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SOME CANCELLATION THEOREMS

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Abstract

If G, H and B are groups such that $G \times B \simeq H \times B$, $G/[G,G].Z(G)$ is free abelian and B is finitely generated abelian, then $G \simeq H$. The equivalence classes of triples (V, ξ, A) where V and A are finitely generated free abelian groups and $\xi: V \otimes V \to A$ is a bilinear form constitute a semigroup $\mathscr{B}$ under a natural external orthogonal sum. This semigroup $\mathscr{B}$ is cancellative. A cancellation theorem for class 2 nilpotent groups is deduced.

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1. Introduction

In an attempt at the problems mentioned in Mislin (1974) we obtained the following results.

THEOREM I. *Let G, H and A be groups such that $G \times A \simeq H + A$. Suppose that $G/[G,G]$. $Z(G)$ is free abelian and A is finitely generated abelian. Then $G \simeq H$.*

THEOREM II. *Let N_1, N_2 and M be finitely generated torsionfree nilpotent groups of nilpotency class $\leqslant 2$. Then $N_1 \times M \simeq N_2 \times M$ implies $N_1 \simeq N_2$.*

In Section 2 we prove Theorem I using Theorem 2.2 and the relation between 2-cocyles and central extensions. Theorem 2.2 is of interest on its own. The key is Lemma 2.1 which can be found, for example, in Mislin (1974). We show (in Theorem 2.5) how it can be utilized effectively in proving various already known cancellation theorems. Theorem 3 of Hirshon (1977) is obtained as a consequence of Theorem I.

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We note that Theorem II follows easily from Theorem I and Theorem 3 of Hirshon (1977). In Section 3 we prove more general results than Theorem II above. Here again we exploit the relation between 2-cocycles and central extensions. We consider certain equivalence classes of triples (V, ξ, A) where V and A are finitely generated free abelian groups and $\xi: V \otimes V \to A$ is a bilinear form. Under a natural external orthogonal sum, these form a semigroup $\mathscr{B}$. We show that this semigroup $\mathscr{B}$ is cancellative (Theorem 3.5).

We are grateful to Mislin for bringing to our notice the works of Hirshon. Thanks are due to the referees for several useful comments.

2. Groups

In this section Theorem I is proved. For a Cartesian product decomposition $G = \times_i G_i$ of a group G let ι_{G_i} denote the inclusion map $G_i \hookrightarrow G$ and let π_{G_i} denote the projection map $G \twoheadrightarrow G_i$. For any group $H = \times_j H_j$ and a map $f: G \to H$ the composite map

$$G_i \xrightarrow{\iota_{G_i}} G \xrightarrow{f} H \xrightarrow{\pi_{H_j}} H_j$$

is denoted by $f(G_i, H_j)$. For any two maps $f, g: G \to H$, where H is a group, the map which associates $x \mapsto f(x).\, g(x^{\pm 1})$ is denoted by $f \pm g$.

LEMMA 2.1. *Let $\lambda: G \times A \xrightarrow{\sim} H \times A$ be an automorphism where G, H and A are groups. Suppose $\lambda(A, A)$ is an automorphism of A. Then $\lambda^{-1}(H, G)$ is an isomorphism with*

$$\alpha = \lambda(G, H) - \lambda(A, H) \circ (\lambda(A, A))^{-1} \circ \lambda(G, A)$$

as its inverse.

PROOF. Since $\lambda \circ \lambda^{-1} = \mathrm{Id}_{H \times A}$, by taking appropriate restrictions, we obtain

$$\lambda(G, A) \circ \lambda^{-1}(H, G) + \lambda(A, A) \circ \lambda^{-1}(H, A) = 0$$

and

$$\lambda(G, H) \circ \lambda^{-1}(H, G) + \lambda(A, H) \circ \lambda^{-1}(H, A) = \mathrm{Id}_H.$$

Thus

$$\begin{aligned} \alpha \circ \lambda^{-1}(H, G) &= \lambda(G, H) \circ \lambda^{-1}(H, G) \\ &\qquad - \lambda(A, H) \circ (\lambda(A, A))^{-1} \circ \lambda(G, A) \circ \lambda^{-1}(H, G) \\ &= \lambda(G, H) \circ \lambda^{-1}(H, G) + \lambda(A, H) \circ \lambda^{-1}(H, A) = \mathrm{Id}_H. \end{aligned}$$

Similarly we can verify that $\lambda^{-1}(H, G) \circ \alpha = \mathrm{Id}_G$.

