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THE EQUATIONS OF VISCOUS INCOMPRESSIBLE NON-HOMOGENEOUS FLUIDS: ON THE EXISTENCE AND REGULARITY

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Abstract

We prove the existence and regularity of the solution of an initial boundary value problem for viscous incompressible non-homogeneous fluids, using a semi-Galerkin approximation and so-called compatibility conditions.

1. Introduction

In [4] A. V. Kajikhov proved, via a Galerkin-type approximation, the existence in the large of at least one weak solution of the equations of the motion of viscous incompressible non-homogeneous fluids. In other words, he proved the existence of a weak solution of the following initial-boundary value problem

$$\begin{aligned} &\rho\partial_t u + \rho u\cdot\nabla u - \mu\Delta u = -\nabla p + \rho f,\\ &\partial_t\rho + u\cdot\nabla\rho = 0, \quad \text{in } Q_T\\ &\nabla\cdot u = 0,\\ &u = 0 \quad \text{on } \Gamma; \qquad u(0) = u_0; \qquad \rho(0) = \rho_0 \quad \text{in } \Omega. \end{aligned} \tag{1.1}$$

Here $Q_T = \Omega\times[0, T]$ with Ω a bounded domain in R^3, $0 < T < \infty$, Γ the boundary of Ω, $\partial_t = \partial/\partial t$; moreover $u = u(t) = u(x, t) = (u_1(x, t), u_2(x, t), u_3(x, t))$ is the velocity, $p = p(t) = p(x, t)$ the pressure, $\rho = \rho(t) = \rho(x, t)$ the density, $f = f(t) = f(x, t) = (f_1(x, t), f_2(x, t), f_3(x, t))$ the external force, and μ the viscosity. In addition, u_0, ρ_0 are the initial velocity and density respectively, and ρ_0 is assumed to satisfy

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$0 < \alpha \le \rho_0 \le \beta$ with α and β positive constants.

Furthermore, in [4], the existence of at least one strong solution, in the small, is proved; i.e., $\rho \in L^\infty(Q_T)$, $u \in L^2(O, T; H^2(\Omega))$; $\partial_t u \in L^2(Q_T)$. However one does not know whether strong solutions are unique. Concerning the uniqueness problem O. A. Ladyzhenskaya and V. A. Solonnikov in [6] prove the solvability of (1.1) in a class of smoother functions; i.e. $u \in W_q^{2,1}(Q_T)$, $\nabla p \in L^q(Q_T)$, $\rho \in C^1(Q_T)$ with $q > 3$ by linearisation and potential theory. In this class, a uniqueness theorem holds. The solvability is proved in the small if u_0 and f are arbitrary, and in a given interval $[0, T]$, $T < \infty$, if u_0 and f are sufficiently small. In [2], J. G. Heywood showed how the Galerkin approach for the existence of the Navier-Stokes equations can be pushed further to give regularity properties of the solutions directly and elegantly. In [2], it is pointed out that norm bounds of higher time derivatives of the Galerkin approximations cannot be obtained without non-local compatibility conditions of various orders for the data at $t = 0$. Heywood avoids, partially, this difficulty by estimating the norms of the time derivatives of any order of the Galerkin approximations in $t = \varepsilon$ with $\varepsilon > 0$, and considers the solution in the interval $[\varepsilon, T]$. We notice that Rautmann in [8] gives an answer to the question, how smooth a Navier-Stokes solution can be at time $t = 0$ without compatibility conditions.

The aim of this paper is to consider regularity properties of solutions of (1.1) in the context of the Heywood results. Continuing the existence theorem, we obtain norm bounds of some time and spatial derivatives of the Galerkin approximations. It seems, however, that the Heywood's norm estimates for the times derivatives of any order cannot be obtained. This is due to the presence of the density.

Consequently, if we are interested in the study of higher regularity we have to introduce the so-called compatibility conditions (see [3], [11]).

The plan of the paper is as follows. Section 2 is devoted to notation and well-known results on differential inequalities. In Section 3, we give preliminaries. In Section 4, we obtain the best possible regularity via Galerkin approximation, without compatibility conditions, and we give a uniqueness theorem.

In Section 5, we consider the cause of the breakdown of the regularity and we derive the compatibility conditions for the solution (u, ρ, p) to be smooth; so we extend to system (1.1) the compatibility conditions derived in [11], [12] for the Navier-Stokes equations.

We notice that the results of Section 3 are important to the study of error estimates of the Galerkin approximations in terms of the eigenvalues of the Stokes operator (see [9]). Regarding the problem (1.1) with $0 \le \rho_0 \le \beta$, see [5] and [10].

2. Statements and notations

Let Ω be an open bounded set of R^3 with boundary Γ and $Q_T = \Omega \times [0, T]$ with T a finite positive number. The motion of a viscous incompressible non-homogeneous fluid of viscosity 1 and subject to the external force f is governed by the system

$$\begin{aligned} &\rho\partial_t u + \rho u \cdot \nabla u - \Delta u = -\nabla p + \rho f, \\ &\partial_t \rho + u \cdot \nabla \rho = 0, \quad \text{in } Q_T \\ &\nabla \cdot u = 0. \end{aligned} \tag{2.1}$$

We complete (2.1) with initial-boundary conditions

$$u(x, 0) = u_0; \qquad \rho(x, 0) = \rho_0 \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \Gamma.$$

In (2.1)

$$\rho u \cdot \nabla u = \sum_{i=1}^{3} \rho u_i \partial_{x_i} u; \qquad \nabla \cdot u = \sum_{i=1}^{3} \partial_{x_i} u_i.$$

Throughout the paper we need the following function spaces (we do not distinguish in our notation whether the functions are R- or R^3-valued):

$D(\Omega) = \{\phi | \phi \in C_0^\infty(\Omega),\ \nabla \cdot \phi = 0\}$;

$H =$ completion of $D(\Omega)$ in $L^2(\Omega)$;

$H^s =$ usual Sobolev spaces of order s on $L^2(\Omega)$;

$V =$ completion of $D(\Omega)$ in $H^1(\Omega)$;

$W_m = \{\phi | \phi \in C([0, T]; H^m \cap V),\ \dfrac{\partial^i \phi}{\partial t^i} \in C([0, T]; H^{m-2i} \cap V)$
with $i = 1, 2, \dots, r$ ($r = [m/2]$ the integer part of $m/2$)$\}$.

