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ONE-SIDED ESTIMATES FOR QUASIMONOTONE INCREASING FUNCTIONS

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Let E be a Banach space ordered by a solid and normal cone K, and normed by the Minkowski functional of an order interval $[-p,p]$, $p \in K^\circ$. We derive global one-sided estimates for quasimonotone increasing functions $f : [0,T) \times E \to E$ with respect to the norm, and the distance to the line generated by p, under conditions on f in direction p.

1. INTRODUCTION

Let E be a real Banach space, ordered by a cone K. A cone K is a closed convex subset of E with $\lambda K \subseteq K$ $(\lambda \geqslant 0)$, and $K \cap (-K) = \{0\}$. As usual $x \leqslant y :\iff y-x \in K$. We shall always assume that K is solid and normal. Since K is solid, the set

$$K^* = \{\varphi \in E^* : \ \varphi(x) \geqslant 0 \ (x \geqslant 0)\}$$

is a cone, the dual cone, in the space of all continuous linear functionals E^*.

A function $f : E \to E$ is *quasimonotone increasing*, in the sense of Volkmann [8], if

$$x, y \in E, \ x \leqslant y, \ \varphi \in K^*, \ \varphi(x) = \varphi(y) \implies \varphi(f(x)) \leqslant \varphi(f(y)),$$

and a function $f : [0,T) \times E \to E$ is called quasimonotone increasing if $x \mapsto f(t,x)$ is quasimonotone increasing for each $t \in [0,T)$. For such a function f we consider the differential equation

$$u'(t) = f(t, u(t)), \quad t \in [0, T_u), \tag{1}$$

and by means of one-sided estimates we shall derive global estimates for solutions and differences of solutions under the assumption that for some $p \in K^\circ$ the behaviour of $f(t,x)$ in direction p is of some quality. For example, if $E = \mathbb{R}^n$ is ordered by the natural

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cone $K_{\mathrm{nat}} = \{x \in \mathbb{R}^n : x_1, \dots, x_n \geqslant 0\}$, and $g_j : [0,T) \times \mathbb{R} \to \mathbb{R}$ is increasing in its second variable ($j = 1, \dots, n$), then $f : [0,T) \times \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$f(t,x) = \begin{pmatrix} g_1(t, x_2 - x_1) \\ g_2(t, x_3 - 2x_2 + x_1) \\ \vdots \\ g_{n-1}(t, x_n - 2x_{n-1} + x_{n-2}) \\ g_n(t, x_{n-1} - x_n) \end{pmatrix} \tag{2}$$

is quasimonotone increasing, and for $p = (1, \dots, 1)$

$$f(t, x + \lambda p) = f(t,x) \quad \big((t,x) \in [0,T) \times E,\ \lambda \in \mathbb{R}\big).$$

From our results it will follow, for example, that f is dissipative with respect to the maximum norm, and with respect to the distance to $[p] := \{\lambda p : \lambda \in \mathbb{R}\}$ (in the maximum norm), which implies that the functions

$$\big\|v(t) - u(t)\big\|, \quad \mathrm{dist}\big(v(t) - u(t), [p]\big)$$

are decreasing in t, for any two solutions u, v of (1).

2. One-sided estimates

Let $q : E \to \mathbb{R}$ be a continuous and sublinear functional, that is

$$q(x+y) \leqslant q(x) + q(y), \quad q(\lambda x) = \lambda q(x) \quad (x, y \in E,\ \lambda \geqslant 0).$$

According to Mazur's results on sublinear functionals [**4**], see also [**3**], the directional derivatives

$$\partial_+ q[x,y] := \lim_{h \to 0+} \frac{q(x+hy) - q(x)}{h} \quad (x, y \in E) \tag{3}$$

exist, and the following functional characterisation is valid:

$$\partial_+ q[x,y] = \max\big\{\varphi(y) : \varphi \in E^*,\ \varphi(\xi) \leqslant q(\xi)\ (\xi \in E),\ \varphi(x) = q(x)\big\}.$$

We fix $p \in K^\circ$ and consider the order interval

$$[-p, p] = \{x \in E : -p \leqslant x \leqslant p\}.$$

The Minkowski functional $\|\cdot\|$ of $[-p,p]$ is a norm which generates the topology of E. Note that $\partial_+ q[x,y]$ for $q = \|\cdot\|$ will, as usual, be denoted by $m_+[x,y]$.

