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ON THE COHOMOLOGICAL DIMENSION OF SOLUBLE GROUPS

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ABSTRACT. It is known that every torsion-free soluble group G of finite Hirsch number hG is countable, and its homological and cohomological dimensions over the integers and rationals satisfy the inequalities

$$hG = hd_{\mathbf{Q}}G = hd_{\mathbf{Z}}G \leq cd_{\mathbf{Q}}G \leq cd_{\mathbf{Z}}G \leq hG+1.$$

We prove that G must be finitely generated if the equality $hG = cd_{\mathbf{Q}}G$ holds. Moreover, we show that if G is a countable soluble group of finite Hirsch number, but not necessarily torsion-free, and if $hG = cd_{\mathbf{Q}}G$, then $h\bar{G} = cd_{\mathbf{Q}}\bar{G}$ for every homomorphic image $\bar{G}$ of G.

1. Introduction

1.1. *The basic inequalities.* Let R denote a commutative ring with $1 \neq 0$, G a soluble group, hG its Hirsch number and $hd_{R}G$, $(cd_{R}G)$ its (co)homological dimension over R. By a result of Stammbach one has $hd_{\mathbf{Q}}G = hG$. In fact, this result remains true if $\mathbf{Q}$ is replaced by any (commutative) $\mathbf{Q}$-algebra R, for one has always $hd_{\mathbf{Q}}G = hd_{R}G$ and $cd_{\mathbf{Q}}G = cd_{R}G$.

If G is torsion-free soluble and of finite Hirsch number, it is countable, ([4], p. 100, Lemma 7.9), and one has:

$$hG = hd_{\mathbf{Q}}G = hd_{\mathbf{Z}}G \leq cd_{\mathbf{Q}}G \leq cd_{\mathbf{Z}}G \leq hG+1, \tag{1.1}$$

(see [4], p. 101, Th. 7.10). The problem of computing the cohomological dimension $cd_{\mathbf{Z}}G$ of a soluble torsion-free group amounts therefore to a characterization of those groups G with $cd_{\mathbf{Z}}G = hG < \infty$. We shall prove, inter alia, that such a group is necessarily finitely generated.

1.2. *The main results.* As motivated above, we can concentrate on countable soluble groups; however, in order to be more flexible, we shall replace $\mathbf{Z}$ by $\mathbf{Q}$, and correspondingly allow G to have torsion. Then the following inequalities are still valid:

$$hG = hd_{\mathbf{Q}}G \leq cd_{\mathbf{Q}}G \leq hG+1.$$

Our first results states that the equality $cd_{\mathbf{Q}}G = hG$ $(=hd_{\mathbf{Q}}G)$ passes to homomorphic images.

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Theorem A. *Let G be a countable soluble group of finite Hirsch number. If $cd_{\mathbf{Q}}G = hG$, then $cd_{\mathbf{Q}}\bar{G} = h\bar{G}$ for every homomorphic image $\bar{G}$ of G.*

It follows, in particular, that for G a group as in Theorem A, the abelianized group $A = G_{ab} = G/[G, G]$ is finitely generated; for, let tor A be the torsion subgroup of A, and let $B \supseteq \text{tor } A$ be a subgroup of A such that $B/\text{tor } A$ is free abelian and A/B is torsion. Then by Theorem A we have firstly that $0 = h(A/B) = cd_{\mathbf{Q}}(A/B)$, thus A/B is finite, ([5], p. 9, Th. C), and $A \simeq (\text{tor A}) \times \mathbf{Z}^{h(A)}$, and so, similarly, tor A is finite. If, for example, G is a nilpotent group of arbitrary cardinality, the above argument implies that $cd_{\mathbf{Q}}G = hG < \infty$ if, and only if, G is finitely generated. This generalizes a result of Gruenberg's ([8], p. 149, Th. 5(2)).