We set

$$(\phi, \psi) = \sum_{i=1}^{3} \int_\Omega \phi_i \psi_i \, dx; \qquad ((\phi, \psi)) = \sum_{i=1}^{3} \int_\Omega \partial_{x_i} \phi \partial_{x_i} \psi \, dx,$$

$$|\phi|^2 = (\phi, \psi); \qquad ||\phi|| = \text{ norm in } H^s; \qquad ||\phi||^2 = ||\phi||^2 = ((\phi, \phi)),$$

and

$|\phi|_q =$ norm in the space $L^q(\Omega)(q > 1)$,

$||\phi||_{s,q} =$ norm in the Sobolev space H_q^s of order s on $L^q(\Omega)$, instead of $L^2(\Omega)$ for H^s.

The spaces $C(\Omega)$, $C^m(\Omega)$, $C_0^m(\Omega)$, $C^\infty(\Omega)$, and their vector-valued analogues are defined as usual.

We assume the boundary Γ is uniformly of class C^k, i.e. first, it is possible to choose local coordinates (y_1, y_2, y_3) in a neighborhood B_ξ of each point $\xi \in \Gamma$ such that $\Gamma \cap B_\xi$ is represented by a function $y_3 = F(y_1, y_2, \xi)$ of class C^k; second, the neighbourhood B_ξ can be chosen as balls, all of the same size, with respective centers ξ and that the derivatives up to order k of each function $F(\cdot, \cdot, \xi)$ are bounded by a constant independent of ξ.

Now we state our results. For simplicity we assume $\beta \leq 1$.

THEOREM 1. *Let Ω be any bounded domain in R^3 with boundary Γ uniformly of class C^3. Let $u_0 \in V$ and $\rho_0 \in L^\infty(\Omega)$ with $0 < \alpha \leq \rho_0 \leq \beta$, and $f \in L^2(Q_T)$. Then there exists an interval $(0, \overline{T})$ and functions $u(x, t)$, $\rho(x, t)$, $p(x, t)$ defined in $Q_{\overline{T}}$ and satisfying the system* (2.1) a.e. *such that*

$$u \in L^\infty(0, T'; V) \cap L^2(0, T'; H^2); \tag{2.3}$$

$$\partial_t u, \nabla p \in L^2(Q_{T'}); \tag{2.4}$$

$$\rho \in L^\infty(Q_{T'}); \qquad \partial_t \rho \in L^\infty(0, T'; H^{-1}); \tag{2.5}$$

$$(\rho(t)u(t) - \rho_0 u_0, v) \to 0 \quad \text{as } t \to 0^+ \quad \forall v \in V; \tag{2.6}$$

with $T' < \overline{T}(\leq T)$. Further, $\overline{T}$ is greater than or equal to a positive number $T(\|u_0\|, \Gamma)$ which depends on $\|u_0\|$, α, β, f, and the C^3-regularity of Γ.

THEOREM 2. *Let $u_0 \in H^2 \cap V$, and $\rho_0 \in C^1(\overline{\Omega})$ with $0 < \alpha \leq \rho_0 \leq \beta$. Γ is as in Theorem 1, and $f \in L^2(Q_T)$, $\partial_t f \in L^2(Q_T)$. Then, the solution (u, ρ) of Theorem 1 satisfies*

$$u \in L^2(0, T'; H^3); \qquad \rho \in C^1(Q_{T'})$$

and it is unique.

THEOREM 3. *We assume Γ of class C^∞ and $\rho_0 \in C^\infty(\overline{\Omega})$ with $0 < \alpha \leq \rho_0 \leq \beta$, $u_0 \in H^m(\Omega) \cap V$, $\frac{\partial^i f}{\partial t^i} \in L^2(0, T'; V)$, $i = 1, 2, \ldots, r$, and (u, ρ, p) is a solution of* (1.1) *in the sense of Theorem* 2 *in the interval $[0, T']$. Then a necessary and sufficient condition for u to belong to W_m is that*

$$\frac{\partial^i u(0)}{\partial t^i} \in V$$

with $i = 1, 2, \ldots, r$ and $m \geq 3$ ($r =$ integer part of $m/2$).

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3. Preliminaries

Let P be the projection operator from $L^2(\Omega)$ in H. In the following we need some preliminary results (see [2]).

LEMMA 1. *Let Ω be an open set of R^3, the boundary Γ of which is uniformly of class C^3. Suppose $u \in V$ is a generalised solution of the Stokes problem*

$$-\Delta u + \nabla p = f, \qquad \nabla \cdot u = 0$$

i.e. $((u, \phi)) = (f, \phi)$ holds $\forall \phi \in V$. Then u possesses second derivatives in $L^2(\Omega)$, and the inequalities

$$\begin{aligned} ||u_2||_2 &\le c_{1\Gamma}(|Pf| + ||u||); \\ ||u||_{1,3} &\le c_{2\Gamma}(|Pf|^{1/2}||u||^{1/2} + ||u||); \\ |u|_\infty &\le c_{3\Gamma}(|Pf|^{1/2}||u||^{1/2} + ||u||) \end{aligned} \tag{3.1}$$

hold with constants depending only on the regularity of Γ.

(Throughout the paper the letter c denotes different constants.)

In the following Lemmas, we assume that $\phi(t)$, $\psi(t)$, $f(t)$, $h(t)$ are smooth non-negative functions defined for all $t \ge 0$.

