Bull. Aust. Math. Soc. **109** (2024), 365–375
doi:10.1017/S0004972723000655

ON CERTAIN CLOSE-TO-CONVEX FUNCTIONS

MD FIROZ ALI and MD NUREZZAMAN

(Received 10 April 2023; accepted 29 May 2023; first published online 17 July 2023)

Abstract

Let $\mathcal{K}_u$ denote the class of all analytic functions f in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$, normalised by $f(0) = f'(0) - 1 = 0$ and satisfying $|zf'(z)/g(z) - 1| < 1$ in $\mathbb{D}$ for some starlike function g. Allu, Sokól and Thomas ['On a close-to-convex analogue of certain starlike functions', *Bull. Aust. Math. Soc.* **108** (2020), 268–281] obtained a partial solution for the Fekete–Szegö problem and initial coefficient estimates for functions in $\mathcal{K}_u$, and posed a conjecture in this regard. We prove this conjecture regarding the sharp estimates of coefficients and solve the Fekete–Szegö problem completely for functions in the class $\mathcal{K}_u$.

2020 *Mathematics subject classification*: primary 30C45; secondary 30C55.

Keywords and phrases: analytic functions, univalent functions, starlike functions, convex functions, close-to-convex function.

1. Introduction

Let $\mathcal{H}$ be the class of all analytic functions in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{B}$ be the subclass of $\mathcal{H}$ consisting of all functions f in $\mathcal{H}$ with $|f(z)| < 1$ for all $z \in \mathbb{D}$, $\mathcal{B}_0$ be the subclass of $\mathcal{B}$ with $f(0) = 0$ and $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of all functions f normalised by $f(0) = f'(0) - 1 = 0$ with the Taylor series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, let $\mathcal{S}$ be the subclass of $\mathcal{A}$ that are univalent (that is, one-to-one) in $\mathbb{D}$. A function $f \in \mathcal{A}$ is called starlike (respectively, convex) if $f(\mathbb{D})$ is a starlike domain (respectively, a convex domain) with respect to the origin. The set of all starlike functions and convex functions in $\mathcal{S}$ are denoted by $\mathcal{S}^*$ and $\mathcal{C}$, respectively. It is well known that a function f in $\mathcal{A}$ is starlike (respectively, convex) if and only if $\operatorname{Re} zf'(z)/f(z) > 0$ (respectively, $\operatorname{Re}(1 + zf''(z)/f'(z)) > 0$) for $z \in \mathbb{D}$. For further information about these classes, we refer to [5, 7].

A function $f \in \mathcal{A}$ is said to be close-to-convex if the complement of the image-domain $f(\mathbb{D})$ in $\mathbb{C}$ is the union of rays that are disjoint (except that the origin of one ray may lie on another one of the rays) and the class of all close-to-convex functions is denoted by $\mathcal{K}$. This class was introduced by Kaplan [10]. A function $f \in \mathcal{A}$ is close-to-convex if and only if there exists a starlike function $g \in \mathcal{S}^*$ and a real number $\alpha \in (-\pi/2, \pi/2)$ such that (see [5, 10])

$$\mathrm{Re}\left(e^{i\alpha}\frac{zf'(z)}{g(z)}\right) > 0, \quad z \in \mathbb{D}.$$

In 1968, Singh [16] introduced and studied the class $\mathcal{S}_u^*$ consisting of functions f in $\mathcal{A}$ such that

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 \quad \text{for } z \in \mathbb{D}.$$

It is easy to see that every function in $\mathcal{S}_u^*$ also belongs to $\mathcal{S}^*$. Singh [16] obtained the distortion theorem, coefficient estimate and radius of convexity for the class $\mathcal{S}_u^*$. Recently, Allu *et al.* [1] introduced a close-to-convex analogue of the class $\mathcal{S}_u^*$ denoted by $\mathcal{K}_u$. A function f in $\mathcal{A}$ belongs to $\mathcal{K}_u$ if there exists a starlike function $g \in \mathcal{S}^*$ such that

$$\left|\frac{zf'(z)}{g(z)} - 1\right| < 1 \quad \text{for } z \in \mathbb{D}.$$

Clearly, every function in $\mathcal{K}_u$ is close-to-convex.