The consequence of finite generation holds under weaker hypotheses than those of nilpotency. Indeed, by invoking a result asserted in [13], p. 79, and proved in the appendix, we can establish the following:

Theorem B. *If G is nilpotent-by-abelian and $cd_{\mathbf{Q}}G = hd_{\mathbf{Q}}G < \infty$, then G is finitely generated.*

Suppose now that G is a torsion-free soluble group of finite Hirsch number. Then G is nilpotent-by-abelian-by-finite, according to a result of Čarin's, ([6] or [1], p. 559, Prop. 5.5 (a)), and hence Theorem B entails

Corollary C. *If G is a torsion-free soluble group and $cd_{\mathbf{Q}}G = hd_{\mathbf{Q}}G < \infty$, then G is finitely generated.*

For certain metabelian groups, there is a connection, not clearly understood in general, between the property of being finitely presented and the cohomological dimension of the group. Let s and t be a pair of integers, $|s| \neq 1 \neq |t|$, and let $G_{s,t}$ denote the semi-direct product $\mathbf{Z}[1/st] \times \langle x \rangle$, where $\langle x \rangle$ is infinite cyclic, and x acts on the underlying abelian group of the ring $\mathbf{Z}[1/st]$ by multiplication by s/t. It is known that $G_{s,t}$ is not finitely presented, and not, for any commutative ring R, of type $(FP)_2$, (see [3]); hence the argument used in [7], Theorem 4, carries over to cohomology over R, and yields $cd_R G_{s,t} = 3$. From Theorem A it then follows that for every torsion-free soluble group G of finite Hirsch number, admitting $G_{s,t}$ as a quotient, one has $cd_{\mathbf{Z}}G = cd_{\mathbf{Q}}G = hG + 1$.

2. ***Proofs***

2.1. *Proof of Theorem A.* If $\bar{G} = G/A$ for some normal subgroup A of G, then A is also soluble, and the form of the assertion of the theorem then makes it clear that it suffices to establish the assertion for A an *abelian* normal subgroup of G. Set $q = hA = hd_{\mathbf{Q}}A$. If W is any left $\mathbf{Q}G$-module on which A acts trivially, the Universal Coefficients Theorem provides isomorphisms

$$\sigma_j : H^j(A, W) \xrightarrow{\simeq} \text{Hom}_{\mathbf{Q}}(H_j(A, \mathbf{Q}), W), \qquad j \geq 0,$$

relating the homology and cohomology groups of A over $\mathbf{Q}$. These isomorphisms are isomorphisms of left $\mathbf{Q}\bar{G}$-modules, with respect to the diagonal action of $\bar{G}$ on the right hand side, the homology group $H_j(A, \mathbf{Q})$ being considered as a right $\bar{G}$-module. It follows, first of all, that $H^{q+1}(A, W)=0$ for every such module W. Next, we analyze $H_q(A, \mathbf{Q})$. By choosing a series

$$\operatorname{tor} A = A_0 < A_1 < \cdots < A_q = A$$

of subgroups of A, with $\operatorname{tor} A$ the torsion subgroup and each A_j/A_{j-1} torsion-free of rank 1, $1 \le j \le q$, and using repeatedly a spectral sequence argument (cf. [4], p. 102, Prop. 7.12), one sees that $H_q(A, \mathbf{Q})$ is a $\mathbf{Q}\bar{G}$-module, whose underlying $\mathbf{Q}$-vector space is one-dimensional. We denote this $\mathbf{Q}\bar{G}$-module by $\tilde{\mathbf{Q}}$. As $\operatorname{Aut}_{\mathbf{Q}}(\mathbf{Q}) \simeq \mathbf{Q}^{\cdot}$, (the multiplicative group of nonzero rational numbers), $\tilde{\mathbf{Q}}$ can at will be considered as a right or a left module, i.e. without switching from g to g^{-1} as it is necessary too for general $\bar{G}$-modules. We have then a $\mathbf{Q}\bar{G}$-module isomorphism

$$\tau : \operatorname{Hom}_{\mathbf{Q}}(\tilde{\mathbf{Q}}, W) \xrightarrow{\simeq} \tilde{\mathbf{Q}} \otimes_{\mathbf{Q}} W$$