LEMMA 2. *Suppose $\phi(0) = \phi_0$ and $\frac{d\phi(t)}{dt} + \psi(t) \le g(\phi(t)) + f(t)$ for $t \ge 0$, where g is a non-negative Lipschitz continuous function defined for $\phi \ge 0$. Then $\phi(t) \le F(t; \phi_0)$ for $t \in [0, T(\phi_0))$ where $F(\cdot; \phi_0)$ is the solution of the initial value problem $\frac{dF(t)}{dt} = g(F(t)) + f(t)$; $F(0) = \phi_0$ and $[0, T(\phi_0))$ is the largest interval to which it can be continued. Also, if g is nondecreasing, then*

$$\int_0^t \psi(\tau)d\tau \le \widetilde{F}(t; \phi_0)$$

with

$$\widetilde{F}(t; \phi_0) = \phi_0 + \int_0^t \Big[g(F(\tau; \phi_0)) + f(\tau)\Big]d\tau.$$

LEMMA 3. *Suppose $\phi(0) = \phi_0$ and $\frac{d\phi(t)}{dt} + \psi(t) \le h(t)\phi(t) + f(t)$ for $t \ge 0$. Then*

$$\phi(t) \le \overline{F}(t; \phi_0); \qquad \int_0^t \psi(\tau)d\tau \le \overline{F}(t; \phi_0) \quad \forall t > 0$$

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where

$$\overline{F}(t\,;\phi_0) = \phi_0 + \int_0^t f(\tau)(\exp\int_0^\tau -h(\sigma)d\sigma)d\tau)\exp\int_0^t h(\tau)d\tau$$

and

$$\overline{F}(t\,;\phi_0) = \phi_0 + \int_0^t [h(\tau)\overline{F}(\tau\,;\phi_0) + f(\tau)]d\tau.$$

Thus estimates for $\phi(t)$, *and* $\int_0^t \psi(\tau)d\tau$ *are obtained from estimates for* ϕ_0, $\int_0^t f(\tau)d\tau$ *and* $\int_0^t f(\tau)d\tau$.

4. Proofs of Theorems 1 and 2

PROOF OF THEOREM 1. We assume for simplicity $f = 0$. Let a^ℓ, λ^ℓ be, respectively, the eigenfunctions and the eigenvalues of the Stokes operator $A = -P\Delta$ in $V \cap H^2$. We take as nth approximation for $t > 0$, the solution

$$u^n = \sum_{\ell=1}^{n} c_\ell^n(t)a^\ell\,; \qquad \rho^n(x\,,t)$$

of the initial value problem for the system of ordinary differential equations

$$(\rho^n\partial_t u^n\,, a^\ell) + (\rho^n u^n\cdot\nabla u^n\,, a^\ell) - (\Delta u^n\,, a^\ell) = 0 \tag{4.1}$$

and for the partial differential equation (the continuity equation)

$$\partial_t\rho^n + u^n\cdot\nabla\rho^n = 0 \tag{4.2}$$

with initial conditions

$$c_\ell^n(0) = (u_0\,, a^\ell)\,; \qquad \rho^n(0) = \rho_0.$$

At first, assuming u^n is known, we can obtain the solution of (4.2) by using the method of the characteristics (see [6]), satisfying

$$0 < \alpha \le \rho^n \le \beta. \tag{4.3}$$

Now we consider the approximation u^n. By standard techniques there exists a solution u^n of (4.1). Now we multiply (4.1) through by $\frac{d}{dt}c_\ell^n$, sum on ℓ and obtain

$$(\rho^n\partial_t u^n\,, \partial_t u^n) + (\rho^n u^n\cdot\nabla u^n\,, \partial_t u^n) - (\Delta u^n\,, \partial_t u^n) = 0. \tag{4.4}$$

Moreover, we multiply (4.1) through by $\lambda^\ell c_\ell^n$, sum on ℓ, and obtain

$$(\rho^n\partial_t u^n\,, Au^n) + (\rho^n u^n\cdot\nabla u^n\,, Au^n) + |Au^n|^2 = 0. \tag{4.5}$$

We sum (4.5) to (4.4) and get

$$\left|\sqrt{\rho^n}\partial_t u^n\right|^2 + \frac{1}{2}\frac{d}{dt}||u^n||^2 + |Au^n|^2$$
$$+ (\rho^n u^n \cdot \nabla u^n, Au^n + \partial_t u^n) + (\rho^n \partial_t u^n, Au^n) = 0;$$

whence

$$\frac{1}{2}\frac{d}{dt}||u^n||^2 + \delta|\sqrt{\rho^n}\partial_t u^n|^2 + \delta|Au^n|^2 \le c_\delta |u^n|_6^2 ||u^n||_{1,3}^2 \tag{4.6}$$

($\delta =$ suitable constant). Thanks to Lemma 1, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}||u^n||^2 + \delta|Au^n|^2 &\le c_{\delta,\Gamma}||u^n||^2(|Au^n|^{1/2}||u^n||^{1/2} + ||u^n||)^2 \\ &\le c_{\delta,\Gamma,\sigma}||u^n||^6 + c_{\delta,\Gamma,\sigma}||u^n||^4 + \sigma|Au^n|^2\end{aligned} \tag{4.7}$$

with $0 < \sigma < \delta$. One clearly has $||u^n(0)|| \le ||u_0||$, then, from Lemma 2, we conclude there exist on some interval $[0, \overline{T})$, continuous functions $F(t)$ and $\widetilde{F}(t)$ such that

$$||u^n(t)|| \le F(t); \qquad \int_0^t |Au^n|^2 d\tau \le \widetilde{F}(t). \tag{4.8}$$

Moreover, it can be assumed $\overline{T} \ge T(||u_0||, \Gamma)$ and

$$F(t) \le F(t; ||u_0||, \Gamma); \qquad \widetilde{F}(t) \le \widetilde{F}(t; ||u_0||, \Gamma)$$

for $t \in [0, T(||u_0||, \Gamma))$ where $T(||u_0||, \Gamma)$, $F(t; ||u_0||, \Gamma)$ and $\widetilde{F}(t; ||u_0||, \Gamma)$ are obtained by integrating (4.7) in the manner of Lemma 2. $T(||u_0||.\Gamma)$ is determined solely by $||u_0||$, α, β and C^3-regularity of Γ. In view of Lemma 2, the inequalities (4.8) imply $\int_0^t ||u^n||_2^2 d\tau \le \widetilde{F}(t)$. Now from (4.6) we obtain

$$\int_0^t |\partial_\tau u^n|^2 d\tau \le G(t) \tag{4.9}$$

(of course $G(t)$ is a continuous function). Thanks to the above estimates, we can choose a subsequence of $(\{\rho^n\}, \{u^n\})$ still denoted by $(\{\rho^n\}, \{u^n\})$ such that with $0 < T' < \overline{T}$

$$\begin{aligned} u^n &\to u && \text{weakly in } L^2(0, T'; V \cap H^2); \\ \partial_t u^n &\to \partial_t u && \text{weakly in } L^2(Q_{T'}); \\ \rho^n &\to \rho && \text{weak}^* \text{ in } L^\infty(Q_{T'}); \\ \partial_t \rho^n &\to \partial_t \rho && \text{weakly in } L^2(0, T'; H^{-1}). \end{aligned} \tag{4.10}$$