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THEOREM 2.2. *Let A and B be abelian groups, X a set and f, g: $X \to B$ set theoretic maps. Suppose λ: $B \oplus A \xrightarrow{\sim} B \oplus A$ is an automorphism such that $\lambda \circ \iota_B \circ f = \iota_B \circ g$. If A is finitely generated then there exists an automorphism λ': $B \xrightarrow{\sim} B$ such that $\lambda' \circ f = g$.*

PROOF. Clearly it suffices to prove the statement when A is either infinite cyclic or isomorphic to $\mathbf{Z}/p^k$, for some prime $p \geqslant 2$ and some $k \geqslant 1$. We first consider a special case: '$\lambda(A, A)$ is an automorphism': The hypothesis $\lambda \circ \iota_B \circ f = \iota_B \circ g$ implies

$$\lambda(B, B) \circ f = g; \quad \lambda(B, A) \circ f = 0. \tag{2.1}$$

Hence if we take $\lambda' = \alpha$ where α is as in Lemma 2.1 (with $G = H = A$) then λ' is as required.

Now suppose $\lambda(A, A)$ is not an isomorphism. Then $\lambda(B, A)$ is nontrivial. Let $K = \ker \lambda(B, A)$. We claim that there exists a subgroup C of B such that $K \oplus C = B$ and $C \simeq A$. If A is infinite cyclic then the exact sequence

$$0 \longrightarrow K \longrightarrow B \longrightarrow A$$

yields a split exact sequence

$$\begin{array}{c} 0 \longrightarrow K \longrightarrow B \underset{\theta}{\rightleftarrows} \mathbf{Z} \longrightarrow 0 \\ \Vert \\ \operatorname{Im} \lambda(B, A) \end{array}$$

On the other hand, suppose $A \simeq \mathbf{Z}/\mathrm{p}^k$. Then $\operatorname{End}(A)$ is a local ring. From $\lambda \circ \lambda^{-1} = \operatorname{Id}_{B \oplus A}$ we obtain

$$\lambda(A, A) \circ \lambda^{-1}(A, A) + \lambda(B, A) \circ \lambda^{-1}(A, B) = \operatorname{Id}_A. \tag{2.2}$$

By assumption $\lambda(A, A)$ is not an automorphism. Hence it follows that $\lambda(B, A) \circ \lambda^{-1}(A, B)$ is an automorphism. Thus $\lambda(B, A)$: $B \to A$ has a right inverse θ: $A \to B$. Taking $C = \operatorname{Im} \theta$ in both the cases it follows that $B = K \oplus C$ and $C \simeq A$. Since $\lambda(K, A) = 0$ (2.2) yields

$$\lambda(A, A) \circ \lambda^{-1}(A, A) + \lambda(C, A) \circ \lambda^{-1}(A, C) = \operatorname{Id}_A. \tag{2.3}$$

Hence there exists an automorphism μ of $C \oplus A$ such that

$$\mu(A, C) = \lambda^{-1}(A, C) \quad \text{and} \quad \mu(A, A) = \lambda^{-1}(A, A).$$

Take $\lambda'' = (I_K, \mu)$. Then λ'' is clearly an automorphism of $B \oplus A$. Since $\lambda(B, A) \circ f = 0$ we have $\operatorname{Im} f \subseteq K$. Hence

$$(I_K, \mu) \circ \iota_B \circ f = (I_K, \mu \circ \iota_C) \circ f = \iota_B \circ f.$$

Hence $\lambda'' \circ \iota_B \circ f = \lambda \circ \iota_B \circ f = \iota_B \circ g$. Finally

$$\begin{aligned}\lambda''(A, A) &= \pi_A \circ \lambda \circ (I_K, \mu) \circ \iota_A \\ &= \lambda(C, A) \circ \mu(A, C) + \lambda(A, A) \circ \mu(A, A) \\ &= \lambda(C, A) \circ \lambda^{-1}(A, C) + \lambda(A, A) \circ \lambda^{-1}(A, A) = \mathrm{Id}_A.\end{aligned}$$

Thus we are back in the special case with λ replaced by λ''. This completes the proof of the theorem.

PROOF OF THEOREM I.

