. 494),

$$(7.10)\qquad \int_{-1}^{+1} [V''(x)]^2\,dx > 9 \int_{-1}^{+1} \frac{[V(x)]^2\,dx}{(1-x^2)^4},$$

which holds for the real function $V(x)$ with two continuous derivatives in the interval $[-1, 1]$, possessing zeros of second order at ± 1. Beesack mentioned that for $V(x) = C(1 - x^2)^{3/2}$, both sides of (7.10) are $+\infty$. However, since $(1 - x^2)^{3/2}$ does not satisfy our hypotheses, (7.10) always holds for the class of functions defined above. Applying (7.10) to $V(x) = U^{(m-2)}(x)$ we obtain

$$(7.10)' \qquad \int_{-1}^{+1} [U^{(m)}(x)]^2\,dx > 9\int_{-1}^{+1} \frac{[U^{(m-2)}(x)]^2}{(1-x^2)^4}\,dx.$$

Proceeding now as before by applying (7.7) successively, (7.6) follows.

Remark. By substituting ρx for x in (7.6) we obtain a modified form of inequality (7.6),

$$(7.6)' \quad \int_{-\rho}^{\rho} [U^{(m)}(x)]^2\,dx > B(2m)\rho^{2m}\int_{-\rho}^{\rho} \frac{[U(x)]^2\,dx}{(\rho^2 - x^2)^{2m}}, \qquad m = 1, 2, \ldots,$$

which holds for the real function $U(x)$ with m continuous derivatives in the interval $[-\rho, \rho]$, possessing zeros of order m at $\pm\rho$.

Proof of Theorem 6. Suppose that the theorem is false and that there exists a solution $y(z)$ with two zeros z_1 and z_2 ($|z_1|, |z_2| < 1$) each of multiplicity m. By a suitable choice of the parameters α and θ in

$$(7.11) \qquad \zeta(z) = e^{i\theta}\frac{(z-\alpha)}{1-\bar{\alpha}z}, \qquad |\alpha| < 1,\ 0 \leqq \theta < 2\pi,$$

it is possible to map $|z| < 1$ onto $|\zeta| < 1$ and z_1 and z_2 on two symmetric points of the real axes $\pm\rho$. By Theorem 4, the differential equation (7.3) is transformed into

$$(7.12) \qquad w^{(2m)}(\zeta) + q(\zeta)w(\zeta) = 0$$

with

$$q(\zeta) = p(z)\left(\frac{dz}{d\zeta}\right)^{2m}.$$

By (6.7) and (7.4), it follows that

$$(7.4)' \qquad |q(\zeta)| = |p(z)|\cdot\left(\frac{1-|z|^2}{1-|\zeta|^2}\right)^{2m} \leqq \frac{B(2m)}{(1-|\zeta|^2)^{2m}}, \qquad |\zeta| < 1.$$

Thus, our assumption that (7.3) has a solution with two zeros of order m implies that (7.12) has a solution $w_1(\zeta)$ possessing two zeros of order m at $\pm\rho$, while (7.4)$'$ holds. We now write (7.12) for $w_1(\zeta)$, multiply by $\overline{w_1(\zeta)}$, and integrate along the real axes. This leads us to

$$\int_{-\rho}^{\rho} w_1^{(2m)}(x)\bar{w}_1(x)\,dx + \int_{-\rho}^{\rho} q(x)|w_1(x)|^2\,dx = 0.$$

By integrating by parts m times, and by noting that all the integrated parts vanish, we obtain

$$(-1)^m\int_{-\rho}^{\rho} |w_1^{(m)}(x)|^2\,dx = -\int_{-\rho}^{\rho} q(x)|w_1(x)|^2\,dx.$$

Hence,

(7.13) $$\int_{-\rho}^{\rho} |w_1^{(m)}(x)|^2\,dx = \left|\int_{-\rho}^{\rho} q(x)|w_1(x)|^2\,dx\right| \leqq \int_{-\rho}^{\rho} |q(x)|\,|w_1(x)|^2\,dx.$$

Writing $w_1(x) = u(x) + iv(x)$, we have that

$$|w_1|^2 = u^2 + v^2, \qquad |w_1^{(m)}|^2 = [u^{(m)}]^2 + [v^{(m)}]^2,$$

and (7.13) takes the form

(7.13)′ $$\int_{-\rho}^{\rho} ([u^{(m)}(x)]^2 + [v^{(m)}(x)]^2)\,dx \leqq \int_{-\rho}^{\rho} |q(x)|[u^2(x) + v^2(x)]\,dx$$

which, by (7.4)′, implies that

(7.14) $$\int_{-\rho}^{\rho} ([u^{(m)}(x)]^2 + [v^{(m)}(x)]^2)\,dx \leqq B(2m)\int_{-\rho}^{\rho} \frac{u^2(x) + v^2(x)}{(1 - x^2)^{2m}}\,dx < B(2m)\rho^{2m}\int_{-\rho}^{\rho} \frac{u^2(x) + v^2(x)}{(\rho^2 - x^2)^{2m}}\,dx.$$

Since $w_1(x) = u(x) + iv(x)$ is supposed to have zeros of order m at $x = \pm\rho$, the same is true for $u(x)$ and $v(x)$ separately. By the remark following Lemma 2 we therefore obtain

(7.15) $$\int_{-\rho}^{\rho} ([u^{(m)}(x)]^2 + [v^{(m)}(x)]^2)\,dx > B(2m)\rho^{2m}\int_{-\rho}^{\rho} \frac{u^2(x) + v^2(x)}{(\rho^2 - x^2)^{2m}}\,dx$$

which contradicts (7.14). Thus, we have proved that no solution of (7.3) can have two zeros of order m in the unit circle if $p(z)$ satisfies (7.4).

Remark 1. For a fourth-order equation ($m = 2$), Theorem 6 is included in (**5**, Theorem 6), while for $m \geqq 3$, our Theorem 6 may serve as a complementary theorem to (**5**, Theorem 6).

Remark 2. With regards to the sharpness of Theorem 6, the question is still open. It seems that for $m = 2$, $B(4) = 9$ is the best constant, while for $m \geqq 3$, $B(2m)$ are not the best.

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