FUNCTIONALS OF BOUNDED FRECHET VARIATION

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1. Introduction. In a series of papers which will follow this paper the authors will present a theory of functionals which are bilinear over a product $A \times B$ of two normed vector[1] spaces A and B. This theory will include a representation[2] theory, a variational theory, and a spectral theory. The associated characteristic equations will include as special cases the Jacobi equations of the classical variational theory when $n = 1$, and self-adjoint integro-differential equations of very general type. The bilinear theory is oriented by the needs of non-linear and non-bilinear analysis in the large.

The object of this paper is the proof of several preliminary but fundamental theorems on functionals k with values $k(s, t)$ defined over a rectangle

$$Q \qquad [a \leqq s \leqq a',\ b \leqq t \leqq b'] \tag{1.1}$$

of the (s, t)-plane. We shall require that the Fréchet[3] variation $P(k)$ (defined in **3**) be finite, and that for the given a and b,

$$k(a, t) = 0, \qquad k(s, b) = 0 \qquad \{t \in [b, b'],\ s \in [a, a']\}. \tag{1.2}$$

The condition (1.2) does not limit the useful generality of the theorems. The theorems to be proved are the analogues of simple theorems on a functional f with values $f(s)$ defined over an interval $[a, a']$ with a finite total variation $T(f)$.

To state these theorems let f^+ and f^- denote functionals defined by f over $[a, a']$ as follows:

$$\begin{aligned} f^+(s) &= \lim_{u \to 0} f(s+u), & [a < s < a', u > 0] \\ f^-(s) &= \lim_{u \to 0} f(s-u), & [a < s < a', u > 0] \\ f^+(a) &= f^-(a) = f(a), & \\ f^+(a') &= f^-(a') = f(a'). & \end{aligned}$$

The theorems to be extended follow:

I. The limits $f(s-)$ and $f(s+)$ exist for $a < s \leqq a'$ and $a \leqq s < a'$ respectively.

II. The points s at which f fails to be continuous are at most countably infinite.

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[1]See Banach, *Théorie des opérations linéaires* (Warsaw, 1932), chap. IV.

[2]Morse and Transue, "Functionals F Bilinear over the Produce $A \times B$ of Two Pseudo-normed Vector Spaces. I. The Representation of F," *Ann. of Math.* (To be published.)

[3]Fréchet, "Sur les fonctionnelles bilinéaires," *Trans. Amer. Math. Soc.*, vol. 16 (1915), 215-234. In this basic memoir Fréchet obtains a representation of any functional K which is bilinear on $C \times C$, in the form of a repeated Stieltjeas integral with a distribution function k of the above type.

III. The functionals f^+ and f^- are of bounded variation and

$$T(f^+) = T(f^-) \leqq T(f).$$

Fréchet has established [cf. **3**] the following for k. For fixed $s \epsilon [a, a']$, the functional on $[b, b']$ whose values are $k(s, t)$ will be denoted by $k(s, \cdot)$. The functional $k(\cdot, t)$ on $[a, a']$ with fixed t is similarly defined. According to Fréchet $T[k(s, \cdot)]$ and $T[k(\cdot, t)]$ are finite with $P(k)$, for fixed s and t respectively. Moreover

$$T[k(s, \cdot)] \leqq P(k), \qquad T[k(\cdot, t)] \leqq P(k). \tag{1.3}$$

The 2-dimensional extensions of I, II, III will be stated in **2**, and proved in the later sections.