The functionals $S : E \to \mathbb{R}$ given by

$$S(x) = \min\{\lambda \in \mathbb{R} : x \leqslant \lambda p\},$$

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and $x \mapsto S(-x)$ are continuous and sublinear, S is increasing, and the chosen norm as well as the distance

$$d(x) := \operatorname{dist}\bigl(x, [p]\bigr) \quad (x \in E)$$

may be represented the following way:

Proposition 1. *For all $x \in E$*

$$\|x\| = \max\bigl\{S(x), S(-x)\bigr\}, \quad d(x) = \frac{1}{2}\bigl(S(x) + S(-x)\bigr).$$

Proof: The first equality follows almost immediately from the definitions of $\|\cdot\|$ and S. For the second, fix $x \in E$, and let $\mu \in \mathbb{R}$. We have

$$\|x - \mu p\| = \max\bigl\{S(x - \mu p), S(-x + \mu p)\bigr\} = \max\bigl\{S(x) - \mu, S(-x) + \mu\bigr\},$$

which is minimal if and only if $S(x) - \mu = S(-x) + \mu$, hence for

$$\mu = \frac{1}{2}\bigl(S(x) - S(-x)\bigr).$$

Therefore

$$d(x) = S(x) - \frac{1}{2}\bigl(S(x) - S(-x)\bigr) = \frac{1}{2}\bigl(S(x) + S(-x)\bigr). \qquad \square$$

Now, let $f : E \to E$ be quasimonotone increasing. Then, the following one-sided estimates are valid:

Theorem 1. *For all $x \in E$*

$$m_+\bigl[x, f(x)\bigr] \leqslant \max\Bigl\{S\Bigl(f\bigl(\|x\|p\bigr)\Bigr), S\Bigl(-f\bigl(-\|x\|p\bigr)\Bigr)\Bigr\}, \tag{4}$$

$$\partial_+ d\bigl[x, f(x)\bigr] \leqslant \frac{1}{2}\Bigl(S\Bigl(f\bigl(S(x)p\bigr)\Bigr) + S\Bigl(-f\bigl(-S(-x)p\bigr)\Bigr)\Bigr). \tag{5}$$

Proof: Fix $x \in E$, and let $\varphi \in E^*$ be such that

$$\varphi(\xi) \leqslant S(\xi) \ (\xi \in E), \quad \varphi(x) = S(x).$$

We have $\varphi(p) \leqslant S(p) = 1$ and $-\varphi(p) \leqslant S(-p) = -1$, hence $\varphi(p) = 1$. Next, let $\xi \leqslant 0$. Then $\xi \leqslant \lambda p$ $(\lambda \geqslant 0)$, thus $\varphi(\xi) \leqslant S(\xi) \leqslant 0$. Therefore we have $\varphi \in K^*$. Moreover

$$x \leqslant S(x)p, \quad \varphi(x) = S(x) = S(x)\varphi(p) = \varphi\bigl(S(x)p\bigr).$$

Since f is quasimonotone increasing we obtain

$$\varphi\bigl(f(x)\bigr) \leqslant \varphi\Bigl(f\bigl(S(x)p\bigr)\Bigr) \leqslant S\Bigl(f\bigl(S(x)p\bigr)\Bigr),$$

hence

$$\partial_+ S\bigl[x, f(x)\bigr] \leqslant S\Bigl(f\bigl(S(x)p\bigr)\Bigr).$$

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Next,

$$\varphi(\xi) \leqslant S(-\xi) \ (\xi \in E), \quad \varphi(x) = S(-x)$$

imply

$$(-\varphi)(\xi) \leqslant S(\xi) \ (\xi \in E),$$

hence, as above,

$$-\varphi \in K^*, \ -\varphi(p) = 1, \ -S(-x)p \leqslant x, \ -\varphi\big(-S(-x)p\big) = -\varphi(x).$$

Therefore

$$\varphi(f(x)) \leqslant \varphi\Big(f\big(-S(-x)p\big)\Big) \leqslant S\Big(-f\big(-S(-x)p\big)\Big).$$