It is well known that if $f \in \mathcal{S}$ is of the form (1.1), then $|a_n| \le n$ for all $n \ge 2$, and equality holds for the rotations of the Koebe function $k(z) = z/(1-z)^2$. Singh [16] proved that if $f \in \mathcal{S}_u^*$, then $|a_n| \le 1/(n-1)$ for all $n \ge 2$, and this inequality is sharp. In 2020, Allu *et al.* [1] studied coefficient bounds for the functions $f(z)$ of the form (1.1) in the class $\mathcal{K}_u$ and obtained the sharp bounds $|a_2| \le 3/2$ and $|a_3| \le 5/3$ and proposed a conjecture that $|a_n| \le (2n-1)/n$ for $n \ge 4$.

The Fekete–Szegö problem is to find the maximum value of the coefficient functional

$$\Phi_\mu(f) = |a_3 - \mu a_2^2|, \quad \mu \in \mathbb{C},$$

when f of the form (1.1) varies over a class of functions $\mathcal{F}$. In 1933, Fekete–Szegö [6] used the Löwner differential method to prove that

$$\max_{f\in\mathcal{S}} \Phi_\mu(f) = \begin{cases} 1 + 2e^{-2\mu/(1-\mu)} & \text{for } 0 \le \mu < 1, \\ 1 & \text{for } \mu = 1. \end{cases}$$

In 1987, Koepf [12] obtained the sharp bound of $\Phi_\mu(f)$ for any $\mu \in \mathbb{R}$ for the class $\mathcal{K}$:

$$\max_{f\in\mathcal{K}} \Phi_\mu(f) = \begin{cases} |3-4\mu| & \text{if } \mu \in \left(-\infty, \frac{1}{3}\right] \cup [1, \infty), \\ \frac{1}{3} + \frac{4}{9\mu} & \text{if } \mu \in \left[\frac{1}{3}, \frac{2}{3}\right], \\ 1 & \text{if } \mu \in \left[\frac{2}{3}, 1\right]. \end{cases}$$

https://doi.org/10.1017/S0004972723000655 Published online by Cambridge University Press

The Fekete–Szegö problem has been studied for different subclasses of $\mathcal{S}$ (see [9, 13–15, 17]). Allu *et al*. [1] considered the class $\mathcal{K}_u$ and obtained an estimate of the Fekete–Szegö functional $|a_3 - \mu a_2^2|$ with $\mu \in \mathbb{R}$. However, they were only able to show sharpness when $\mu \leq 0$, $2/3 \leq \mu \leq 1$ and $\mu \geq 10/9$.

Let $\mathcal{LU}$ denote the subclass of $\mathcal{H}$ consisting of all locally univalent functions in $\mathbb{D}$, that is, $\mathcal{LU} := \{f \in \mathcal{H} : f'(z) \neq 0 \text{ for all } z \in \mathbb{D}\}$. For a locally univalent function $f \in \mathcal{LU}$, the pre-Schwarzian derivative is defined by

$$P_f(z) = \frac{f''(z)}{f'(z)},$$

and the pre-Schwarzian norm (the hyperbolic sup-norm) is defined by

$$\|P_f\| = \sup_{z\in\mathbb{D}}(1 - |z|^2)|P_f(z)|.$$

This norm has significant meaning in the theory of Teichmüller spaces. For a univalent function f, it is well known that $\|P_f\| \leq 6$ and the estimate is sharp. However, if $\|P_f\| \leq 1$, then f is univalent in $\mathbb{D}$ (see [2, 3]). In 1976, Yamashita [18] proved that $\|P_f\|$ is finite if and only if f is uniformly locally univalent in $\mathbb{D}$. Moreover, if $\|P_f\| < 2$, then f is bounded in $\mathbb{D}$ (see [11]). We will obtain results related to the pre-Schwarzian norm for functions $f \in \mathcal{K}_u$.