2. The principal theorems. The statement of our theorems requires certain definitions. We shall use the notation f^+, f^- in the sense of **1**. For $s \epsilon [a, a']$ set $k(s, \cdot) = f_s$. For fixed s and $t \epsilon [b, b'], f_s^+$ is well defined. Set

$$f_s^+(t) = k^2(s, t), \qquad f_s^-(t) = k^{-2}(s, t) \qquad [(s, t) \epsilon Q].$$

With $k(\cdot, t) = \phi_t$, for fixed t set

$$\phi_t^+(s) = k^1(s, t), \qquad \phi_t^-(t) = k^{-1}(s, t) \qquad [(s, t) \epsilon Q].$$

Under our hypothesis that $P(k)$ is finite the functionals with values $k^\mu(s, t)$ are well defined over Q for $\mu = \pm 1, \pm 2$. Granting the result of Lemma 3.3 that

$$P(k^\mu) \leqq P(k)$$

and Fréchet's result (1.3), the functionals $(k^\mu)^\nu$ are likewise well defined over Q for $\mu, \nu = \pm 1, \pm 2$. As a matter of notation set $(k^\mu)^\nu = k^{\mu, \nu}$. At inner points (s, t) of Q, for $\mu = 1$, $\nu = 2$ for example,

$$k^{1,2}(s, t) = \lim_{v=0+}\left[\lim_{u=0+} k(s + u, t + v)\right].$$

In the (u, v)-plane let the four quadrants on which $uv \neq 0$ be designated as follows:

$H_{1,2}$	$[u > 0, v > 0]$,	1st quadrant,
$H_{-1,2}$	$[u < 0, v > 0]$,	2nd quadrant,
$H_{-1,-2}$	$[u < 0, v < 0]$,	3rd quadrant,
$H_{1,-2}$	$[u > 0, v < 0]$,	4th quadrant.

Thus the sign preceding a subscript 1 indicates the sign of u in a quadrant $H_{\mu,\nu}$, while the sign preceding a subscript 2 indicates the sign of v in a quadrant $H_{\mu,\nu}$. It will be convenient to let $H_{\nu,\mu}$ indicate the same quadrant as $H_{\mu,\nu}$ so that

$$H_{\mu,\nu} = H_{\nu,\mu} \qquad [\mu, \nu = \pm 1, \pm 2, |\mu| \neq |\nu|].$$

With this understood, let (s, t) be an inner point of Q and set

$$\lim_{(u,v)\to(0,0)} k(s + u, t + v) = k^{(\mu,\nu)}(s, t), \qquad [(u, v) \epsilon H_{\mu,\nu}] \tag{2.1}$$

granting the existence of the limit, as affirmed in Theorem 5.1. This is a limit of k at (s, t) on approaching (s, t) from a specified quadrant with vertex at (s, t). The relation (2.1) defines $k^{(\mu,\nu)}$ at *inner* points of Q. On the edges of Q, $k^{\mu,\nu}$ is well defined, and we shall complete the definition of $k^{(\mu,\nu)}$ by setting

(2.2) $$k^{(\mu,\nu)}(s, t) = k^{\mu,\nu}(s, t)$$

for (s, t) on any edge of Q.

In order to state the full analogue of I, **1**, we need to define an *extension* $\bar{k}$ of k over the whole (s, t)-plane. If L is any line parallel to the coordinate axes and intersecting Q, we require that $\bar{k}$ be constant on the closure of each of the two segments of L exterior to Q. On the closure of each of the four regions in the (s, t)-plane on which $\bar{k}$ is as yet undefined, we require that $\bar{k}$ be constant. These requirements on $\bar{k}$ are consistent as is readily seen.

The generalizations of I, II, and III of **1** can be stated now.

It is assumed throughout that the Fréchet variation $P(k)$ is finite and that k vanishes on the lower and left edges of Q.

THEOREM 2.1. *The limit functionals $k^{\mu,\nu}$ exist at each point of Q. The limits defined by* (2.1) *exist not only at each inner point of Q, but if k is extended over the (s, t)-plane as above, at each point of the (s, t)-plane.*

THEOREM 2.2. *The points of discontinuity of k can be covered by a countable number of lines parallel to the coordinate axes.*

THEOREM 2.3. *On Q, for $\mu, \nu = \pm 1, \pm 2$ with $|\mu| \neq |\nu|$,*

$$k^{(\mu,\nu)}(s, t) = k^{\mu,\nu}(s, t) = k^{\nu,\mu}(s, t),$$

while $P\,[k^{(\mu,\nu)}(s, t)]$ is independent of admissible μ, ν, and at most $P(k)$.