Thus $q(x) = S(-x)$ satisfies

$$\partial_+ q\big[x, f(x)\big] \leqslant S\Big(-f\big(-S(-x)p\big)\Big).$$

Now (5) follows by Proposition 1 and (3). To see (4) we consider three cases. If $S(x) > S(-x)$ then, according to Proposition 1,

$$\begin{aligned} m_+\big[x, f(x)\big] &= \lim_{h \to 0+} \frac{S(x + hf(x)) - S(x)}{h} \\ &= \partial_+ S\big[x, f(x)\big] \leqslant S\Big(f\big(S(x)p\big)\Big) = S\Big(f\big(\|x\|p\big)\Big). \end{aligned}$$

Analogously $S(x) < S(-x)$ implies

$$m_+\big[x, f(x)\big] \leqslant S\Big(-f\big(-S(-x)p\big)\Big) = S\Big(-f\big(-\|x\|p\big)\Big).$$

In case $S(x) = S(-x)$ we have, dependent on $h > 0$,

$$\frac{\|x + hf(x)\| - \|x\|}{h} = \frac{S(x + hf(x)) - S(x)}{h}$$

or

$$\frac{\|x + hf(x)\| - \|x\|}{h} = \frac{S(-x - hf(x)) - S(-x)}{h}.$$

Therefore, in this case also

$$\begin{aligned} m_+\big[x, f(x)\big] &\leqslant \max\Big\{S\Big(f\big(S(x)p\big)\Big), S\Big(-f\big(-S(-x)p\big)\Big)\Big\} \\ &= \max\Big\{S\Big(f\big(\|x\|p\big)\Big), S\Big(-f\big(-\|x\|p\big)\Big)\Big\}. \end{aligned}$$

Alltogether (4) is valid. ∎

In the sequel let always $f : [0, T) \times E \to E$ be quasimonotone increasing. By means of Theorem 1 we are able to derive from properties of f with respect to p one-sided estimates valid on E, and one-sided estimates lead to estimates for solutions of the corresponding equation (1): If $q : E \to \mathbb{R}$ is any continuous and sublinear functional, and if $u : [0,\tau) \to E$ is differentiable, then $t \mapsto q\big(u(t)\big)$ is differentiable from the right on $[0,\tau)$, and

$$(6) \qquad \big(q(u)\big)'_+(t) = \partial_+ q\big[u(t), u'(t)\big] \quad \big(t \in [0,\tau)\big),$$

see for example [**3**].

Corollary 1. *Let there exist functions $\alpha : [0,T) \times \mathbb{R} \to \mathbb{R}$, and $b : [0,T) \to E$ such that*

$$f(t,\lambda p) \leqslant \alpha(t,\lambda)p + b(t) \quad \big(t \in [0,T),\ \lambda \geqslant 0\big),$$
$$f(t,\lambda p) \geqslant \alpha(t,\lambda)p + b(t) \quad \big(t \in [0,T),\ \lambda \leqslant 0\big).$$

Then

$$m_+\big[x, f(t,x)\big] \leqslant \max\Big\{\alpha\big(t,\|x\|\big), -\alpha\big(t,-\|x\|\big)\Big\} + \big\|b(t)\big\|$$

for all $(t,x) \in [0,T) \times E$.

Proof: According to (4) we have

$$\begin{aligned} m_+\big[x, f(t,x)\big] &\leqslant \max\Big\{S\Big(\alpha\big(t,\|x\|\big)p + b(t)\Big), S\Big(-\alpha\big(t,-\|x\|\big)p - b(t)\Big)\Big\} \\ &= \max\Big\{\alpha\big(t,\|x\|\big) + S\big(b(t)\big), -\alpha\big(t,-\|x\|\big) + S\big(-b(t)\big)\Big\} \\ &\leqslant \max\Big\{\alpha\big(t,\|x\|\big), -\alpha\big(t,-\|x\|\big)\Big\} + \big\|b(t)\big\|. \end{aligned}$$

∎

Remark. Corollary 1 can be used to obtain stability results for equation (1), for example. For the natural cone in $\mathbb{R}^n$ related stability conditions for critical points of autonomous quasimonotone increasing systems were derived by Rautmann [**5**] by comparison methods.