We first prove the conjecture $|a_n| \leq (2n-1)/n$ for $n \geq 2$ for functions in $\mathcal{K}_u$ as proposed by Allu *et al*. [1]. We next obtain the sharp estimate of the Fekete–Szegö functional $\Phi_\mu(f)$ for the class $\mathcal{K}_u$ for any $\mu \in \mathbb{R}$. Finally, we obtain estimates of the pre-Schwarzian norm for functions in $\mathcal{K}_u$.

2. Main results

Before stating our main results, we will discuss some preliminaries which will help us to prove our results. The first lemma is part of a result proved by Choi *et al*. [4].

LEMMA 2.1. *For $A, B \in \mathbb{C}$ and $K, L, M \in \mathbb{R}$, let*

$$\Omega(A, B, K, L, M) = \max_{\substack{|u_1|\leq 1\\ |v_1|\leq 1}}(|A|(1 - |u_1|^2) + |B|(1 - |v_1|^2) + |Ku_1^2 + Lv_1^2 + 2Mu_1v_1|).$$

Further consider the following four conditions involving A, B, K, L, M:

(A1) $|A| \geq \max\left\{|K|\sqrt{1 - \frac{M^2}{KL}}, |M| - |K|\right\}$;

(A2) $|K| + |M| \leq |A| < |K|\sqrt{1 - \frac{M^2}{KL}}$;

(B1) $|B| \geq \max\left\{|L|\sqrt{1 - \frac{M^2}{KL}}, |M| - |L|\right\}$;

(B2) $|L| + |M| \leq |B| < |L|\sqrt{1 - \frac{M^2}{KL}}$.

https://doi.org/10.1017/S0004972723000655 Published online by Cambridge University Press

If $KL \geq 0$ *and* $D = (|K| - |A|)(|L| - |B|) - M^2$, *then*

$$\Omega(A,B,K,L,M) = \begin{cases} |A| + |L| - \dfrac{M^2}{|K| - |L|} & \text{if } |A| > |M| + |K| \text{ and } D < 0, \\ |B| + |K| - \dfrac{M^2}{|L| - |B|} & \text{if } |B| > |M| + |L| \text{ and } D < 0, \\ |K| + 2|M| + |L| & \text{otherwise.} \end{cases}$$

If $KL < 0$, *then* $\Omega(A,B,K,L,M) = |A| + |B| + \max\{0, R\}$, *where*

$$R = \begin{cases} |K| - |A| + \dfrac{M^2}{|B| + |L|}, & \text{when (B1) holds but (A1) and (A2) do not hold,} \\ |L| - |B| + \dfrac{M^2}{|A| + |K|}, & \text{when (A1) holds but (B1) and (B2) do not hold.} \end{cases}$$

For two functions f and g in $\mathcal{H}$, we say that $f(z)$ is majorised by $g(z)$ if $|f(z)| \leq |g(z)|$ for all $z \in \mathbb{D}$ or equivalently, if there exists $\omega \in \mathcal{B}$ such that $f(z) = \omega(z)g(z)$. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $F(z) = \sum_{n=0}^{\infty} A_n z^n$ be two power series convergent in some disk $E_R = \{z : |z| < R,\ R > 0\}$. We say that $f(z)$ is dominated by $F(z)$ and we write $f(z) \ll F(z)$ if for any integer $n \geq 0$, $|a_n| \leq |A_n|$.

LEMMA 2.2 [8, Theorem 6.7]. *If* $f(z) = \sum_{n=1}^{\infty} a_n z^n$, $z \in \mathbb{D}$, *is majorised by* g *and* $g \in \mathcal{S}^*$, *then* $|a_n| \leq n$ *for all* $n \geq 1$, *that is,* $f(z) \ll k(z)$, *where* $k(z) = z/(1-z)^2$ *is the Koebe function.*

Our first result confirms the conjecture of Allu *et al.* in [1].