That $P(k)$ can be finite while the classical total variation $V(k)$ of Vitali is infinite has been shown by example by Clarkson and Adams.[4] The difficult problem which Clarkson and Adams solve corresponds to a general problem which we shall solve in the bilinear theory. In particular it is possible to replace the Clarkson and Adams k with its infinitely many points of discontinuity by a k which is absolutely continuous in each variable separately with $P(k)$ finite and $V(k)$ still infinite. Such an example is essential for the theory of functionals bilinear on $L^p \times L^q$, $p \geqq 1$, $q \geqq 1$.

It is easy to produce an example in which the limit functionals defined by k,

(2.3) $$k^{1,2},\ k^{-1,2},\ k^{-1,-2},\ k^{1,-2}$$

have four different values at some point of Q. One can, for example, suppose that $k(s, t) = 0$ on Q except on some rectangle Q_1 interior to Q. We admit only those rectangles whose edges are parallel to the coordinate axes. Let Q_1 be divided into four congruent rectangles by lines parallel to its edges. On these four rectangles let k have the values 1, 2, 3, 4. The assignment of the dividing lines to the four rectangles is immaterial. At the centre (s_0, t_0) of Q_1 the functionals (2. 3) take on the values 1, 2, 3, 4 in some order.

The fact that the Vitali variation $V(k)$ is in general infinite prohibits the

[4]Clarkson and Adams, "On Definitions of Bounded Variation for Functions of Two Variables," *Trans. Amer. Math. Soc.*, vol. 35 (1933), 824-854.

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use of the Fubini theorem to justify the interchange of the order of integration in the repeated Lebesgue Stieltjes integrals which we shall use in the representation theory. However the equality

$$k^{\mu,\nu}(s,t) = k^{\nu,\mu}(s,t) \qquad [(s,t) \in Q]$$

fills the gap, so that the desired interchange of order of integration will be seen to be permissible.

3. Definitions and first properties of *P(k)*. Let $[s, s']$ and $[t, t']$ be subintervals of $[a, a']$ and $[b, b']$ respectively. The "*mixed difference*" determined by k and the rectangle $[s, s'] \times [t, t']$ will be denoted by $[k: s, s': t, t']$. It is given by the equation

$$[k: s, s': t, t'] = k(s', t') - k(s', t) - k(s, t') + k(s, t).$$

Let $s_0, s_1, \ldots, s_n$ and $t_0, t_1, \ldots, t_p$ be values of s and t such that

$$\begin{aligned} a &= s_0 \leqq s_1 \leqq \ldots \leqq s_n = a', \\ b &= t_0 \leqq t_1 \leqq \ldots \leqq t_p = b'. \end{aligned} \tag{3.1}$$

With (3.1) *associate* numbers $e_1, \ldots, e_n$ and $e_1', \ldots, e_p'$ equal to ± 1. For $0 < i \leqq n$, and $0 < j \leqq p$, let

$$\Delta_{ij}(k) = [k : s_i, s_{i-1} : t_j, t_{j-1}]. \tag{3.2}$$

The subdivision of Q defined by the set of lines $s = s_i$ and $t = t_j$ will be called the *partition* π of Q defined by (3.1). Using the summation convention of tensor algebra, let

$$\sup e_i e_j' \Delta_{ij}(k) = P(k), \qquad [i = 1, \ldots, n;\ j = 1, \ldots, p] \tag{3.3}$$

taking the sup over all partitions of Q and associated sets (e) and (e'). We shall term $P(k)$ the *Fréchet variation* of k. We are assuming that $P(k)$ is finite.

The following lemma is due to Fréchet.