Corollary 2. *Let there exist a function $\alpha : [0,T) \times [0,\infty) \to \mathbb{R}$ such that*

$$f(t, x + \lambda p) - f(t,x) \leqslant \alpha(t,\lambda)p \quad \big((t,x) \in [0,T) \times E,\ \lambda \geqslant 0\big).$$

Then

$$m_+\big[y - x, f(t,y) - f(t,x)\big] \leqslant \alpha\big(t, \|y - x\|\big) \quad \big((t,x),(t,y) \in [0,T) \times E\big).$$

Proof: Fix $x, y \in E$, and consider $g : [0,T) \times E \to E$ given by

$$g(t,z) = f(t, z + x) - f(t,x),$$

which is quasimonotone increasing. We obtain

$$g(t,\lambda p) \leqslant \alpha(t,\lambda)p\ (\lambda \geqslant 0), \quad -g(t,\lambda p) \leqslant \alpha(t,-\lambda)p\ (\lambda \leqslant 0),$$

and by means of (4)

$$m_+[z,g(t,z)] \leqslant \max\Big\{S\big(g(t,\|z\|p)\big), S\big(-g(t,-\|z\|p)\big)\Big\}$$
$$\leqslant \max\Big\{\alpha(t,\|z\|), \alpha\big(t,-(-\|z\|)\big)\Big\} = \alpha(t,\|z\|).$$

For $z = y - x$ this means

$$m_+[y-x, f(t,y) - f(t,x)] = m_+[y-x, g(t,y-x)] \leqslant \alpha(t,\|y-x\|). \qquad \square$$

REMARKS.

1. For the case $\alpha(t,\lambda) = L\lambda$ a result related to Corollary 2 was proved in [**2**] by different methods. For linear operators compare [**1**].
2. According to a Theorem of Martin [**3**, p. 238], a one sided Lipschitz condition is a well posedness condition for ordinary differential equations in Banach spaces: Let $f : [0,T) \times E \to E$ be continuous, such that $f(I \times B)$ is bounded for $B \subseteq E$ bounded and $I \subseteq [0,T)$ compact. If $m_+[y-x, f(t,y) - f(t,x)] \leqslant \mu(t)\|y-x\|$ on $[0,T) \times E$, for some $\mu \in C([0,T), \mathbb{R})$, then each initial value problem $u'(t) = f(t,u(t))$, $u(0) = u_0$ is uniquely solvable on $[0,T)$, and the solution depends continuously on the initial value.

COROLLARY 3. *Let there exist functions $\alpha : [0,T) \times \mathbb{R} \to \mathbb{R}$, $\omega : [0,T) \times [0,\infty) \to \mathbb{R}$, and $b : [0,T) \to E$ such that*

$$f(t,\lambda p) = \alpha(t,\lambda)p + b(t) \quad \big(t \in [0,T),\ \lambda \in \mathbb{R}\big),$$

and

$$\alpha(t,\lambda) - \alpha(t,\mu) \leqslant \omega(t,\lambda - \mu) \quad \big(t \in [0,T),\ \mu \leqslant \lambda\big).$$

Then

$$\partial_+ d[x, f(t,x)] \leqslant \frac{1}{2}\omega\big(t, 2d(x)\big) + d\big(b(t)\big) \quad \big((t,x) \in [0,T) \times E\big).$$

PROOF: By means of (5) we obtain

$$2\partial_+ d[x, f(t,x)] \leqslant S\Big(f\big(t, S(x)p\big)\Big) + S\Big(-f\big(t, -S(-x)p\big)\Big)$$
$$= \alpha\big(t, S(x)\big) + S\big(b(t)\big) - \alpha\big(t, -S(-x)\big) + S\big(-b(t)\big)$$
$$\leqslant \omega\big(t, S(x) + S(-x)\big) + S\big(b(t)\big) + S\big(-b(t)\big) = \omega\big(t, 2d(x)\big) + 2d\big(b(t)\big). \qquad \square$$

COROLLARY 4. *Let there exist functions $\alpha : [0,T) \times \mathbb{R} \to \mathbb{R}$, and $\omega : [0,T) \times [0,\infty) \to \mathbb{R}$ such that*

$$f(t, x + \lambda p) - f(t,x) \leqslant \alpha(t,\lambda)p \quad \big((t,x) \in [0,T) \times E,\ \lambda \in \mathbb{R}\big),$$

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and

$$\alpha(t,\lambda)+\alpha(t,\mu)\leqslant\omega(t,\lambda+\mu)\quad\big(t\in[0,T),\ \mu+\lambda\geqslant 0\big).$$

Then

$$\partial d_+\big[y-x,f(t,y)-f(t,x)\big]\leqslant\frac{1}{2}\omega\big(t,2d(y-x)\big)$$

for all $(t,x),(t,y)\in[0,T)\times E$.