By virtue of the compactness theorem in [7] (page 58), one has

$$\begin{aligned} u^n &\to u && \text{strongly in } L^2(Q_{T'}); \\ \rho^n &\to \rho && \text{strongly in } L^2(0, T', H^{-1}) \end{aligned}$$

and consequently

$$\rho^n u^n \to \rho u \quad \text{weakly in } L^2(Q_{T'}).$$

Now, if ϕ^m is any function of the form $\phi^m = \sum_{\ell=1}^m b_\ell(t)a^\ell(x)$ with continuous coefficients in $[0, T']$, (4.1) implies

$$\int_0^{T'} (\rho^n \partial_t u^n - \Delta u^n + \rho^n u^n \cdot \nabla u^n, \phi^m)dt = 0.$$

for all $n > m$. One can easily pass to the limit for $n \to \infty$ obtaining

$$\int_0^{T'} (\rho \partial_t u - \Delta u + \rho u \cdot \nabla u, \phi^m)dt = 0.$$

Now

$$\rho \partial_t u - \Delta u + \rho u \cdot \nabla u \in L^2(Q_{T'})$$

since the functions ϕ^m are dense in $L^2(0, T'; H)$; furthermore there exists a function $p = p(x, t)$ with $\nabla p \in L^2(Q_{T'})$ such that

$$\rho \partial_t u - \Delta u + \rho u \cdot \nabla u = -\nabla p \quad \text{almost everywhere in } Q_{T'}.$$

Now, from the continuity equation, we get $\forall \phi \in C_0^1(O, T'; H^1)$

$$\int_0^{T'} \{(\rho^n, \partial_\tau \phi) + (u^n \rho^n, \nabla \phi)\}d\tau = 0$$

and passing to the limit for $n \to \infty$ we obtain

$$\int_0^{T'} \{(\rho, \partial_\tau \phi) + (\rho u, \nabla \phi)\}d\tau.$$

To complete the proof, we need only show

$$(\rho(t)u(t) - \rho_0 u_0, \phi) \to 0 \quad \text{as } t \to 0^+$$

for each function $\phi \in V$, or equivalently,

$$(\rho(t)u(t) - \rho_0 u_0, a^\ell) \to 0 \quad \text{as } t \to 0^+$$

for each basic function a^ℓ. This requires several observations.

First, notice that

$$\begin{aligned} |(\rho^n(t)u^n(t) - \rho^n(0)u^n(0), a^\ell)| &= \left| \int_0^t \partial_\tau (\rho^n(\tau)u^n(\tau), a^\ell)d\tau \right| \\ &= \left| \int_0^t ((\Delta u^n - \rho^n u^n \cdot \nabla u^n - \nabla \cdot (\rho^n u^n)u^n), a^\ell)d\tau \right| \to 0 \end{aligned} \tag{4.11}$$

uniformly in n as $t \to 0^+$.

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Next, thanks to (4.10), we have $\partial_t(\rho^n u^n) \in L^2(0, T; H^{-1})$ uniformly with respect to n; consequently, for any fixed $t \in [0, T']$

$$((\rho(t)u(t) - \rho^n(t)u^n(t), a^\ell) \to 0 \quad \text{as } n \to \infty. \tag{4.12}$$

And finally, we note

$$\begin{aligned}(\rho^n(0)u^n(0) - \rho_0 u_0, a^\ell) &= (\rho_0(u^n(0) - u_0), a^\ell) \\ &= (u^n(0) - u_0, \rho_0 a^\ell) \to 0 \quad \text{as } n \to \infty.\end{aligned} \tag{4.13}$$

Then (4.11), (4.12), (4.13) imply (2.6). This completes the proof.

Proof of Theorem 2. Now, we shall prove further estimates for the approximations u^n, ρ^n given in Theorem 1. We will show there exist continuous functions $F_1(t, \varepsilon)$, $\widetilde{F}_1(t, \varepsilon)$ of $t \in [\varepsilon, T']$ such that

$$|\partial_t u^n| \le F_1(t, \varepsilon); \qquad |Au^n(t)| \le \widetilde{F}_1(t, \varepsilon)$$

for $t \in [\varepsilon, T']$ and every $\varepsilon > 0$. The function $F_1(t, \varepsilon)$, $\widetilde{F}_1(t, \varepsilon)$ will depend on given functions as in the Lemma 2.

Differentiating (4.1) with respect to t, we obtain

$$\begin{aligned}&(\partial_t \rho^n \partial_t u^n, a^\ell) + (\rho^n \frac{\partial^2 u^n}{\partial t^2}, a^\ell) \\ &\quad - (\partial_t \Delta u^n, a^\ell) + (\partial_t(\rho^n u^n \cdot \nabla u^n), a^\ell) = 0.\end{aligned} \tag{4.14}$$

Multiplying (4.14) through by $\frac{d}{dt} c^n_\ell$ and summing on ℓ, one obtains

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left|\sqrt{\rho^n}\partial_t u^n\right|^2 + \|\partial_t u^n\|^2 = &-(\partial_t \rho^n u^n \cdot \nabla u^n, \partial_t u^n) - (\rho^n \partial_t u^n \cdot \nabla u^n, \partial_t u^n) \\ &- (\rho^n u^n \cdot \nabla \partial_t u^n, \partial_t u^n) + \frac{1}{2}(\partial_t \rho^n \partial_t u^n, \partial_t u^n).\end{aligned}$$