LEMMA 3.1. *If $P(k)$ is finite, then for fixed s and t respectively on $[a, a']$ and $[b, b']$*

$$T[k(s, \cdot)] \leqq P(k), \qquad T[k(\cdot, t)] \leqq P(k). \tag{3.4}$$

Using the partition (3.1) set

$$k(s_i, \cdot) - k(s_{i-1}, \cdot) = \Delta_i^1(k), \qquad [i = 1, \ldots, n]$$

and observe[5] that

$$P(k) \geqq \sup T\,[e_i \Delta_i^1(k)], \tag{3.5}$$

where the sup is taken over all partitions of $[a, a']$ and corresponding sets (e) with $e_i = \pm 1$. Since the functional in the bracket [ ] in (3.5) has a null value when $t = b$, it follows that for each t in $[b, b']$, on setting $\Delta_i^1\, k(\cdot, t) = k(s_i \cdot, t) - k(s_{i-1}, t)$,

$$\left| e_i \Delta_i^1\, k(\cdot, t) \right| \leqq T[e_i \Delta_i^1 (k)] \leqq P(k).$$

Hence the second inequality in (3.4) holds. Interchanging the roles of s and t the first inequality in (3.4) follows similarly.

[5]The equality holds in (3.5).

Lower semi-continuity. (Written l.s.c.) Let G be the class of all functionals which map Q into R_1, the space of real numbers. Convergence of a sequence of elements $g^{(n)} \epsilon G$ to $g \epsilon G$ shall mean pointwise convergence of $g^{(n)}$ to g, that is that

$$\lim_{n\to\infty} g^{(n)}(s, t) = g(s, t) \qquad [(s, t) \epsilon Q].$$

Let R_1, extended by adding the value $+\infty$, be denoted by R_1^+. Let F map G into R_1^+. The value of F at g is denoted by $F(g)$. By definition F is l.s.c. if, whenever $g^{(n)} \to g$ in G,

$$\liminf_{n\to\infty} F[g^{(n)}] \geqq F(g).$$

A sufficient condition that F be l.s.c. is given in the following.

(A). *For each element a in a range (a) let F_a be a l.s.c. map of G into R_1. If for every g in G,*

$$F(g) = \sup_{(a)} F_a(g),$$

then F is l.s.c. over G. Cf. McShane.[6]

LEMMA 3.2. *The Fréchet variation P is l.s.c. over G.*

To prove this lemma we apply (A). To this end let a symbolize any partition (3.1) of Q together with associated sets

$$(e) = (e_1, \ldots, e_{n_a}), \qquad (e') = (e'_1, \ldots, e'_{p_a}).$$

For this partition and sets (e) and (e') set

$$F_a(k) = \left| e_i e'_j \Delta_{ij}(k) \right| \qquad [i = 1, \ldots, n_a;\ j = 1, \ldots, p_a].$$

It is clear that for a fixed a, F_a is l.s.c. over G, in fact continuous. Since

$$P(k) = \sup_{(a)} F_a(k)$$

by definition of $P(k)$, the lemma follows from (A).

LEMMA 3.3. $P(k^\mu) \leqq P(k)$ *for* $\mu = \pm 1, \pm 2$.

Since $P(k)$ is finite the limit functional k^μ exists by Lemma 3.1. To continue we suppose $\mu = 1$.

For each positive integer n, let ϕ_n be a homeomorphic mapping of $[a, a']$ onto $[a, a']$ leaving $s = a$ and $s = a'$ invariant, and such that

$$0 < \phi_n(s) - s < \frac{1}{n} \qquad [a < s < a'].$$

Such a mapping is seen to exist. In terms of k set

(3.6) $$k[\phi_n(s), t] = k_n(s, t) \qquad [n = 1, 2, \ldots].$$

It is clear that

(3.7) $$\lim_{n\to\infty} k_n(s, t) = k^1(s, t).$$

For each n however

(3.8) $$P(k) = P(k_n).$$

[6] McShane, *Integration* (Princeton, 1944), 41.