$$f \longrightarrow 1 \otimes f(1),$$

where the action by $\bar{G}$ is understood to be diagonal on both sides. Let now $p = h(\bar{G}) = hd_{\mathbf{Q}}(\bar{G})$. Then $cd_{\mathbf{Q}}(\bar{G}) \le p+1$, by the inequalities stated in 1.2, whereas, by the above, $H^j(A, W)=0$ whenever $j>q$ and W is a $\mathbf{Q}G$-module with trivial A-action. By the usual corner argument, the spectral sequence associated with the extension $A \lhd G \twoheadrightarrow \bar{G}$, gives isomorphisms

$$H^{p+1+q}(G, W) \simeq E_2^{p+1,q}(W) \simeq H^{p+1}(\bar{G}, H^q(A, W)) \simeq H^{p+1}(\bar{G}, \tilde{\mathbf{Q}} \otimes_{\mathbf{Q}} W).$$

Let $\tilde{\mathbf{Q}}^{-1}$ denote the one-dimensional $\mathbf{Q}$-vector space equipped with the inverse G-action, so that the diagonal action on $\tilde{\mathbf{Q}} \otimes_{\mathbf{Q}} \tilde{\mathbf{Q}}^{-1}$ is the trivial one. We conclude that for every $\mathbf{Q}\bar{G}$-module V,

$$H^{p+1}(\bar{G}, V) \simeq H^{p+1}(\bar{G}, \tilde{\mathbf{Q}} \otimes_{\mathbf{Q}} (\tilde{\mathbf{Q}}^{-1} \otimes_{\mathbf{Q}} V)) \simeq H^{p+1+q}(G, \tilde{\mathbf{Q}}^{-1} \otimes_{\mathbf{Q}} V).$$

In particular, $cd_{\mathbf{Q}} G \le p+q$ implies that $cd_{\mathbf{Q}}(\bar{G}) \le p$, as asserted. □

2.2. *Proof of Theorem B.* We first notice that G is finitely generated if every countable subgroup U of G, with $hU = hG$, is finitely generated. Since for such a subgroup U, we have

$$hU \le cd_{\mathbf{Q}} U \le cd_{\mathbf{Q}} G = hG = hU,$$

it follows that it will suffice to prove the assertion under the additional hypothesis that G is countable.

Let N denote the commutator subgroup $[G, G]$ of G, and set $A = N_{ab} = N/[N, N]$. Then, conjugation turns A into a $\mathbf{Z}G_{ab}$-module. By Section 1.2 we know that G_{ab} is a finitely generated abelian group. We now claim that A is a

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finitely generated $\mathbf{Z}G_{ab}$-module. To prove this, we first note that there exists a finitely generated $\mathbf{Z}G_{ab}$-submodule A_1 of A, with $hA_1 = hA$, and it suffices to show that $B = A/A_1$ if finitely generated. Suppose B is not a finitely generated $\mathbf{Z}G_{ab}$-module. Since B is countable, we can then write it as the union of a strictly increasing countable chain:

$$B_1 < B_2 < \cdots < B_n < \cdots$$

of finitely generated $\mathbf{Z}G_{ab}$-submodules B_n. Using the fact that every finitely generated $\mathbf{Z}G_{ab}$-module is residually finite, ([9], pp. 597 and 611–613), we prove, by induction on n, that there is another chain

$$C_1 \le C_2 \le \cdots \le C_n \le \cdots$$

of submodules of B, such that for all n, $C_n < B_n$, B_n/C_n is finite, $C_{n+1} \cap B_n \subset C_n$ and $B_n + C_{n+1} \neq B_{n+1}$. (The property of residual finiteness is applied to B_{n+1}/C_n to yield C_{n+1}). Let C be the union of the chain $C_1 \le C_2 \le \cdots$. Then

$$(*) \qquad B_1/C_1 \rightarrowtail B_2/C_2 \rightarrowtail B_3/C_3 \rightarrowtail \cdots$$

is a strictly increasing chain of embeddings, with direct limit isomorphic to B/C. Let M be the inverse image of C under the projection: $N \twoheadrightarrow A \twoheadrightarrow B$. Then G/M is an extension of the locally finite group B/C by the finitely generated (abelian) group G_{ab}, and hence locally polycyclic. As $hd\, G/M = cd_{\mathbf{Q}}\, G/M < \infty$, the Corollary to the Theorem in section 3.3 shows that G/M is actually polycyclic. However, this implies that the chain (*) becomes stationary: a contradiction.