THEOREM 2.3. *Let* $f \in \mathcal{K}_u$ *be of the form (1.1). Then,*

$$|a_n| \leq \frac{2n-1}{n} \quad \text{for all } n \geq 2.$$

Moreover, the estimate is sharp.

PROOF. Let $f \in \mathcal{K}_u$ be of the form (1.1). Then there exists a starlike function $g \in \mathcal{S}^*$ such that

$$\left|\frac{zf'(z)}{g(z)} - 1\right| < 1.$$

Further, there exists a function $\omega(z) \in \mathcal{B}_0$ such that

$$zf'(z) = g(z)(1 + \omega(z)),$$

that is,

$$zf'(z) = g(z) + zg(z)\omega_1(z) \tag{2.1}$$

for some $\omega_1(z) \in \mathcal{B}$. Since, $g(z)\omega_1(z)$ is majorised by $g(z)$ and $g \in \mathcal{S}^*$, by Lemma 2.2, the function $g(z)\omega_1(z)$ is dominated by $k(z)$, that is, $g(z)\omega_1(z) \ll k(z)$. Thus, from (2.1),

$$zf'(z) \ll k(z) + zk(z),$$

https://doi.org/10.1017/S0004972723000655 Published online by Cambridge University Press

and consequently,

$$|a_n| \le \frac{2n-1}{n}.$$

The estimate is sharp for the function $f_1 \in \mathcal{K}_u$ given by

$$f_1(z) = \frac{2z}{1-z} + \log(1-z). \qquad \square$$

For functions in $\mathcal{K}_u$, Allu *et al.* [1] obtained an estimate of the Fekete–Szegö functional $|a_3 - \mu a_2^2|$ with $\mu \in \mathbb{R}$. The result is sharp only when $\mu \le 0$, $2/3 \le \mu \le 1$ and $\mu \ge 10/9$. In the next theorem, we will give the sharp bounds of $|a_3 - \mu a_2^2|$ for all values of $\mu \in \mathbb{R}$. Our proof is completely different from that in [1]. Our main tool to get the sharp bound is Lemma 2.1.

THEOREM 2.4. *Let $f \in \mathcal{K}_u$ be given by (1.1). Then for every $\mu \in \mathbb{R}$,*

$$|a_3 - \mu a_2^2| \le \begin{cases} \dfrac{5}{3} - \dfrac{9}{4}\mu & \text{if } \mu \le 0, \\ \dfrac{4(5-3\mu)}{3(4+3\mu)} & \text{if } 0 \le \mu \le \dfrac{2}{3}, \\ \dfrac{2}{3} & \text{if } \dfrac{2}{3} \le \mu \le 1, \\ \dfrac{3\mu-5}{3(3\mu-4)} & \text{if } 1 \le \mu \le \dfrac{10}{9}, \\ \dfrac{9}{4}\mu - \dfrac{5}{3} & \text{if } \mu \ge \dfrac{10}{9}. \end{cases}$$

Moreover, all the inequalities are sharp.

PROOF. Let $f \in \mathcal{K}_u$ be of the form (1.1). Then there exists a starlike function $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ in $\mathcal{S}^*$ such that

$$\left| \frac{zf'(z)}{g(z)} - 1 \right| < 1.$$

Thus, there exists $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ in $\mathcal{B}_0$ such that

$$f'(z) = \frac{g(z)}{z}(1 + \omega(z)). \tag{2.2}$$

From (2.2), comparing the coefficients of z^2 and z^3 on both sides,

$$a_2 = \frac{b_2}{2} + \frac{c_1}{2} \quad \text{and} \quad a_3 = \frac{c_2}{3} + \frac{b_3}{3} + \frac{1}{3}b_2 c_1. \tag{2.3}$$

Since $g \in \mathcal{S}^*$, it follows that there exists another $\rho \in \mathcal{B}_0$ of the form $\rho(z) = \sum_{n=1}^{\infty} d_n z^n$ such that

$$\frac{zg'(z)}{g(z)} = \frac{1+\rho(z)}{1-\rho(z)}. \tag{2.4}$$

On comparing the coefficients of z^2 and z^3 on both sides,

$$b_2 = 2d_1 \quad \text{and} \quad b_3 = d_2 + 3d_1^2. \tag{2.5}$$

From (2.3) and (2.5),

$$a_2 = d_1 + \frac{c_1}{2} \quad \text{and} \quad a_3 = \frac{c_2}{3} + \frac{d_2}{3} + d_1^2 + \frac{2}{3}d_1c_1.$$