To establish (3.8) let π be a partition of Q given by (3.1) with (e) and (e') corresponding sets and Δ_{ij}^{π} the corresponding mixed difference operator. Let π_n be a partition in which s_i in (3.1) is replaced by $\phi_n(s_i)$ and t_j remains unchanged, while $\Delta_{ij}^{\pi_n}$ is the corresponding mixed difference operator. With k_n defined by (3.6) it is clear that

$$e_i\, e_j'\Delta_{ij}^{\pi}(k) = e_i\, e_j'\Delta_{ij}^{\pi_n}(k_n).$$

Relation (3.8) follows.

Since $k_n \to k^1$ pointwise, and $P(g)$ is l.s.c.

$$\liminf_{n\to\infty} P(k_n) \geqq P(k^1).$$

From (3.8) therefore, $P(k) \geqq P(k^1)$.

The cases in which $\mu = -1$, or ± 2 are similar.

THEOREM 3.1. *The limit functionals $k^{\mu,\nu}$ exist for $\mu, \nu = \pm 1, \pm 2$, and $P(k^{\mu,\nu}) \leqq P(k)$.*

When $P(k)$ is finite k^{μ} exists by Lemma 3.1, and $P(k^{\mu}) \leqq P(k)$ by Lemma 3.3. Hence by Lemma 3.1, $(k^{\mu})^{\nu} = k^{\mu,\nu}$ exists, and by Lemma 3.3

$$P(k^{\mu,\nu}) \leqq P(k).$$

A second definition of $P(k)$. Corresponding to the partition (3.1) of Q, let

$$(\eta_1, \ldots, \eta_n), \quad (\eta_1', \ldots, \eta_p')$$

be sets of constants at most 1 in absolute value, in particular admitting the value 0. Taking the sup over all partitions (3.1) of Q and corresponding sets (η) and (η'), set

$$P'(k) = \sup \eta_i\eta'_j\Delta_{ij}(k) \quad [i = 1, \ldots, n;\, j = 1, \ldots, p]. \tag{3.9}$$

LEMMA 3.4. $P(k) = P'(k)$.

It is immediately clear that $P(k) \leqq P'(k)$. It remains to show that $P'(k) \leqq P(k)$. To that end set

$$e_i = \text{sign}\,[\eta_j'\Delta_{ij}(k)]\,.$$

It then appears that for a partition (3.1) of Q

$$\eta_i\eta_j'\Delta_{ij}(k) \leqq e_i\eta_j'\Delta_{ij}(k).$$

Similarly if

$$e_j' = \text{sign}\,[e_i\Delta_{ij}(k)]\,,$$

$$e_i\eta_j'\Delta_{ij}(k) \leqq e_ie_j'\Delta_{ij}(k) \leqq P(k)$$

from which it follows that $P'(k) \leqq P(k)$.

4. A fundamental lemma. Let (σ, τ) be a point interior to Q. Let

$$\begin{gathered} a < s_1 < s_1' < s_2 < s_2' < \ldots \\ b < t_1 < t_1' < t_2 < t_2' < \ldots \end{gathered} \tag{4.1}$$

be increasing sequences of values of s and t respectively converging to σ and τ. Let r and m be arbitrary positive integers. Denote the mixed difference determined by k and the rectangle $[s_r, s_r'] \times [t_m, t_m']$ by $[k: r, m]$.

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Let δ be a positive constant. With $[s_r, s'_r]$ fixed, for $\tau - t_m$ sufficiently small and for $t_m < t'_m < \tau$,

(4.2) $$[k: r, m] < \delta.$$

This follows from the existence of the limits

$$k(s_r, \tau-), \quad k(s'_r, \tau-),$$

recalling that these limits exist since $P(k)$ is finite. Similarly, for fixed $[t_m, t'_m]$, for $\sigma - s_r$ sufficiently small, and for $s_r < s'_r < \sigma$, (4.2) again holds.

Let (s, t) and (s', t') be points of Q for which

(4.3) $$a < s < s' < \sigma, \quad b < t < t' < \tau,$$

and as previously let $[k: s, s': t, t']$ be the mixed difference determined by k and the rectangle $[s, s'] \times [t, t']$. The following lemma is fundamental.