Clearly $Z(G) \times A \simeq Z(H) \times A$ and hence $Z(G) \simeq Z(H) = B$ say. Also we have

$$G/Z(G) \simeq G \times A/Z(G) \times A \simeq H \times A/Z(H) \times A \simeq H/Z(H) = K$$

say. Without loss of generality we can assume that G and H are given by central extensions

$$E \colon B \rightarrowtail G \twoheadrightarrow K,$$

$$F' \colon B \rightarrowtail H \twoheadrightarrow K$$

corresponding to $[E]$, $[F'] \in H^2(K; B)$, with $Z(G) = B = Z(H)$. Thus an isomorphism $\psi \colon G \times A \xrightarrow{\sim} H \times A$ yields an isomorphism of central extentions

$$\begin{array}{ccccc} B \times A & \rightarrowtail & G \times A & \twoheadrightarrow & K \\ \downarrow \wr \lambda & & \downarrow \wr \psi & & \downarrow \wr \varphi \\ B \times A & \rightarrowtail & H \times A & \twoheadrightarrow & K \end{array}$$

which is equivalent to say $\lambda_* \circ \iota_{B*}([E]) = \iota_{B*} \varphi^*([F']) = \iota_{B*}([F])$ where $\varphi^*([F']) = [F]$ say. For abelian group L, by the Universal Coefficient Theorem, we have a natural exact sequence

$$\mathrm{Ext}(H_1(K), L) \rightarrowtail H^2(K, L) \twoheadrightarrow \mathrm{Hom}(H_2(K); L).$$

Since $H_1(K) = ab(K) \simeq G/[G, G] . Z(G)$ is free abelian it follows that

$$h \colon H^2(K, L) \xrightarrow{\sim} \mathrm{Hom}(H_2(K); L)$$

is an isomorphism. Thus we obtain $\lambda \circ \iota_B \circ h([E]) = \iota_B \circ h([F])$ for maps

$$h([E]), h([F]) \colon H_2(K) \to B.$$

From Theorem 2.2 there is an automorphism λ' of B such that $\lambda' \circ h([E]) = h([F])$. This in turn implies $\lambda'_*([E]) = [F] = \varphi*([F'])$ in $H^2(K; B)$ and hence yields an

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isomorphism of central extensions

$$\begin{array}{rccccc} E\colon & B & \rightarrowtail & G & \twoheadrightarrow & K \\ & \downarrow\wr\lambda' & & \downarrow\wr & & \downarrow\wr\varphi \\ F'\colon & B & \rightarrowtail & H & \twoheadrightarrow & K \end{array}$$

In particular, we have proved that $G \simeq H$.

COROLLARY 2.4. (Hirshon (1977).) *Let $G \times A \simeq H \times A$ where A is a finitely generated abelian group and G is a finitely generated tosion free nilpotent group of nilpotency class 2.Then $G \simeq H$.*

PROOF. Since $\mathrm{nil}(G) \leqslant 2$, $[G, G] \subseteq Z(G)$. Since G is finitely generated torsion free $G/[G, G]$. $Z(G) = G/Z(G)$ is also a finitely generated torsion free abelian group and hence free. We can now apply the above theorem.

We shall now illustrate how Lemma 2.1 can be effectively employed in cancelling an infinite cyclic factor:

THEOREM 2.5. *Let $G \times \mathbf{Z} \simeq H \times \mathbf{Z}$. Then $G \simeq H$ if one of the following conditions holds:*

(a) *$Z(G)$ is periodic* [Walker (1956)].
(b) *$Z(G)$ is divisible* [Hirshon (1977)].
(c) *G is nilpotent and $Z(G)$ is infinite cyclic* [Hirshon (1975), Baumslag (1975)].