So, we can find a finite set X in N, such that A is generated as a ZG_{ab}-module by the image of X, and a finite set Y in G, whose image in G_{ab} generates this abelian group. Then $X \cup Y$ is a finite set generating G.

3. **Appendix.**

3.1. Let a group G be the union of a chain $G_1 < G_2 < G_3 < \cdots$ of subgroups. Our aim is to compute $cd_{\mathbf{R}}G$ in terms of the numbers $cd_{\mathbf{R}}G_n$, subject to suitable restrictions, which hold, e.g., if the G_n are polycyclic and of fixed Hirsch number, and $R = \mathbf{Q}$.

For any left RG-module W and $K \ge 0$, the restriction maps: $H^k(G, W) \to H^k(G_{n_1}, W) \to H^k(G_{n_2}, W)$, where $n_1 > n_2$, induce a canonical map

$$\beta^k : H^k(G, W) \to \varprojlim\{\cdots \to H^k(G_{n_1}, W) \to H^k(G_{n_2}, W) \to \cdots\}.$$

It is always surjective, and it is injective whenever all the restrictions: $H^{k-1}(G_n, W) \to H^{k-1}(G_{n-1}, W)$ are surjective, as can be seen from the exact sequence:

(3.1)
$$0 \to \varprojlim{}^1 H^{k-1}(G_n, W) \to H^k(G, W) \xrightarrow{\beta^k} \varprojlim H^k(G_n, W) \to 0,$$

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involving the inverse limit $\varprojlim$ and its derived functor $\varprojlim^1$. (This exact sequence can be obtained from the explicit description of $\varprojlim^1$ in [10], §2, and the Mayer–Vietoris sequence associated with the direct limit (tree product) G of the groups G_n; the assertions can also be proved by using, in the manner of [11], the homogeneous non-normalized bar-resolutions associated with G and the subgroups G_n.) Assume now that the G_n all have the same cohomological dimension, say $h = cd_R G_n$ for all natural numbers n, and that $H^k(G_n, F) = 0$ for every free RG-module F, every n, and every $k < h$. Then $cd_R G \le h+1$, (see e.g. (3.1)), $H^k(G, F) = 0$ for every free RG-module F and every $k < h$, and

$$H^h(G, F) \simeq \varprojlim_n H^h(G_n, F).$$

If $H^h(G, F)$ can be shown to be trivial for every free RG-module F, then $cd_R G$ must actually be $h+1$.

3.2. In this section we recall some relevant facts that can be found, for instance, in [4], 5.1, 5.2, and 5.3. Let T be an arbitrary group, $\mathbf{P} \twoheadrightarrow R$ an RT-projective resolution of R, and W a left RT-module. Then there exist canonical maps

$$\phi^k : H^k(T, RT) \otimes_{RT} W = H_k(\mathbf{P}^*) \otimes_{RT} W \to H_k(\mathbf{P}^* \otimes_{RT} W) \to H^k(T, W),$$

(cf. [4], p. 67), where P^* is short for $\mathrm{Hom}_{RT}(P, RT)$. If W is a free RT-module and $\mathbf{P}$ is made up of finitely generated projectives, these maps are bijective. Next, let S be a subgroup of finite index in T, and let

$$T = St_1 \dot\cup St_2 \dot\cup \cdots \dot\cup St_m$$

be a coset decomposition of T. If V is a right and W a left RT-module, there exists a transfer map:

$$tr : V \otimes_{RT} W \to V \otimes_{RS} W$$

taking $v \otimes w$ to $\sum_i vt_i^{-1} \otimes t_i w$. Moreover, the canonical projection $\pi : RT \to RS$, sending t to t if $t \in S$, and to 0 otherwise, induces for every left RT-module P an isomorphism of right RS-modules:

$$\sigma : \mathrm{Hom}_{RT}(P, RT) \xrightarrow{\text{res}} \mathrm{Hom}_{RS}(P, RT) \xrightarrow{\mathrm{Hom}(1,\pi)} \mathrm{Hom}_{RS}(P, RS).$$