LEMMA 4.1. *Lim* $[k: s, s': t, t'] = 0$ *as* $(s, t) \to (\sigma, \tau)$ *with* (4.3.) *holding.*

Given $e > 0$, the lemma affirms in effect that for $\sigma - s$ and $\tau - t$ sufficiently small and positive, and with (4.3) holding

(4.4) $$|[k: s, s': t, t']| \leqq e.$$

Let e_{rm} be a double sequence of positive constants such that

$$\Sigma\, e_{rm} < e \qquad [r, m = 1, 2, \ldots].$$

Assume the lemma false. Then for some $e > 0$, (4.4) will fail to hold subject to (4.3), for $\sigma - s$ and $\tau - t$ sufficiently small. One can accordingly choose the numbers

(4.5) $$[s_m, s'_m, t_m, t'_m]$$

in the order of increasing m so that (4.1) holds, so that $s_m \to \sigma$, $t_m \to \tau$, and so that the following conditions are successively satisfied. For $m = 1$, the set (4.5) is to be such that

$$|[k: 1, 1]| > e.$$

For $m = 2$, the set (4.5) is to be such that the order (4.1) holds and

(4.6) $$|[k: 2, 2]| > e; \quad |[k: 1, 2]| < e_{12}; \quad |[k: 2, 1]| < e_{21}.$$

The last two conditions will be fulfilled if $\sigma - s_2$ and $\tau - t_2$ are sufficiently small and positive, as has been remarked in connection with (4.2). For a general m the set (4.5) is to be such that (4.1) holds,

$$|[k: m, m]| > e,$$

and for $r = 1, 2, \ldots, m-1$,

$$|[k: r, m]| < e_{rm}, \quad |[k: m, r]| < e_{mr}.$$

A partition $\pi^{(m)}$ of Q will be defined now by the points of division

(4.7) $$a = s_0 < s_1 < s'_1 < s_2 < s'_2 < \ldots < s_m < s'_m < a',$$

(4.8) $$b = t_0 < t_1 < t'_1 < t_2 < t'_2 < \ldots < t_m < t'_m < b'.$$

To the $2m + 1$ subintervals into which $[a, a']$ and $[b, b']$ are thereby divided, constants

$$[\eta_1, \ldots, \eta_{2m+1}], \quad [\eta'_1, \ldots, \eta'_{2m+1}]$$

with $|\eta_i| \leqq 1$, $|\eta'_j| \leqq 1$, will be respectively assigned. In accordance with the definition (3.9) of $P'(k) = P(k)$,

$$P(k) \geqq \eta_i \eta'_j \Delta_{ij}(k) \qquad [i, j = 1, 2, \ldots, 2m + 1]$$

for the partition $\pi^{(m)}$ and sets (η) and (η'). We set

$$(\eta) = [0, \operatorname{sign}[k: 1, 1], 0, \operatorname{sign}[k: 2, 2], \ldots, \operatorname{sign}[k: m, m], 0],$$

$$(\eta') = [0, 1, 0, 1, \ldots, 1, 0].$$

For the partition $\pi^{(m)}$, and above choice of (η) and (η')

$$\eta_i \eta'_j \Delta_{ij}(k) = \Sigma \eta_i \eta'_i \Delta_{ii}(k) + \sum_{i \neq j} \eta_i \eta'_j \Delta_{ij}(k)$$
$$= \sum_{r=1}^{m} |[k: r, r]| + R,$$

where

$$|R| < \Sigma e_{rp} < e, \qquad [r, p = 1, \ldots, m]$$

so that

$$\eta_i \eta'_j \Delta_{ij}(k) > me - e.$$

We conclude that $P(k) > (m - 1)e$. Since m is an arbitrary integer $P(k) = \infty$. From this contradiction we infer the truth of the lemma.