PROOF. We shall prove (c) here. The proof of (a) and (b) will be similar and simpler. So let $\lambda\colon G \times \mathbf{Z} \xrightarrow{\sim} H \times \mathbf{Z}$ be an isomorphism and G be a nilpotent group with $Z(G) \simeq \mathbf{Z}$. This induces isomorphisms

$$\lambda_1\colon Z(G) \times \mathbf{Z} \xrightarrow{\sim} Z(H) \times \mathbf{Z}, \quad \lambda_2\colon [G, G] \to [H, H]$$

and

$$\lambda_3\colon Z(G) \cap [G, G] \to Z(H) \cap [H, H]$$

by restriction. Set

$$V = Z(G)/Z(G) \cap [G, G] \quad \text{and} \quad W = Z(H)/Z(H) \cap [H, H].$$

Now λ_1 and λ_3 together induce an isomorphism $\bar{\lambda}\colon V \times \mathbf{Z} \to W \times \mathbf{Z}$ such that $\bar{\lambda}(\mathbf{Z}, \mathbf{Z}) = \lambda(\mathbf{Z}, \mathbf{Z})$. Since G is nilpotent $Z(G) \cap [G, G]$ is nontrivial and since $Z(G) \simeq \mathbf{Z}$, it follows that V is finite. Hence $\bar{\lambda}(V, \mathbf{Z}) \equiv 0$. Thus $\lambda(\mathbf{Z}, \mathbf{Z})$ is an isomorphism. Now we use Lemma 2.1 to conclude that $G \simeq H$.

Finally we state and prove a lemma which we are going to use in the next section.

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LEMMA 2.6. *Let* λ: $G \times H \to G \times H$ *be an automorphism where* G *and* H *are arbitrary groups. Suppose* $\lambda(H,H)$ *and* $\lambda^{-1}(H,H)$ *are trivial. Then* $\lambda(G,G)+\lambda(H,G)\circ\lambda(G,H)$ *is an automorphism of* G *with* $\lambda^{-1}(G,G)+\lambda^{-1}(H,G)\circ\lambda^{-1}(G,H)$ *as its inverse.*

PROOF. Using the fact that λ is an automorphism and that elements of G and H commute one can check in a straightforward manner that $\lambda(G,G)+\lambda(H,G)\circ\lambda(G,H)$ is a homomorphism. The rest of the proof is similar to the arguments in Lemma 2.1.

3. Bilinear forms

We consider the family of triples (V,ξ,A) where V and A are finitely generated abelian groups and ξ: $V\otimes V\to A$ is a bilinear form. The external orthogonal sum $(V_1,\xi_1,A_1)\oplus(V_2,\xi_2,A_2)$ is defined to be the triple (W,θ,B) with $W=V_1\oplus V_2$, $B=A_1\oplus A_2$ and $\theta=(\iota_{A_1}\circ\xi_1)\perp(\iota_{A_2}\circ\xi_2)$ where $\perp$ denotes the usual orthogonal sum. A triple (V,ξ,A) is said to be 'decomposable' if there exist triples (V_i,ξ_i,A_i), $i=1,2$, such that

$$(V,\xi,A)=(V_1,\xi_1,A_1)\oplus(V_2,\xi_2,A_2)$$

and at least one of V_i or A_i is nontrivial; otherwise it is said to be 'indecomposable'. (Note that unfortunately this terminology differs from the usual meaning of decomposable bilinear forms over a ring.) We introduce an equivalence relation $\sim$, in this family by saying $(V,\xi,A)\sim(W,\theta,B)$ if and only if there are isomorphisms f: $V\to W$ and λ: $A\to B$ such that $\lambda\circ\xi=\theta\circ(f\otimes f)$. We denote the equivalence class represented by (V,ξ,A) by $[V,\xi,A]$. Clearly $\oplus$ is well defined on these equivalence classes and forms a (commutative) semigroup which we denote by $\mathscr{B}$. The zero element of this semigroup is $[0,\mathscr{O},0]$. Throughout this section $\mathscr{O}$ will denote a trivial form, the domain and the range of it being understood from the context. The following propositions are easily seen.

PROPOSITION 3.1. (V,ξ,A) *is indecomposable if* ξ *is anisotropic and* $A\simeq\mathbf{Z}$.

PROPOSITION 3.2. *If* (V,ξ,A) *is indecomposable and* ξ *is nontrivial then* ξ *is anisotropic and* $\operatorname{rank}(\operatorname{Im}\xi)=\operatorname{rank}(A)$.

PROPOSITION 3.3. *Every triple* (V,ξ,A) *is the external orthogonal sum of finitely many indecomposable ones.*

Before actually going to the main result of this section we will state and prove a special case of it, using arguments similar to that in Theorem 2.2.