Therefore, for any $\mu \in \mathbb{R}$,

$$a_3 - \mu a_2^2 = Ac_2 + Bd_2 + Kc_1^2 + Ld_1^2 + 2Mc_1d_1,$$

where

$$A = \frac{1}{3}, \quad B = \frac{1}{3}, \quad K = -\frac{\mu}{4}, \quad M = \frac{2-3\mu}{6}, \quad L = 1 - \mu.$$

Thus,

$$\begin{aligned}|a_3 - \mu a_2^2| &\le |A||c_2| + |B||d_2| + |Kc_1^2 + Ld_1^2 + 2Mc_1d_1| \\ &\le |A|(1-|c_1|^2) + |B|(1-|d_1|^2) + |Kc_1^2 + Ld_1^2 + 2Mc_1d_1|.\end{aligned}$$

Now, we have to find the maximum value of $|a_3 - \mu a_2^2|$ when $|c_1| \le 1$, $|d_1| \le 1$. To do this, we will use Lemma 2.1 and consider the following five cases.

Case 1: Let $\mu \le 0$. A simple calculation shows that

$$KL = -\frac{\mu(1-\mu)}{4} \ge 0, \quad D = -\frac{2-3\mu}{6} < 0, \quad |A| \le |M| + |K|, \quad |B| \le |M| + |L|.$$

Therefore, from Lemma 2.1,

$$|a_3 - \mu a_2^2| \le |K| + 2|M| + |L| = \tfrac{5}{3} - \tfrac{9}{4}\mu.$$

The inequality is sharp and the equality holds for the function $f \in \mathcal{K}_u$ given by (2.2) and (2.4) with $\omega(z) = z$ and $\rho(z) = z$, that is,

$$f(z) = \frac{2z}{1-z} + \log(1-z) = z + \frac{3}{2}z^2 + \frac{5}{3}z^3 + \cdots.$$

Case 2: Let $0 \le \mu \le 2/3$. A simple calculation shows that

$$KL = -\frac{\mu(1-\mu)}{4} < 0.$$

Thus, from Lemma 2.1,

$$|a_3 - \mu a_2^2| \le |A| + |B| + \max\{0, R\}, \tag{2.6}$$

where R can be obtained from Lemma 2.1. For $0 \le \mu \le \frac{2}{3}$,

$$|M| - |K| = \frac{4-9\mu}{12} \le \frac{1}{3} = |A|$$

https://doi.org/10.1017/S0004972723000655 Published online by Cambridge University Press

and

$$|K|\sqrt{1-\frac{M^2}{KL}} \le |A| \iff \frac{\mu}{4}\sqrt{1+\frac{(2-3\mu)^2}{9\mu(1-\mu)}} \le \frac{1}{3}$$
$$\iff 3\mu^2 - 20\mu + 16 \ge 0,$$

which is true for all $\mu \in [0,\ 2/3]$. Thus, the condition (A1) of Lemma 2.1 is satisfied.

Again, for $0 \le \mu \le 2/3$,

$$|M| - |L| = \frac{3\mu - 4}{6} \le -\frac{1}{3} \le |B|$$

and

$$|L|\sqrt{1-\frac{M^2}{KL}} \le |B| \iff (1-\mu)\sqrt{1+\frac{(2-3\mu)^2}{9\mu(1-\mu)}} \le \frac{1}{3}$$
$$\iff 3\mu^2 - 8\mu + 4 \le 0,$$

which is not true for any $\mu \in [0,\ 2/3]$. Thus, the condition (B1) of Lemma 2.1 is not satisfied. Further, for $0 \le \mu \le 2/3$,

$$|L| + |M| = \frac{4 - 3\mu}{4} \ge \frac{1}{2} \ge |B|$$

and so, the condition (B2) of Lemma 2.1 is not satisfied.