5. The limit functions $k^{(\mu, \nu)}$. Given values of s and t

$$\text{(5.1)} \qquad \begin{aligned} a &< s_1 < s_2 < s_3 < a', \\ b &< t_1 < t_2 < t_3 < b', \end{aligned}$$

we shall establish the formula

$$\text{(5.2)} \qquad \begin{aligned} [k: a, s_3: b, t_3] &= [k: a, s_2: b, t_2] + [k: a, s_1: t_2, t_3] \\ &+ [k: s_2, s_3: b, t_1] + [k: s_1, s_3: t_1, t_3] \\ &- [k: s_1, s_2: t_1, t_2]. \end{aligned}$$

This formula results from the fact that mixed differences are additive functions of intervals (2-dimensional).

To establish (5.2) we first represent the rectangle $[a, s_2] \times [b, t_2]$ as the union of four non-overlapping rectangles,

$$\begin{aligned} [a, s_2] \times [b, t_2] &= [a, s_1] \times [b, t_1] + [a, s_1] \times [t_1, t_2] \\ &+ [s_1, s_2] \times [b, t_1] + [s_1, s_2] \times [t_1, t_2], \end{aligned}$$

corresponding to which one has the relation,

$$\text{(5.3)} \qquad \begin{aligned} [k: a, s_2: b, t_2] &= [k: a, s_1: b, t_1] + [k: a, s_1: t_1, t_2] \\ &= [k: s_1, s_2: b, t_1] + [k: s_1, s_2: t_1, t_2]. \end{aligned}$$

A decomposition of $[a, s_3] \times [b, t_3]$ gives the relation

$$\text{(5.4)} \qquad \begin{aligned} [k: a, s_3: b, t_3] &= [k: a, s_1: b, t_1] + [k: a, s_1: t_1, t_2] \\ &+ [k: a, s_1: t_2, t_3] + [k: s_1, s_2: b, t_1] \\ &+ [k: s_2, s_3: b, t_1] + [k: s_1, s_3: t_1, t_3]. \end{aligned}$$

If one adds $[k: s_1, s_2: t_1, t_2]$ to both sides of (5.4), and makes use of (5.3) in simplifying the resulting right member, one has

$$\begin{aligned} &[k: a, s_3: b, t_3] + [k: s_1, s_2: t_1, t_2] \\ &= [k: a, s_2: b, t_2] + [k: a, s_1: t_2, t_3] \\ &+ [k: s_2, s_3: b, t_1] + [k: s_1, s_3: t_1, t_3]. \end{aligned} \tag{5.5}$$

Relation (5.5) differs from (5.2) only in the transposition of one term.

Lemma 5.1. *At each interior point* (σ, τ) *of* Q

$$\lim_{(u,v)\to(0,0)} k(\sigma - u, \tau - v) \qquad [u > 0, v > 0]$$

exists.

To prove the lemma the values appearing in (5.1) will be taken so that

$$\begin{aligned} a &< s_1 < s_2 < s_3 < \sigma, \\ b &< t_1 < t_2 < t_3 < \tau. \end{aligned} \tag{5.6}$$

Let e be a positive constant. Referring to (5.2), choose s_1 and t_1 respectively so near σ and τ, say

$$0 < \sigma - s_1 < \delta_1, \quad 0 < \tau - t_1 < \delta_1,$$

that

$$|[k: s_1, s_3: t_1, t_3]| < e, \quad |[k: s_1, s_2: t_1, t_2]| < e. \tag{5.7}$$

This is possible by virtue of Lemma 4.1. With (s_1, t_1) so chosen and held fast, choose s_2 and t_2 respectively so near σ and τ, say

$$0 < \sigma - s_2 < \delta_2, \quad 0 < \tau - t_2 < \delta_2,$$

that

$$|[k: a, s_1: t_2, t_3]| < e, \quad |[k: s_2, s_3: b, t_1]| < e. \tag{5.8}$$

This is possible because of the existence of the limits $k(s, \tau-)$ and $k(\sigma-, t)$ respectively. With these choices of s_1, s_2, t_1, t_2 it follows from (5.2) that