Using the relation

$$(1/2)(\partial_t \rho^n \partial_t u^n, \partial_t u^n) = -(1/2)(u^n \cdot \nabla \rho^n \partial_t u^n, \partial_t u^n)$$

and bearing in mind Lemma 1, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left|\sqrt{\rho^n}\partial_t u^n\right|^2 + \|\partial_t u^n\|^2 &\le |(\nabla \cdot (\rho^n u^n) u^n \cdot \nabla u^n, \partial_t u^n)| \\ &\quad + |(\rho^n \partial_t u^n \cdot \nabla u^n, \partial_t u^n)| \\ &\le c|u^n|_\infty |\nabla u^n| \, \|u^n\|_{1,3} \|\partial_t u^n\| + c|\nabla u^n|^2 |Au^n| \, \|\partial_t u^n\| \\ &\quad + |\nabla u^n|^2 |Au^n| \, \|\partial_t u^n\| + c_\delta |Au^n|^2 \partial_t u^n|^2 + \delta \|\partial_t u^n\|^2 \\ &\le c_\delta(|\nabla u^n|^4 |Au^n|^2 + |\nabla u^n|^5 |Au^n| + \left|\sqrt{\rho^n}\partial_t u^n\right|^2 |Au^n|^2).\end{aligned} \tag{4.15}$$

Now from (4.9), we deduce that for each approximation u^n, there exists a number τ_n satisfying $\varepsilon < \tau_n < 2\varepsilon$ such that (ε is a positive number)

$$|\partial_t u^n(\tau_n)|^2 \le \varepsilon^{-1} G(\varepsilon, 0). \tag{4.16}$$

Thanks to Lemma 3 and (4.16), (4.15) implies

$$|\partial_t u^n| \le F_1(t, \varepsilon); \qquad \int_\varepsilon^t \|\partial_t u^n\|^2 dt \le \widetilde{F}_1(\varepsilon, t)$$

for $t \in [\varepsilon, T']$ where $F_1(\cdot, t)$, $\widetilde{F}_1(\cdot, t)$ are continuous functions of t, and T' is as in Theorem 1.

Now, if the initial velocity u_0 possesses second derivatives in $L^2(\Omega)$ i.e. $u_0 \in V \cap H^2$, then there exist continuous functions $F_1(t)$, $\widetilde{F}_1(t)$ such that

$$|\partial_t u^n| \le F_1(t); \qquad \int_0^t \|\partial_t u^n\|^2 d\tau \le \widetilde{F}_1(t) \tag{4.17}$$

for $t \in [0, T']$. The functions $F_1(t)$, $\widetilde{F}_1(t)$ depend on the functions $F(t)$, $\widetilde{F}(t)$ appearing in (4.7), and on u_0, $\rho(0)$.

In fact, first we note $|Au^n(0)| \le |Au_0|$. By (4.1) we have

$$\begin{aligned}|\partial_t u^n(0)|^2 &\le c_{\alpha,\Gamma}\|u_0^n\|(|Au_0^n|^{1/2}\|u_0^n\|^{1/2} + \|u_0^n\|)|\partial_t u^n(0)| + c_\alpha|\partial_t u^n(0)||Au^n(0)| \\ &\le c_{\alpha,\Gamma}|\partial_t u^n(0)|(|Au^n(0)|^2 + \|u^n(0)\|^2 + \|u^n(0)\|^3).\end{aligned}$$

Now we reconsider (4.14) in the following form

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left|\sqrt{\rho^n}\partial_t u^n\right|^2 + \|\partial_t u^n\|^2 &\le c_{\Gamma,\delta}|\nabla u^n|^4 |Au^n|^2 \\ &\quad + c_{\Gamma,\delta}|\nabla u^n|^5|Au^n| + c_{\Gamma,\delta}\left|\sqrt{\rho^n}\partial_t u^n\right|^2.\end{aligned} \tag{4.18}$$

Integrating (4.18) in the manner of Lemma 3, we have $\forall t \in [0, T']$

$$|\partial_t u^n| \le F_1(t); \qquad \int_0^t \|\partial_t u^n\|^2 d\tau \le F_1(t). \tag{4.19}$$

Multiplying (4.14) through by $\lambda_\ell c_\ell^n$ and summing on ℓ, we have

$$|Au^n|^2 = (\rho^n \partial_t u^n, Au^n) + (\rho^n u^n \cdot \nabla u^n, Au^n).$$

By

$$|\rho^n u^n \cdot \nabla u^n| \le \frac{1}{2}|Au^n| + c_\Gamma\|u^n\|^3 + c_\Gamma\|u^n\|,$$

we get

$$\frac{1}{2}|Au^n| \le c|\partial_t u^n| + c_\Gamma\|u^n\|^3 + c_\Gamma\|u^n\|^2$$

whence

$$|Au^n| \le G(t) \quad \text{for } t \in [0, T']. \tag{4.20}$$

Now let P_n and Q_n be the orthogonal projection of $L^2(\Omega)$ onto spans of $(a^1(x), \dots, a^n(x))$ and of $(a^{n+1}(x), a^{n+2}(x), \dots)$ respectively. For $\phi \in H$ let us write $\phi = P_n\phi + Q_n\phi = \phi^n + Q_n\phi$. Also for $\phi \in V$

$$(\nabla u^n, \nabla\phi) = (\nabla u^n, \nabla\phi^n).$$

Thanks to (4.1), we have, for any $\phi \in D(\Omega)$,

$$(\nabla u^n, \nabla\phi) = -(P_n(\rho^n\partial_t u^n + \rho^n u^n \cdot \nabla u^n), \phi) = (g^n, \phi) \tag{4.21}$$

with $g^n = -P_n(\rho^n\partial_t u^n + \rho^n u^n \cdot \nabla u^n)$. From (4.19) and (4.20), g^n belongs to $L^2(0, T'; L^6(\Omega))$, uniformly with respect to n. Whence, by Cattabriga's results (see [1]), one has

$$\int_0^t \|u^n\|_{2,6}^2 \le c\widetilde{G}(t). \tag{4.22}$$

The function $\widetilde{G}(t)$ depends on the functions $F_1(t)$ and $G(t)$ appearing in (4.19) and (4.20) respectively.

From Sobolev's embedding theorems, we have

$$\int_0^t \|\nabla u^n\|_{C(\Omega)}^2 d\tau \le cK(t) \tag{4.23}$$

where the function $K(t)$ depends on $\widetilde{G}(t)$.

From the continuity equation, we have (see [6])

$$\|\rho^n\|_{C^1(Q_{T'})} \le \|\rho_0\|_{C^1(\overline{\Omega})} e^{c\widetilde{K}(t)} \tag{4.24}$$

where, of course, the function $\widetilde{K}(t)$ depends on $K(t)$.