In cohomology, it yields isomorphisms

$$\sigma^k_{T,S} : H^k(T, RT) \xrightarrow{\simeq} H^k(S, RS)$$

of right RS-modules (cf. [4], p. 73). The various maps defined above fit into a commutative diagram:

$$\begin{array}{ccccc}
H^k(T, RT) \otimes_{RT} W & \xrightarrow{\text{tr}} & H^k(T, RT) \otimes_{RS} W & \xrightarrow{\sigma_{T,S} \otimes 1} & H^k(S, RS) \otimes_{RS} W \\
\downarrow \phi^k & & & & \downarrow \phi^k \\
H^k(T, W) & & \xrightarrow{\text{res}} & & H^k(S, W)
\end{array}$$

3.3. Using the tools prepared in 3.1 and 3.2 we are now able to prove the result announced:

THEOREM. *Let G be the union of a chain of subgroups:*

$$G_1 \leq G_2 \leq G_3 \leq \cdots \tag{3.2}$$

such that the indices $[G_n : G_{n-1}]$ are all finite, and let h be a fixed non-negative integer. Suppose that each G_n is of type (FP), of cohomological dimension $cd_R G_n = h$, and such that $H^k(G_n, RG_n) = 0$ for $0 \leq k < h$. Then $cd_R G$ is h or $h+1$, depending on whether G is finitely generated or not.

COROLLARY. *Every locally polycyclic group G with $cd_R G = hG < \infty$ is finitely generated.*

Proof of the Corollary. As it suffices to prove that every countable subgroup G_0 of G, with $hG_0 = hG$, is finitely generated, we may as well assume G to be countable. Then G is the union of a chain: $G_1 \leq G_2 \leq G_3 \leq \cdots$ of polycyclic subgroups with $hG_n = hG$ for all n, and the indices $[G_n : G_{n-1}]$ are automatically finite. Because every polycyclic group of finite cohomological dimension is a duality group, (cf. [4], p. 140, Th. 9.2 and p. 157, Th. 9.9 and Th. 9.10), all the hypotheses of the Theorem are fulfilled, and we conclude that G must be finitely generated. □

Proof of the Theorem. If G is finitely generated, then $G = G_n$ for some n, and $cd_R G = h$. In the contrary case we may assume that the subgroups in (3.2) are all distinct. Choose for each $n \geq 2$, elements g_α, $\alpha \in G_{n-1} \backslash G_n$, such that $G_n = \dot{\cup}\{G_{n-1} \cdot g_\alpha \mid \alpha \in G_{n-1} \backslash G_n\}$ is a coset decomposition of G_n. Similarly, choose for each $n \geq 1$, elements $h_\beta^{(n)}$, $\beta \in G_n \backslash G$, such that $G = \dot{\cup}\{G_n \cdot h_\beta^{(n)} \mid \beta \in G_n \backslash G\}$ is a coset decomposition of G.

By 3.1, it suffices to prove that the inverse limit of the diagram:

$$\cdots \to H^h(G_n, F) \to H^h(G_{n-1}, F) \to \cdots \to H^h(G_2, F) \to H^h(G_1, F)$$

is zero whenever F is a free RG-module. By 3.2, this diagram is isomorphic to the diagram

$$\cdots \longrightarrow H^h(G_n, RG_n) \otimes_{RG_n} F \xrightarrow{\tau_n} H^h(G_{n-1}, RG_{n-1}) \otimes_{RG_{n-1}} F \longrightarrow \cdots \longrightarrow H^h(G_1, RG_1) \otimes_{RG_1} F,$$

where

$$\tau_n(c \otimes f) = \sum_\alpha (\sigma^h_{G_n, G_{n-1}}(c \cdot g_\alpha^{-1})) \otimes g_\alpha \cdot f.$$