PROOF: Fix $x,y\in E$, and consider $g:[0,T)\times E\to E$ defined by

$$g(t,z)=f(t,z+x)-f(t,x),$$

which is quasimonotone increasing. We obtain

$$g(t,\lambda p)\leqslant\alpha(t,\lambda)p,\quad -g(t,\lambda p)\leqslant\alpha(t,-\lambda)p\quad(\lambda\in\mathbb{R}).$$

By means of (5)

$$\begin{aligned}2\partial d_+\big[z,g(t,z)\big]&\leqslant S\Big(g\big(t,S(z)p\big)\Big)+S\Big(-g\big(t,-S(-z)p\big)\Big)\\&\leqslant\alpha\big(t,S(z)\big)+\alpha\big(t,S(-z)\big)\leqslant\omega\big(t,S(z)+S(-z)\big)=\omega\big(t,2d(z)\big).\end{aligned}$$

Again by setting $z=y-x$ we get

$$\partial d_+\big[y-x,f(t,y)-f(t,x)\big]=\partial d_+\big[y-x,g(t,y-x)\big]\leqslant\frac{1}{2}\omega\big(t,2d(y-x)\big).\qquad\square$$

3. Examples and Applications

We shall first study example (2) from the introduction. If $\mathbb{R}^n$ is ordered by K_{nat} and $p=(1,\dots,1)$, then $\|\cdot\|$ is the maximum norm and

$$d(x)=\frac{1}{2}\big(\max\{x_1,\dots,x_n\}-\min\{x_1,\dots,x_n\}\big).$$

By applying Corollary 1 we obtain ($\alpha=0$, $b(t)=f(t,0)$)

$$m_+\big[x,f(t,x)\big]\leqslant\|f(t,0)\|,$$

and, in case $t\mapsto f(t,0)$ is continuous, by means of (6)

$$\big\|u(t)\big\|\leqslant\big\|u(0)\big\|+\int_0^t\big\|f(\tau,0)\big\|\,d\tau\quad\big(t\in[0,T_u)\big)$$

for every solution u of (1).

By application of Corollary 2 we get ($\alpha=0$)

$$m_+\big[y-x,f(t,y)-f(t,x)\big]\leqslant 0,$$

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and by means of (6) we find that $\|v(t) - u(t)\|$ is decreasing in t for any two solutions u, v of (1).

Application of Corollary 3 leads to ($\alpha = 0$, $\omega = 0$, $b(t) = f(t, 0)$)

$$\partial d_+ [x, f(t, x)] \leqslant d(f(t, 0)),$$

and again, if $t \mapsto f(t, 0)$ is continuous, by means of (6)

$$d(u(t)) \leqslant d(u(0)) + \int_0^t d(f(\tau, 0))\, d\tau \quad (t \in [0, T_u))$$

for every solution u of (1).

Finally by applying Corollary 4 we obtain ($\alpha = 0$, $\omega = 0$)

$$\partial d_+ [y - x, f(t, y) - f(t, x)] \leqslant 0,$$

and by means of (6) we find that $d(v(t) - u(t))$ is decreasing in t for any two solutions u, v of (1).

Next, consider $\mathbb{R}^3$ ordered by the ice-cream cone $K_{ice} = \{(x, y, z) : z \geqslant \sqrt{x^2 + y^2}\}$. For $p = (0, 0, 1) \in K^\circ$ we obtain

$$\|(x, y, z)\| = |z| + \sqrt{x^2 + y^2}, \quad d(x, y, z) = \sqrt{x^2 + y^2}.$$

A characterisation of the linear quasimonotone increasing mappings can be found in [**6**], and by linearisation it is easy to check that the following function $g : \mathbb{R}^3 \to \mathbb{R}^3$ is quasimonotone constant, that is g and $-g$ are quasimonotone increasing:

$$g(x, y, z) = (2xz + y, 2yz - x, x^2 + y^2 + z^2).$$

Thus, for any continuous function $h : [0, \infty) \to [-1, 1]$, the function

$$f(t, x, y, z) = h(t) g(x, y, z)$$

is quasimonotone increasing. We have

$$f(t, \lambda p) = h(t) \lambda^2 p \quad (t \geqslant 0,\ \lambda \in \mathbb{R}),$$

and according to Corollary 1 ($\alpha(t, \lambda) = h(t)\lambda^2$, $b(t) = 0$)

$$m_+ [(x, y, z), f(t, x, y, z)] \leqslant |h(t)|\ \|(x, y, z)\|^2 \leqslant \|(x, y, z)\|^2.$$