Proposition 3.4. *Let ξ_i: $V \otimes V \to A$ be bilinear forms, $i = 1, 2$ where V and A are free abelian groups. Suppose there is an automorphism f: $V \oplus \mathbf{Z} \to V \oplus \mathbf{Z}$ such that $\xi_1 \oplus \mathcal{O} = (\xi_2 \oplus \mathcal{O}) \circ (f \otimes f)$ where $\mathcal{O}$: $\mathbf{Z} \otimes \mathbf{Z} \to A$ is the trivial form. Then there exists an automorphism g: $V \xrightarrow{\sim} V$ such that $\xi_1 = \xi_2 \circ (g \otimes g)$.*

Proof. First consider the special case where: '$f(\mathbf{Z}, \mathbf{Z})$ is an isomorphism'. Then as in the proof of Theorem 2.2

$$g = f(V, V) - f(\mathbf{Z}, V) \circ (f(\mathbf{Z}, \mathbf{Z}))^{-1} \circ f(V, \mathbf{Z})$$

is an automorphism of V. Moreover $\xi_1 \oplus \mathcal{O} = (\xi_2 \oplus \mathcal{O}) \circ (f \otimes f)$ implies, by restriction, that

$$\xi_1 = \xi_2 \circ (f(V, V) \otimes f(V, V))$$

and

$$\mathcal{O} = \xi_2 \circ (f(\mathbf{Z}, V) \otimes f(\mathbf{Z} \otimes V))$$

Hence $\xi_2 \circ (g \otimes g) = \xi_1$ as required.

Now suppose $f(\mathbf{Z}, \mathbf{Z})$ is not an isomorphism. Then $f(\mathbf{Z}, V)$ is a nontrivial homomorphism and hence we can write $V = V_1 \oplus V_2$ with $V_2 \simeq \mathbf{Z}$ and $V_2 \supseteq \operatorname{Im} f(\mathbf{Z}, V) \simeq \mathbf{Z}$. We first claim that $\xi_2(v_2 \otimes v) = 0 = \xi_2(v \otimes v_2)$ for every $v_2 \in V_2$, $v \in V$. By choice, there exists a nonzero integer n such that $nv_2 = f(\mathbf{Z}, V)(1)$. Then

$$n\xi_2(v_2 \otimes v) = \xi_2(nv_2 \otimes v) = \xi_2(f(\mathbf{Z}, V)(1) \otimes v)$$
$$= (\xi_2 \perp \mathcal{O})(f(1) \otimes v) = (\xi_1 \perp \mathcal{O})(1 \otimes f^{-1}(v)) = 0.$$

(Here $1 \in \mathbf{Z} \subset V \oplus \mathbf{Z}$ is the generator of the second factor $\mathbf{Z}$). Hence $\xi_2(v_2 \otimes v) = 0$. Similarly, $\xi_2(v \otimes v_2) = 0$. Thus for any automorphisms γ of $V_2 \oplus \mathbf{Z}$ it follows that

$$(\xi_2 \perp \mathcal{O}) \circ ((\mathrm{Id}_{V_1}, \gamma) \otimes (\mathrm{Id}_{V_1}, \gamma)) = \xi_2 \perp \mathcal{O}.$$

Since f is an isomorphism, by arguments similar to that in Theorem 2.2, it follows that there exists an automorphism γ of $V_2 \oplus \mathbf{Z}$ such that if $f' = (\mathrm{Id}_{V_2}, \gamma) \circ f$, then $f'(\mathbf{Z}, \mathbf{Z}) = \mathrm{Id}_{\mathbf{Z}}$. Moreover

$$\xi_2 \circ (f \otimes f) = \xi_2 \circ ((\mathrm{Id}_{V_1}, \gamma) \otimes (\mathrm{Id}_{V_1}, \gamma)) \circ (f \otimes f) = \xi_2 \circ (f \otimes f) = \xi_1$$

and hence we are back in the special case with f replaced by f'. This completes the proof of 3.4.

We shall now state and prove the main result of this section.

Theorem 3.5. *$\mathscr{B}$ is cancellative. In other words,*

$$[V_1, \xi_1, A_1] \oplus [W, \theta, B] = [V_2, \xi_2, A_2] \oplus [W, \theta, B]$$
$$\Rightarrow [V_1, \xi_1, A_1] = [V_2, \xi_2, A_2].$$

More generally, we have the following theorem, from which Theorem 3.5 follows as an immediate corollary.