Therefore, by Lemma 2.1,

$$R = |L| - |B| + \frac{M^2}{|A| + |K|} = \frac{2}{3} - \mu + \frac{(2-3\mu)^2}{4(4+3\mu)}$$

and consequently, from (2.6),

$$|a_3 - \mu a_2^2| \le \frac{4(5-3\mu)}{3(4+3\mu)}.$$

The inequality is sharp and the equality holds for the function $f \in \mathcal{K}_u$ given by (2.2) and (2.4) with

$$\omega(z) = \frac{az(z + \bar{a}v_1)}{1 + a\bar{v}_1 z} \quad \text{and} \quad \rho(z) = z,$$

where

$$v_1 = \frac{2(2-3\mu)}{4+3\mu} \quad \text{and} \quad a = \frac{v_2}{1 - v_1^2} \quad \text{with} \quad v_1^2 + v_2 = 1,$$

that is,

$$f(z) = \int_0^z \frac{1 + (a\bar{v}_1 + v_1)t + at^2}{(1-t)^2(1 + a\bar{v}_1 t)}\,dt = z + \frac{6}{4+3\mu}z^2 + \frac{80 + 120\mu - 36\mu^2}{3(4+3\mu)^2}z^3 + \cdots.$$

Case 3: Let $2/3 \le \mu \le 1$. It is easy to show that $KL = -\frac{1}{4}\mu(1-\mu) < 0$. So, from Lemma 2.1,

$$|a_3 - \mu a_2^2| \le |A| + |B| + \max\{0, R\}, \tag{2.7}$$

where R can be obtained from Lemma 2.1. Proceeding as in Case 2, we can verify that the condition (A1) holds but (B1) and (B2) of Lemma 2.1 do not hold. Therefore,

$$R = |K| - |A| + \frac{M^2}{|B| + |L|} = \frac{\mu - 1}{4 - 3\mu} \le 0 \quad \text{for } \frac{2}{3} \le \mu \le 1$$

and consequently, from (2.7),

$$|a_3 - \mu a_2^2| \le \tfrac{1}{3} + \tfrac{1}{3} = \tfrac{2}{3}.$$

The inequality is sharp and the equality holds for the function $f \in \mathcal{K}_u$ given by (2.2) and (2.4) with $\omega(z) = z^2$ and $\rho(z) = z^2$, that is,

$$f(z) = \log \frac{1+z}{1-z} - z = z + \frac{2}{3}z^3 + \cdots.$$

Case 4: Let $1 \le \mu \le 10/9$. A simple calculation shows that

$$KL = -\frac{\mu(1-\mu)}{4} \ge 0, \quad D = -\frac{1-\mu}{3} < 0 \quad \text{and} \quad |B| > |M| + |L|.$$

Thus, from Lemma 2.1,

$$|a_3 - \mu a_2^2| \le |B| + |K| - \frac{M^2}{|L| - |B|} = \frac{5 - 3\mu}{3(4 - 3\mu)}.$$

The inequality is sharp and the equality holds for the function $f \in \mathcal{K}_u$ given by (2.2) and (2.4) with

$$\omega(z) = z \quad \text{and} \quad \rho(z) = \frac{az(z + \bar{a}v_1)}{1 + a\bar{v}_1 z},$$

where

$$v_1 = \frac{3\mu - 2}{8 - 6\mu} \quad \text{and} \quad a = -\frac{v_2}{1 - v_1^2} \quad \text{with} \quad v_1^2 + v_2 = 1,$$

that is,

$$\frac{g(z)}{z} = \exp\bigg(\int_0^z \frac{2(v_1 + at)}{1 + a\bar{v}_1 t - v_1 t - at^2}\, dt\bigg) = 1 + 2v_1 z + (4v_1^2 - 1)z^2 + \cdots$$

and

$$f(z) = \int_0^z \frac{g(t)}{t}(1+t)\, dt = z + \frac{1}{4 - 3\mu}z^2 + \frac{30\mu - 20 - 9\mu^2}{3(4 - 3\mu)^2}z^3 + \cdots.$$