Now, the approximations (u^n, ρ^n) satisfy (4.19), (4.20), (4.22), ..., (4.24); a subsequence can be chosen, which we again denote by (u^n, ρ^n) so that u^n and $\partial_t u^n$ converge to u and $\partial_t u$ weakly in $L^2(0, T'; V \cap H_6^2)$ and $L^2(Q_{T'})$ respectively, and ρ^n converges to ρ weakly in $H^1(Q_{T'})$. So $u \in L^2(0, T; H_6^2 \cap V) \cap H^1(Q_{T'})$, and consequently $\rho \in C^1(Q_{T'})$ (see [6]). Now (u, ρ) satisfies

$$\begin{aligned} &\rho\partial_t u - \Delta u + \rho u \cdot \nabla u + \nabla p = 0 \quad \text{almost everywhere in } Q_{T'}; \\ &\partial_t u + u \cdot \nabla\rho = 0. \end{aligned} \tag{4.25}$$

Now, we consider the equation

$$\Delta u = \rho\partial_t u + \rho u \cdot \nabla u + \nabla p. \tag{4.26}$$

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Given any $\phi \in D(\Omega)$, we can multiply (4.26) for ϕ and integrate over Ω, we have $(\Delta u, \phi) = (g, \phi)$ where $g = \rho\partial_t u + \rho u \cdot \nabla u$. From (4.19), ... ,(4.24), we have $g \in L^2(0, T'; H^1)$.

Given any $\phi \in D(\Omega)$, one can mulitply (4.26) by $\partial_{x_i}\phi$ and integrate over $\Omega(\partial_{x_i}$ = derivative respect the spatial variable x_i). After integration by parts, the result is

$$(\nabla\partial_{x_i} u, \nabla\phi) = -(\partial_{x_i} g, \phi).$$

Now utilising the results of Cattabriga in [1], we have

$$u \in L^2(0, T; H^3 \cap V).$$

The function (u, ρ) belongs to the class of functions where the uniqueness is proved (see [6]). This completes the proof.

5. Succesive regularity and compatibility conditions at $t = 0$ and proof of Theorem 3

Now we shall study higher regularity properties of the solution of Theorem 2 assuming that the data u_0, ρ_0, f possess further regularity properties. It is known that, for an initial-boundary value problem for the Navier-Stokes equations, the solution may not be smooth near $t = 0$ even if the data are $C^\infty(\overline{\Omega})$. The breakdown of the regularity is due to the presence of the compatibility conditions (see [2]). Naturally, one expects that analogous considerations hold also for the system (1.1); in addition, it seems that we cannot avoid the compatibility conditions considering initial estimates in $t = \varepsilon$ as in [2]. This is due to the presence of the density.

Now we examine the cause of the breakdown of the regularity for the system (1.1) with the assumptions of Theorem 2. We shall show that the problem

$$\begin{aligned} \Delta p_0 - \frac{\Delta\rho_0}{\rho_0}\nabla p_0 &= \rho_0\nabla\cdot(f_0 - u_0\cdot\nabla u_0) - \frac{\nabla\rho_0}{\rho_0}\Delta u_0 \quad \text{in } \Omega \\ \partial_\nu p_0 &= (\Delta u_0 + \rho_0 f_0 - \rho_0 u_0\cdot\nabla u_0)\cdot\nu \quad \text{on } \Gamma \end{aligned} \tag{5.1}$$

determines the initial pressure p_0. In (4.1), ν is the outside unit normal to Γ.

It is shown in [12] that the mapping $u \to u\cdot\nu$ is defined and continuous from $\overline{H} = \{u | u \in L^2(\Omega).\ \nabla\cdot u \in L^2(\Omega)\}$ to $H^{-1/2}(\Gamma)(H^{-1/2}(\Gamma)$ is the dual of $H^{1/2}(\Gamma))$. Furthermore $\Delta u_0 \in H$ and $u_0\cdot\nabla u_0 \in H^1$. Then one has

$$(\Delta u_0 + \rho_0 f_0 - \rho_0 u_0\cdot\nabla u_0)\cdot\nu \in H^{-1/2}(\Gamma),$$

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and

$$\rho_0 \nabla \cdot (f_0 - u_0 \cdot \nabla u_0) - \frac{\nabla \rho_0}{\rho_0} \Delta u_0 \in L^2(\Omega).$$

We note that the compatibility conditions for (5.1) are satisfied i.e. the adjoint homogeneous problem of (5.1)

$$\begin{aligned} -\Delta \tilde{p}_0 - \nabla \cdot \left(\frac{\nabla \rho_0}{\rho_0} \tilde{p}_0 \right) = 0 \quad \text{in } \Omega, \\ \partial_\nu \tilde{p}_0 + \frac{\nabla p_0}{\rho_0} \tilde{p}_0 \nu = 0 \quad \text{on } \Gamma \end{aligned} \tag{5.2}$$

has an unique linearly independent solution: $\tilde{p}_0 = \frac{1}{\rho_0}$; then the compatibility conditions

$$\int_\Omega \left(\nabla \cdot (f_0 - u_0 \cdot \nabla u_0) - \frac{\nabla \rho_0}{\rho_0^2} \Delta u_0 \right) dx$$
$$= \int_\Gamma \frac{1}{\rho_0} (\Delta u_0 + \rho_0 f_0 - \rho_0 u_0 \cdot \nabla u_0) \nu \, d\Gamma$$

hold. From this, it follows that the problem (5.1) is uniquely solvable for a generalised solution $p_0 \in H^1/R$. Thanks to (4.24), we can prove the following proposition. In the following we shall denote T' by T.