THEOREM 3.6. *Let A, B, V and W be free abelian groups and ξ_1: $V \otimes V \to A$, $i = 1, 2$, and θ: $W \otimes W \to B$ be any bilinear forms. Suppose there are automorphisms f: $V \oplus W \xrightarrow{\sim} V \oplus W$ and λ: $A \oplus B \xrightarrow{\sim} A \oplus B$ such that*

$$\lambda \circ (\iota_A \circ \xi_1 \perp \iota_B \circ \theta) = (\iota_A \circ \xi_2 \perp \iota_B \circ \theta) \circ (f \otimes f) \tag{3.1}$$

and that W and B are finitely generated. Then there exists automorphism g: $V \xrightarrow{\sim} V$ and μ: $A \xrightarrow{\sim} A$ such that $\mu \circ \xi_1 = \xi_2 \circ (g \otimes g)$.

PROOF. By Proposition 3.3 and a simple induction argument we can assume that (W, θ, B) is indecomposable.

Suppose first that θ: $W \otimes W \to B$ is the trivial form. Then either $W = 0$ and $(B \simeq \mathbf{Z})$ or $B = 0$ (and $W \simeq \mathbf{Z}$) since (W, θ, B) is assumed to be indecomposable. The conclusion of the theorem would then follow by appealing to either Theorem 2.2 or to Proposition 3.4 respectively.

So from now on we shall assume that θ is nontrivial also. By taking appropriate restriction and projections the condition (3.1) in the statement of the theorem is seen to be equivalent to the following set of conditions:

$$\begin{cases} \lambda(A, A) \circ \xi_1 \perp \lambda(B, A) \circ \theta = (\xi_2 \perp \mathcal{O}) \circ (f \otimes f), \\ \lambda(A, B) \circ \xi_2 \perp \lambda(B, B) \circ \theta = (\mathcal{O} \perp \theta) \circ (f \otimes f) \end{cases} \tag{3.2}$$

which in turn is equivalent to

$$\begin{cases} \lambda(A, A) \circ \xi_1 = \xi_2 \circ (f(V, V) \otimes f(V, V)); \\ \lambda(B, A) \circ \theta = \xi_2 \circ (f(W, V) \otimes f(W, V)), \\ \lambda(A, B) \circ \xi_1 = \theta \circ (f(V, W) \otimes f(V, W)); \\ \lambda(B, B) \circ \theta = \theta \circ (f(W, W) \otimes f(W, W)) \end{cases} \tag{3.3}$$

Also (3.1) is equivalent to

$$\lambda^{-1} \circ (\iota_A \circ \xi_2 \perp \iota_B \circ \theta) = (\iota_A \circ \xi_1 \perp \iota_B \circ \theta) \circ (f^{-1} \otimes f^{-1}) \tag{3.1^{-1}}$$

which in turn, is equivalent to statements $(3.2)^{-1}$ and $(3.3)^{-1}$ analogous to (3.2) and (3.3).

We first claim that $\lambda(B, B)$ is injective or $\lambda(B, A)$ is injective. Assuming on the contrary, we shall prove that (W, θ, B) is decomposable which contradicts our assumption. So let $B_1 = \operatorname{Ker} \lambda(B, A)$, $B_2 = \operatorname{Ker} \lambda(B, B)$, $W_1 = \operatorname{Im} f^{-1}(W, W)$ and $W_2 = \operatorname{Im} f^{-1}(V, W)$. We shall show that

$$(W, \theta, B) = (W_1, \theta_1, B_1) \oplus (W_2, \theta_2, B_2) \quad (\text{for some } \theta_i,\ i = 1, 2).$$

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Clearly $W = W_1 + W_2$. By $(3.2)^{-1}$ we have

$$\lambda^{-1}(A,B)\circ\xi_2\perp\lambda^{-1}(B,B)\circ\theta = (\mathcal{O}\perp\theta)\circ(f^{-1}\otimes f^{-1})$$

and hence for any $w_1 = f^{-1}(W,W)(w')$, $w'\in W$ and $w_2 = f^{-1}(V,W)(v)$, $v\in V$, we have

$$\theta(w_1\otimes w_2) = (\mathcal{O}\perp\theta)(f^{-1}(w')\otimes f^{-1}(v))$$
$$= (\lambda^{-1}(A,B)\circ\xi_2\perp\lambda^{-1}(B,B)\circ\theta)(w'\otimes v) = 0.$$

Similarly $\theta(w_2\otimes w_1) = 0$, for any $w_i\in W_i$, $i = 1,2$. By the anisotropy of θ it now follows that $W_1\cap W_2 = 0$. Hence $W_1\oplus W_2 = W$.