Case 5: Let $\mu \ge 10/9$. A simple calculation shows that

$$KL = -\frac{\mu(1-\mu)}{4} \ge 0, \quad D = -\frac{1-\mu}{3} < 0, \quad |A| \le |M| + |K|, \quad |B| \le |M| + |L|.$$

https://doi.org/10.1017/S0004972723000655 Published online by Cambridge University Press

Thus, from Lemma 2.1,

$$|a_3 - \mu a_2^2| \leq |K| + 2|M| + |L| = \frac{9\mu}{4} - \frac{5}{3}.$$

The inequality is sharp and the equality holds for the function $f \in \mathcal{K}_u$ given by (2.2) and (2.4) with $\omega(z) = z$ and $\rho(z) = z$, that is,

$$f(z) = \frac{2z}{1-z} + \log(1-z) = z + \frac{3}{2}z^2 + \frac{5}{3}z^3 + \cdots. \qquad \square$$

Finally, we establish a result related to the pre-Schwarzian norm for functions in $\mathcal{K}_u$. We first note that a function f in $\mathcal{A}$ belongs to $\mathcal{K}_u$ if there exists a function $g \in \mathcal{S}^*$ such that $|zf'(z)/g(z) - 1| < 1$. In other words, if there exists a convex function $h \in C$ with $g(z) = zh'(z)$ such that

$$\left|\frac{f'(z)}{h'(z)} - 1\right| < 1.$$

THEOREM 2.5. *Let $f \in \mathcal{K}_u$ and $h \in C$ be the associated convex function. Then,*

$$|\,\|P_f\| - \|P_h\|\,| \leq 2,$$

and the estimate is sharp. Further, $\|P_f\| \leq 6$.

PROOF. Let $f \in \mathcal{K}_u$ and $h \in C$ be the associated convex function such that

$$\left|\frac{f'(z)}{h'(z)} - 1\right| < 1.$$

Then there exists a function $\omega(z) \in \mathcal{B}_0$ such that

$$\frac{f'(z)}{h'(z)} = 1 + \omega(z).$$

Taking the logarithmic derivative on both sides,

$$\frac{f''(z)}{f'(z)} - \frac{h''(z)}{h'(z)} = \frac{\omega'(z)}{1+\omega(z)}$$

and so,

$$\left|\frac{f''(z)}{f'(z)}\right| - \left|\frac{h''(z)}{h'(z)}\right| \leq \left|\frac{f''(z)}{f'(z)} - \frac{h''(z)}{h'(z)}\right| = \left|\frac{\omega'(z)}{1+\omega(z)}\right|.$$

Thus,

$$\begin{aligned} |\,\|P_f\| - \|P_h\|\,| &= \left|\sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| - \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{h''(z)}{h'(z)}\right|\right| \\ &\leq \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\left(\left|\frac{f''(z)}{f'(z)}\right| - \left|\frac{h''(z)}{h'(z)}\right|\right)\right| \end{aligned}$$

https://doi.org/10.1017/S0004972723000655 Published online by Cambridge University Press

$$\leq \sup_{z\in\mathbb{D}}(1-|z|^2)\left|\frac{f''(z)}{f'(z)}-\frac{h''(z)}{h'(z)}\right|$$
$$= \sup_{z\in\mathbb{D}}(1-|z|^2)\frac{|\omega'(z)|}{|1+\omega(z)|}.$$

Since $\omega(z) \in \mathcal{B}_0$, by the Schwarz–Pick lemma,

$$|\omega'(z)| \leq \frac{1-|\omega(z)|^2}{1-|z|^2}.$$

Therefore,

$$|\,\|P_f\| - \|P_h\|\,| \leq \sup_{z\in\mathbb{D}} \frac{1-|\omega(z)|^2}{|1+\omega(z)|} \leq 2.$$

The above inequality is sharp for the functions

$$f(z) = -\log(1-z) \quad \text{and} \quad h(z) = \frac{z}{1-z}.$$

It is well known that $\|P_h\| \leq 4$ for $f \in C$ (see [19]), and so $\|P_f\| \leq 6$. □

References

[1] V. Allu, J. Sokól and D. K. Thomas, 'On a close-to-convex analogue of certain starlike functions', *Bull. Aust. Math. Soc.* **108** (2020), 268–281.