Proposition 5.1. *Let $u_0 \in V \cap H^2$ and ρ_0, f, Γ be smooth enough. Let (u, ρ, p) the solution of (1.1) in the sense of Theorem 2. Then the pressure $p(x, t)$ tends to the solution p_0 (4.1) in the sense that*

$$|\nabla(p(t) - p_0)| \to 0 \quad \textit{as } t \to 0.$$

For the proof, by Theorem 2, we have $u \in C([0, T); H^2)$; then $u(t) \to u_0$ in $H^2(\Omega)$ as $t \to 0$ because $u(t) \to u_0$ at least weakly in $L^2(\Omega)$ from (2.6), (4.19), (4.24). Now, $p(x, t)$ is a generalised solution in H^1/R of

$$\begin{aligned} \Delta p - \frac{\nabla \rho}{\rho} \nabla p = \rho \nabla \cdot (f - u \cdot \nabla u) - \frac{\nabla \rho}{\rho} \Delta u \quad \text{in } \Omega, \\ \partial_\nu p = (\Delta u + \rho f - \rho u \cdot \nabla u) \cdot \nu \quad \text{on } \Gamma \end{aligned} \tag{5.4}$$

From what was said above, it is easily seen that

$$(\Delta u + \rho f - \rho u \cdot \nabla u) \nu \to (\Delta u_0 + \rho_0 f_0 - \rho_0 u_0 \cdot \nabla u_0) \cdot \nu$$

strongly in $H^{-1/2}(\Gamma)$, and

$$\rho \nabla \cdot (f - u \cdot \nabla u) + \frac{\nabla \rho}{\rho} \to \rho_0 \nabla \cdot (f_0 - u_0 \cdot \nabla u_0) + \frac{\nabla \rho_0}{\rho_0}$$

strongly in $L^2(\Omega)$ as $t \to 0$. This implies (5.4). The proof of the proposition is completed.

From the convergence of $u(\cdot, t)$ to u_0 and of $p(\cdot, t)$ to p_0, it follows that

$$\partial_t u(\cdot, t) \to \frac{\nabla p_0}{\rho_0} + \frac{\Delta u_0}{\rho_0} - u_0 \cdot \nabla u_0 + f_0 \tag{5.5}$$

strongly in $L^2(\Omega)$. This suggests that, generally, $\partial_t u(\cdot, 0) \neq 0$ on Γ; consequently $|\nabla \partial_t u| \to \infty$ as $t \to 0$, since the boundary condition implies $\partial_t u = 0$. This explains the cause for the breakdown of regularity in solutions without additional conditions on the data.

In [2], Heywood avoids this difficulty, for the Navier-Stokes equations, estimating the norms of the time derivatives of any order of the Galerkin approximation in $t = \varepsilon$, and considering as initial time $t = \varepsilon$. This procedure, for the system (1.1), is inhibited by the presence of the density ρ. In other words, to obtain estimates of time derivatives of any order of the Galerkin approximation u^n in $t = \varepsilon$, we need time derivatives estimates of ρ^n in $[0, \varepsilon]$ and these last depend on time derivatives estimates of u^n in $[0, \varepsilon]$. So, to obtain more regularity for (u, ρ), with respect to Theorem 2, we need compatibility conditions of the data. To prove this theorem we need, at every step, further regularity of u with respect to x to obtain more regularity of $\partial_t \rho$. For this reason we consider the equation (4.25) instead of (4.1). Furthermore, to avoid tedious calculations and notations, we work directly with the derivatives with respect to t of u instead of its differential quotients.

Proof of Theorem 3. Now we prove that, if $\partial^i u(0)/\partial t^i \in V$ $(i = 1, 2, \ldots, r)$, we have $u \in W_m$. First we prove that, if $u \in L^2(0, T; H^m(\Omega) \cap V)$, and $\partial^i u/\partial t^i \in L^2(0, T : H^{m-2i}(\Omega) \cap V)$ for $i = 1, 2, \ldots, [m/2]$, then

$$\rho \in L^\infty(0, T; H^m(\Omega)), \ \frac{\partial^i \rho}{\partial t^i} \in L^\infty(0, T; H^{m-2i+1}) \cap L^2(0, T; H^{m-2i+2}), \\ \frac{\partial^{i+1} \rho}{\partial t^{i+1}} \in L^2(0, T; H^{m-2i}). \tag{5.6}$$

In fact, applying the operator D_x^γ to the continuity equation

$$\partial_t \rho + u \cdot \nabla \rho = 0 \tag{5.7}$$

and multiplying through by $D_x^\gamma \rho$ and integrating over Ω, we obtain

$$\frac{1}{2}\frac{d}{dt}|D_x^\gamma \rho|^2 = (D_x^\gamma(u \cdot \nabla \rho), D_x^\gamma \rho)$$

(here $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is a multi-index with

$$|\gamma| \le m \quad \text{and} \quad D_x^\gamma = \partial^\gamma / \partial x_1^{\gamma_1} \partial x_2^{\gamma_2} \partial x_3^{\gamma_3}).$$

By adding for $|\gamma| \leq m$ and bearing in mind $(u \cdot \nabla D_x^\gamma \rho, D_x^\gamma \rho) = 0$, one gets

$$\frac{1}{2}\frac{d}{dt}||\rho||_m^2 \leq c||\rho||_m^2||u||_m.$$

Then, from Gronwall's lemma, we have $\rho \in L(0, T; H^m(\Omega)$. From (5.7), we have immediately $\partial_t \rho \in L^\infty(0, T; H^{m-1}(\Omega))$. Now differentiating with respect to t (5.7), we have

$$\frac{\partial^2 \rho}{\partial t^2} + u \cdot \nabla \partial_t \rho + \partial_t u \cdot \nabla \rho = 0.$$

Bearing in mind that

$$u \cdot \nabla \partial_t \rho, \partial_t u \cdot \nabla \rho \in L^\infty(0, T; H^{m-3}(\Omega)) \cap L^2(0, T; H^{m-2})$$

we have

$$\frac{\partial^2 \rho}{\partial t^2} \in L^\infty(0, T; H^{m-3}(\Omega)) \cap L^2(0, T; H^{m-2}).$$

Continuing, for every $i \in (3, 4, \dots, r)$, we have

$$\frac{\partial^i \rho}{\partial t^i} = \sum_{j=0}^{i-1} \binom{i-1}{j} \frac{\partial^j u}{\partial t^j} \cdot \nabla \frac{\partial^{i-j^{-1}} \rho}{\partial t^{i-j}}$$
$$\in L^\infty(0, T; H^{m-2i+1}(\Omega)) \cap L^2(0, T; H^{m-2i+2})$$

and, from (5.7), we get

$$\frac{\partial^{i+1} \rho}{\partial t^{i+1}} \in L^2(0, T; H^{m-2i}).$$

Now we pause to consider the derivatives of u. For $i = 1$, the equation

$$\partial_t \rho \partial_t u + \rho \partial_t(\partial_t u) + \partial_t(\rho u \cdot \nabla \rho) + \nabla \partial_t p - \Delta \partial_t u = \partial_t(\rho f) \tag{5.8}$$

together with $\partial_t u(0) \in V$ allows us to show that

$$\partial_t u \in L^2(0, T; H^2 \cap V) \cap C([0, T]; V); \qquad \partial_t \rho \in L^\infty(0, T; H^1).$$