Let $\theta_i = \theta/W_i\otimes W_i$, $i = 1,2$. We now claim that $\operatorname{Im}\theta_i\subseteq B_i$, $i = 1,2$. So let $w_1 = f^{-1}(W,W)(w')$ for some $w'\in W$. Then for any $x\in W_1$,

$$\lambda(B,A)\circ\theta_1(w_1\otimes x) = \lambda(B,A)\circ\theta(f^{-1}(W,W)(w')\otimes x)$$
$$= (\lambda(A,A)\circ\xi_1\perp\lambda(B,A)\circ\theta)(f^{-1}(w')\otimes x)$$
$$= (\xi_2\perp\mathcal{O})(w'\otimes f(x)) = 0.$$

It follows that $\lambda(B,A)\circ\theta_1\equiv 0$, i.e. $\operatorname{Im}\theta_1\subseteq B_1$. Similar arguments yield that $\operatorname{Im}\theta_2\subseteq B_2$. Hence $\operatorname{Im}\theta = \operatorname{Im}\theta_1 + \operatorname{Im}\theta_2\subseteq B_1 + B_2$. Clearly $B_1\cap B_2 = 0$. Thus in order to prove that $B_1\oplus B_2 = B$ it suffices to prove that $B_1 + B_2$ is a pure subgroup of B, since by Proposition 3.2

$$\operatorname{rank}(B)\geqslant\operatorname{rank}(B_1+B_2)\geqslant\operatorname{rank}(\operatorname{Im}\theta) = \operatorname{rank}(B).$$

So let $nb\in B_1 + B_2$ for some $n\neq 0$, $b\in B$ any. Let $nb = b_1 + b_2$, $b_i\in B_i$, $i = 1,2$, say. Then

$$n\lambda(b) = \lambda(nb) = \lambda(b_1) + \lambda(b_2)$$

with $\lambda(b_1)\in B$ and $\lambda(b_2)\in A$. If $\lambda(b) = b_3 + a_3$ for some $b_3\in B$, $a_3\in A$ then $nb_3 = \lambda(b_1)$ and $na_3 = \lambda(b_2)$. Taking $x = \lambda^{-1}(b_3)$ and $y = \lambda^{-1}(a_3)$ it follows that $nx = b_1$ and $ny = b_2$. Since B_1 and B_2 are pure subgroups of $A + B$ it follows that $x\in B_1$ and $y\in B_2$. Now $nb = n(x+y)$ and hence $b = x + y$. Hence $b\in B_1 + B_2$. Thus we have shown that $B_1 + B_2$ is a pure subgroup of B, and hence $B = B_1\oplus B_2$. This completes the proof of our claim that $\lambda(B,B)$ or $\lambda(B,A)$ is injective.

We shall now prove the theorem considering these two cases separately.

Case 1. $\lambda(B,B)$ is injective. We shall show that $f(V) = V$ (that is, $f(V,W)\equiv 0$) and hence defines an automorphism g of V, namely $g = f(V,V)$. Then from (3.3) we have

$$\lambda(A,A)\circ\xi_1 = \xi_2\circ(f(V,V)\otimes f(V,V)) = \xi_2\circ(g\otimes g)$$

and

$$\lambda(A,B)\circ\xi_1 = \theta\circ(f(V,W)\otimes f(V,W)) = 0 \quad (\text{since } f(V,W)\equiv 0).$$

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Hence $\lambda \circ \iota_A \circ \xi_1 = \iota_A \circ \xi_2 \circ (g \otimes g)$. By applying Theorem 2.2 we obtain an automorphism μ of A such that $\mu \circ \xi_1 = \xi_2 \circ (g \times g)$ as desired.