[2] J. Becker, 'Löwnersche Differentialgleichung und quasikonform fortsetzbare schlichte Funktionen', *J. reine angew. Math* **255** (1972), 23–43.

[3] J. Becker and C. Pommerenke, 'Schlichtheitskriterien und Jordangebiete', *J. reine angew. Math.* **354** (1984), 74–94.

[4] N. E. Choi, Y. C. Kim and T. Sugawa, 'A general approach to the Fekete–Szegö problem', *J. Math. Soc. Japan* **59** (2007), 707–727.

[5] P. L. Duren, *Univalent Functions*, Grundlehren der mathematischen Wissenschaften, 259 (Springer-Verlag, New York–Berlin–Heidelberg–Tokyo, 1983).

[6] M. Fekete and G. Szegö, 'Eine Bemerkung über ungerade schlichte Funktionen', *J. Lond. Math. Soc.* **8** (1933), 85–89.

[7] A. W. Goodman, *Univalent Functions*, Vol. I and II (Mariner Publishing Co., Tampa, FL, 1983).

[8] D. J. Hallenbeck and T. H. Macgregor, *Linear Problems and Convexity Techniques in Geometric Function Theory*, Monographs and Studies in Mathematics, 22 (Pitman, Boston, MA, 1984).

[9] S. Kanas and A. Lecko, 'On the Fekete–Szegö problem and the domain of convexity for a certain class of univalent functions', *Folia Sci. Univ. Tech. Resoviensis* **73** (1990), 49–57.

[10] W. Kaplan, 'Close-to-convex schlicht functions', *Michigan Math. J.* **1** (1952), 169–185.

[11] Y. C. Kim and T. Sugawa, 'Growth and coefficient estimates for uniformly locally univalent functions on the unit disk', *Rocky Mountain J. Math.* **32** (2002), 179–200.

[12] W. Koepf, 'On the Fekete–Szegö problem for close-to-convex functions', *Proc. Amer. Math. Soc.* **101** (1987), 89–95.

[13] W. Koepf, 'On the Fekete–Szegö problem for close-to-convex functions II', *Arch. Math.* **49** (1987), 420–433.

[14] B. Kowalczyk and A. Lecko, 'Fekete–Szegö problem for a certain subclass of close-to-convex functions', *Bull. Malays. Math. Sci. Soc.* **38**(2) (2015), 1393–1410.

[15] R. R. London, 'Fekete–Szegö inequalities for close-to-convex functions', *Proc. Amer. Math. Soc.* **117** (1993), 947–950.

[16] R. Singh, ‘On a class of starlike functions’, *Compos. Math.* **19** (1968), 78–82.
[17] R. Singh and V. Singh, ‘On a class of bounded starlike functions’, *Indian J. Pure Appl. Math.* **5**(8) (1974), 733–754.
[18] S. Yamashita, ‘Almost locally univalent functions’, *Monatsh. Math.* **81** (1976), 235–240.
[19] S. Yamashita, ‘Norm estimates for function starlike or convex of order alpha’, *Hokkaido Math. J.* **28**(1) (1999), 217–230 (summary in English).

MD FIROZ ALI, National Institute of Technology Durgapur,
Mahatma Gandhi Road, Durgapur, Durgapur-713203, West Bengal, India
e-mail: ali.firoz89@gmail.com, firoz.ali@maths.nitdgp.ac.in

MD NUREZZAMAN, National Institute of Technology Durgapur,
Mahatma Gandhi Road, Durgapur, Durgapur-713203, West Bengal, India
e-mail: nurezzaman94@gmail.com