In particular each solution u of (1) satisfies

$$(\|u\|)'_+(t) \leqslant \|u(t)\|^2 \text{ so } \|u(t)\| \leqslant \frac{\|u(0)\|}{1 - \|u(0)\| t}$$

on $\left[0, 1/\|u(0)\|\right) \subseteq [0, T_u)$, if $[0, T_u)$ is the right maximal interval of existence.

Finally we shall apply our results to certain integro-differential equations. Let $\Omega \subseteq \mathbb{R}^n$ be compact, and consider $E = C(\Omega, \mathbb{R})$ endowed with the topology of uniform convergence, and ordered by the cone K of all nonnegative functions in E. Let

$$g : [0, T) \times \Omega \times \Omega \times \mathbb{R} \to \mathbb{R}, \quad p : \Omega \to (0, \infty)$$

be continuous functions, let g be monotone increasing in its fourth variable, and let $f : [0, T) \times E \to E$ be defined by

$$\big(f(t, x)\big)(s) = \int_\Omega g\big(t, s, \sigma, p(s)x(\sigma) - p(\sigma)x(s)\big)\, d\sigma.$$

Then f is continuous. To prove that f is quasimonotone increasing, it is sufficient to check the definition of quasimonotonicity for a subset $M \subseteq K^* \setminus \{0\}$ with the property that

$$\{x \in K : \ \exists \varphi \in M : \varphi(x) = 0\}$$

is dense in the boundary of K, according to a result of Uhl [**7**, Theorem 2]. Here we can choose

$$M = \{\varphi_s : \varphi_s(x) = x(s), \ s \in \Omega\}.$$

Let $x \leqslant y$, $\varphi_{s_0} \in M$, and $x(s_0) = \varphi_{s_0}(x) = \varphi_{s_0}(y) = y(s_0)$. Then

$$\begin{aligned}
\varphi_{s_0}\big(f(t, x)\big) &= \int_\Omega g\big(t, s_0, \sigma, p(s_0)x(\sigma) - p(\sigma)x(s_0)\big)\, d\sigma \\
&\leqslant \int_\Omega g\big(t, s_0, \sigma, p(s_0)y(\sigma) - p(\sigma)x(s_0)\big)\, d\sigma \\
&= \int_\Omega g\big(t, s_0, \sigma, p(s_0)y(\sigma) - p(\sigma)y(s_0)\big)\, d\sigma = \varphi_{s_0}\big(f(t, y)\big).
\end{aligned}$$

Hence f is quasimonotone increasing.

We have $p \in K^\circ$, and the corresponding norm and distance are

$$\|x\| = \max_{s \in \Omega} \frac{|x(s)|}{p(s)}, \quad d(x) = \frac{1}{2}\left(\max_{s \in \Omega} \frac{x(s)}{p(s)} - \min_{s \in \Omega} \frac{x(s)}{p(s)}\right).$$

Next,

$$f(t, x + \lambda p) - f(t, x) = 0 \quad \big((t, x) \in [0, T) \times E, \ \lambda \in \mathbb{R}\big),$$

and according to Corollary 2 and Corollary 4 we conclude that f is dissipative with respect to $\|\cdot\|$ and d, respectively. In particular, again $\|v(t) - u(t)\|$ and $d\big(v(t) - u(t)\big)$ are decreasing in t for any two solutions u, v of (1). Moreover, according to Remark 2. following Corollary 2, each initial value problem

$$u'(t) = f\big(t, u(t)\big), \quad u(0) = u_0 \in E$$

is uniquely solvable on $[0, T)$, the solution depends continuously on u_0, and it also depends increasing on u_0, according to the classical results on differential inequalities for quasimonotone increasing functions [**8**].

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