Let σ_n be short for $\sigma^h_{G_n, G_{n-1}}$, and let $\mathscr{L}$ be a basis for F as an RG-module. Then $\mathscr{L}_n = \{h_\beta^{(n)} \cdot b \mid \beta \in G_n \backslash G, b \in \mathscr{L}\}$ is a basis for F as an RG_n-module, and so every element of $H^n(G_n, RG_n)\otimes_{RG_n} F$ has a unique representation of the form

$$x = \sum \{c_{\beta,b} \otimes h_\beta^{(n)} b \mid h_\beta^{(n)} \cdot b \in \mathcal{L}_n\},$$

where, of course, supp $x = \{h_\beta^{(n)} b \in \mathcal{L}_n \mid c_{\beta,b} \neq 0\}$ is finite. The image of x under τ_n is then

$$\tau_n(x) = \sum \{\sigma_n(c_{\beta,b} \cdot g_\alpha^{-1}) \otimes g_\alpha h_\beta^{(n)} b \mid \alpha \in G_{n-1}\backslash G_n, h_\beta^{(n)} b \in \mathcal{L}_n\}. \tag{3.3}$$

Now, the $g_\alpha h_\beta^{(n)}$ constitute a transversal of G_{n-1} in G, although not necessarily the one originally chosen. However, we see from the form of the "monomial" matrix, describing the change from the basis $\{g_\alpha h_\beta^{(n)} b\}$ to the basis $\{h_\beta^{(n-1)} b\}$, that the number of terms in the expression (3.3) for $\tau_n(x)$, associated with the first basis, is the same as the number of terms in the analogous expression for $\tau_n(x)$, associated with the second basis. Explicitly,

$$\#(\text{supp. } \tau_n(x)) = [G_n : G_{n-1}].\#\text{supp. } (x).$$

But this then implies that all the τ_n are injective, and for every non-zero $x \in H^h(G_1, RG_1)\otimes_{RG_1} F$, there exists an integer n_0 such that x is not in the image of the map:

$$H^h(G_{n_0}, RG_{n_0})\otimes_{RG_{n_0}} F \to \cdots \to H^h(G_1, RG_1)\otimes_{RG_1} F,$$

whence $\varprojlim_n H^h(G_n, RG_n)\otimes_{RG_n} F = 0$, as desired. □

References

1. R. Baer and H. Heineken, *Radical groups of finite abelian subgroup rank.* Illinois J. Math. **16** (1972), 533–580.

2. R. Bieri, *Über die cohomologische Dimension der auflösbaren Gruppen.* Math. Z. **128** (1972), 235–243.

3. R. Bieri and R. Strebel, *Almost finitely presented soluble groups.* Comment. Math. Helv. **53** (1978), 258–278.

4. R. Bieri, *Homological dimension of discrete groups.* Queen Mary College Lecture Notes, QMC (University of London) 1976.

5. D. E. Cohen, *Groups of cohomological dimension one.* Lecture Notes in Maths. Springer-Verlag. Berlin-Heidelberg-New York, 1972.

6. V. S. Čarin, *On soluble groups of type A_4.* Mat. Sb. **52** (94) (1960), 895–914.

7. D. Gildenhuys, *Classification of soluble groups of cohomological dimension two.* Math. Z. **166,** (1979), 21–25.

8. K. W. Gruenberg, *Cohomological topics in group theory.* Springer Lecture Notes in Mathematics **143**. Springer-Verlag, Berlin-Heidelberg-New York, 1970.

9. P. Hall, *On the finiteness of certain soluble groups.* Proc. London Math. Soc. (3) **9** (1959), 592–622.

10. C. U. Jensen, *Les foncteurs dérivés de $\varprojlim$ et leurs applications en theorie des modules.* Springer Lecture Notes in Mathematics, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

11. D. J. S. Robinson, *On the cohomology of soluble groups of finite rank.* J. Pure and Applied Alg. **6** (1975), 155–164.

12. U. Stammbach, *On the weak homological dimension of the group algebra of soluble groups.* J. London Math. Soc. (2) **2** (1970), 657–570.

13. R. Strebel, *On homological duality*. J. Pure and applied Alg. **8** (1976), 75–96.

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