So, in order to prove that $f(V, W) \equiv 0$ we first observe that $f(W, W)$: $W \rightarrowtail W$. For any $w \in W$, suppose $f(w) \in V$. Then

$$\lambda(B, B) \circ \theta(w \otimes x) = \theta(f(W, W)(w) \otimes f(W, W)(x)) = 0$$

for every $x \in W$. Since $\lambda(B, B)$ is injective and θ is anisotropic, this implies $w = 0$, that is $f(W, W)$: $W \rightarrowtail W$. Since W is finitely generated it follows that for every $w \in W$ there is an integer $n \neq 0$ and $w' \in W$ such that $f(W, W)(w') = nw$. Now for any v,

$$\begin{aligned} n\theta(f(V, W)(v) \otimes w) &= \theta(f(V, W)(v) \otimes f(W, W)(w')) \\ &= (\mathcal{O} \perp \theta)(f(v), f(w')) \\ &= (\lambda(A, B) \circ \xi_1 \perp \lambda(B, B) \circ \theta)(v \otimes w)) = 0. \end{aligned}$$

Hence $\theta(f(V, W)(v) \otimes w) = 0$ for every $w \in W$. Since θ is anisotropic, this implies $f(V, W) \equiv 0$, as claimed.

Case 2. $\lambda(B, A)$ is injective. We first make the following two observations:

(a) $\lambda(B, A)$ is injective and the hypothesis (3.1) imply $f^{-1}(W, W) \equiv 0$.
(b) $f^{-1}(W, W) \equiv 0$ and the hypothesis $(3.1)^{-1}$ imply $\lambda^{-1}(B, B) \equiv 0$.

Proof of (a). Fix $w \in W$. Then for every $w' \in W$

$$\begin{aligned} \lambda(B, A) \circ \theta(f^{-1}(W, W)(w) \otimes w') &= (\lambda(A, A) \circ \xi_1 \perp \lambda(B, A) \circ \theta)(f^{-1}(w) \otimes w') \\ &= (\xi_2 \perp \mathcal{O})(w \otimes f(w')) = 0 \end{aligned}$$

and hence $\theta(f^{-1}(W, W)(w) \otimes w') = 0$. By the anisotropy of θ it follows that $f^{-1}(W, W) \equiv 0$.

Proof of (b). From $(3.1)^{-1}$ we have

$$\lambda^{-1}(B, B) \circ \theta = \theta \circ (f^{-1}(W, W) \otimes f^{-1}(W, W)) = 0$$

and hence $\lambda^{-1}(B, B)(\operatorname{Im} \theta) = 0$. Since $\operatorname{rank}(\operatorname{Im} \theta) = \operatorname{rank}(B)$ it follows that $\lambda^{-1}(B, B) \equiv 0$.

From (a) and (b) it follows that under the hypothesis (3.1) (which is equivalent to $(3.1)^{-1}$) we have the following chain of implications:

$$\lambda(B, A) \text{ is injective} \Rightarrow f^{-1}(W, W) \equiv 0 \Rightarrow \lambda^{-1}(B, B) \equiv 0$$

and hence in particular,

$$\lambda^{-1}(B, A) \text{ is injective} \Rightarrow f(W, W) \equiv 0 \Rightarrow \lambda(B, B) \equiv 0.$$

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Hence, from Lemma (2.6)

$$g = f(V,V) + f(W,W) \circ f(V,W) \quad \text{and} \quad \mu = \lambda(A,A) + \lambda(B,A) \circ \lambda(A,B)$$

are automorphisms of V and A respectively. Finally, using (3.2) it is easily verified that $\mu \circ \xi_1 = \xi_2 \circ (g \otimes g)$.

This completes the proof of Theorem 3.6.

REMARK 3.7. If L and K are abelian groups, the exact sequence

$$0 \longrightarrow \operatorname{Ext}(K, L) \longrightarrow H^2(K; L) \xrightarrow{h} \operatorname{Hom}(H_2(K); L) \longrightarrow 0$$

can be identified with the canonical exact sequence

$$0 \longrightarrow \operatorname{Ext}(K, L) \longrightarrow H^2(K; L) \xrightarrow{\alpha} \operatorname{Hom}(\Lambda^2 K; L) \longrightarrow 0$$

where α is given by the commutator form (see, for example Warfield (1976), p. 29). Thus when K is free abelian we obtain a canonical isomorphism

$$H^2(K; L) \xrightarrow{\sim} \operatorname{Hom}(\Lambda^2 K; L).$$

By using standard arguments as in the proof of Theorem I it is not hard to see that Theorem II is equivalent to Theorem 3.5 restricted to alternating forms. Finally it is also clear how one can obtain slightly more general result than Theorem II out of Theorem 3.6.

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