In fact, we multiply (5.8) with $A\partial_t u$ in $L^2(\Omega)$ and obtain

$$\begin{aligned}(\partial_t \rho \partial_t u, A\partial_t u) + \left(\rho \frac{\partial^2 u}{\partial t^2}, A\partial_t u\right) + |A\partial_t u|^2 + (\partial_t \rho u \cdot \nabla u, A\partial_t u) \\ + (\rho \partial_t u \cdot \nabla u, A\partial_t u) + (\rho u \cdot \nabla \partial_t u, \partial_t u) = (\partial_t(\rho f), A\partial_t u).\end{aligned} \tag{5.9}$$

Now, multiplying (5.8) with $\frac{\partial^2 u}{\partial t^2}$ in $L^2(\Omega)$, we get

$$\begin{aligned}\left(\partial_t \rho \partial_t u, \frac{\partial^2 u}{\partial t^2}\right) + \left|\sqrt{\rho}\frac{\partial^2 u}{\partial t^2}\right|^2 + \frac{1}{2}\frac{d}{dt}||\partial_t u||^2 + \left(\partial_t \rho u \cdot \nabla u, \frac{\partial^2 u}{\partial t^2}\right) \\ + \left(\rho \partial_t u \cdot \nabla u, \frac{\partial^2 u}{\partial t^2}\right) + \left(\rho u \cdot \nabla \partial_t u, \frac{\partial^2 u}{\partial t^2}\right) = \left(\partial_t(\rho f), \frac{\partial^2 u}{\partial t^2}\right).\end{aligned} \tag{5.10}$$

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Adding (5.10) to (5.9) and after some calculations, we have

$$\left|\sqrt{\rho}\frac{\partial^2 u}{\partial t^2}\right|^2 + \frac{1}{2}\frac{d}{dt}\|\partial_t u\|^2 + |A\partial_t u|^2 + (\partial_t\rho\partial_t u\,,\,\partial_t(\partial_t u + Au))$$
$$= -(\partial_t\rho u\cdot\nabla u\partial_t(Au+\partial_t u)) + (\rho\partial_t u\cdot\nabla u\,,\,\partial_t(\partial_t u + Au))$$
$$+ (\rho u\cdot\nabla\partial_t u\,,\,\partial_t(\partial_t u + Au)) + (\partial_t(\rho f)\,,\,\partial_t(\partial_t u + Au))$$

which can be reduced to

$$\left|\sqrt{\rho}\frac{\partial^2 u}{\partial t^2}\right|^2 + \frac{1}{2}\frac{d}{dt}\|\partial_t u\|^2 + \delta|A\partial_t u|^2 \le c_\delta|\partial_t\rho\partial_t u|^2 + |\rho\partial_t u\cdot\nabla u|^2$$
$$+ c_\delta|\partial_t\rho u\cdot\nabla u|^2 + c_\delta|\rho u\cdot\nabla\partial_t u|^2.$$

Now, from Lemma 1, we have

$$|\partial_t\rho\partial_t u| \le |\partial_t\rho|_3^2\|\partial_t u\|^2\,,$$
$$|\rho\partial_t u\cdot\nabla u|^2 \le c_{2\Gamma}\|\partial_t u\|^2(|Au|^{1/2}\|u\|^{1/2} + \|u\|)^2.$$

Thanks to (4.19), (4.20), and (5.6) we have

$$\partial_t u \in L^2(0\,,\,T\,;\,H^2\cap V)\cap C([0\,,\,T]\,,\,V)\,, \qquad \frac{\partial^2 u}{\partial t^2} \in L^2(Q_T).$$

Now using the Cattabriga's results [1], one has

$$u \in L^2(0\,,\,T\,;\,H^4\cap V).$$

We continue by induction; once we establish

$$\frac{\partial^i u}{\partial t^i} \in L^2(0\,,\,T; H^2\cap V)\cap C([0\,,\,T];\,V) \quad \text{for } i = 1\,,\,2\,,\,\dots\,,\,r-1\,,$$
$$\frac{\partial^{i+1}u}{\partial t^{i+1}} \in L^2(Q_T)$$

we consider the equation

$$\frac{\partial^{r-1}(\rho\partial_t u)}{\partial t^{r-1}} - \Delta\frac{\partial^{r-1}}{\partial t^{r-1}} + \frac{\partial^{r-1}}{\partial t^{r-1}}(\rho u\cdot\nabla u) + \nabla\frac{\partial^{r-1}}{\partial^{r-1}} = \frac{\partial^{r-1}}{\partial t^{r-1}}(\rho f).$$

Now

$$\frac{\partial^{r-1}}{\partial t^{r-1}}(\rho u\cdot\nabla u) = \sum_{j=0}^{r-1}\sum_{k=0}^{j}\binom{r-1}{j}\binom{j}{k}\frac{\partial^{r-j-1}\rho}{\partial t^{r-j}}\,,$$
$$\frac{\partial^{j-k}u}{\partial t^{j-k}}\cdot\nabla\frac{\partial^k u}{\partial t^k} \in L^2(Q_T).$$

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Then, repeating the arguments used for (5.8), we obtain

$$\frac{\partial^r u}{\partial t^r} \in L^2(0, T; H^2 \cap V) \cap C(0, T; V).$$

Now, by Cattabriga's results [1], we have

$$\frac{\partial^{r-1} u}{\partial t^{r-1}} \in L^2(0, T; H^4 \cap V), \qquad \frac{\partial^{r-2} u}{\partial t^{r-2}} \in L^2(0, T; H^6 \cap V),$$

and so on up to $u \in L^2(0, T; H^m \cap V)$.

About the proof of the necessary condition, one can proceed, inductively on $\partial^i u(0)/\partial t^i$, applying at every step the arguments used in Proposition (5.1), and in the proof of (5.5). This completes the proof of the theorem.

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