$$|[k: a, s_3: b, t_3] - [k: a, s_2: b, t_2]| < 4e. \tag{5.9}$$

Recall that $[k: a, s: b, t] = k(s, t)$ since k vanishes on the lower and left edges of Q. It follows from (5.9) that if

$$\begin{aligned} 0 < \sigma - s < \delta_2, &\quad 0 < \sigma - s' < \delta_2, \\ 0 < \tau - t < \delta_2, &\quad 0 < \tau - t' < \delta_2, \end{aligned}$$

then

$$|k(s, t) - k(s', t')| < 8e. \tag{5.10}$$

The lemma is a ready consequence of (5.10).

Lemma 5.2. *At each interior point* (σ, τ) *of* Q

$$\lim_{(u,v)\to(0,0)} k(\sigma + u, \tau + v) \qquad [u > 0, v > 0]$$

exists.

Lemma 5.2 can be deduced from Lemma 5.1. To that end make a transformation

$$s - a = a' - s', \quad t - b = b' - t' \tag{5.11}$$

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of Q onto Q, interchanging the upper and lower, and the right and left edges of Q. Under (5.11) set

$$k(s, t) = h(s', t').$$

We have $P(k) = P(h)$, but cannot apply Lemma 5.1 to h since $h(a, t')$ and $h(s', b)$ are not necessarily zero for $t' \epsilon [b, b']$ and $s' \epsilon [a, a']$. We accordingly introduce the functional g with values

$$g(s', t') = h(s', t') - h(a, t') - h(s', b) + h(a, b),$$

for which $g(a, t') = g(s', b) = 0$ as desired. Observe also that $P(g) = P(h)$.

If then (σ', τ') is any interior point of Q, it follows from the preceding lemma that the

$$\lim_{(u,v) \to (0,0)} g(\sigma' - u, \tau' - v) \qquad [u > 0, v > 0] \tag{5.12}$$

exists. Since the limits $h(a, \tau-)$ and $h(\sigma-, b)$ exist, it follows from (5.12) that a limit similar to (5.12) exists for h. But for the image (σ, τ) of (σ', τ') under (5.11),

$$\lim_{(u,v) \to (0,0)} h(\sigma' - u, \tau' - v) = \lim_{(u,v) \to (0,0)} k(\sigma + u, \tau + v) \qquad [u > 0, v > 0]$$

and the proof of the lemma is complete.

On using the transformation

$$s - a = a' - s', \quad t = t',$$

the existence of the limit

$$\lim_{(u,v) \to (0,0)} k(\sigma + u, \tau - v) \qquad [u > 0, v > 0]$$

is similarly deduced from Lemma 5.1. The transformation

$$s = s', \qquad t - b = b' - t',$$

is similarly used to establish the limit

$$\lim_{(u,v) \to (0,0)} k(\sigma - u, \tau + v) \qquad [u > 0, v > 0].$$

The limits defined in (2.1) have thus been proved to exist at each interior point of Q.

If moreover k is given an extension $\bar{k}$ over the whole (s, t)-plane as in **2**, the limits defined in (2.1) exist at each point of the plane. To see this, observe that the mixed difference determined by $\bar{k}$ and any admissible rectangle X (with edges parallel to the coordinate axes) vanishes when the interior of X is on the complement of Q. Hence if Q' is any admissible rectangle which contains Q in its interior, the Fréchet variation of $\bar{k}$ over Q' equals $P(k)$ taken over Q. The edges of Q are interior to Q' so that the limits (2.1) taken for $\bar{k}$ exist at points on these edges. We thus have the theorem.

THEOREM 5.1. *The limits* (2.1), *evaluated for the extension* $\bar{k}$ *of* k *over the* (s, t)*-plane, exist at each point of the* (s, t)*-plane.*

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6. The derived theorems. We begin with the lemma.

LEMMA 6.1. *At each interior point (s, t) of Q*

$$k^{(\mu,\nu)}(s,t) = k^{\mu,\nu}(s,t) \qquad [\mu, \nu = \pm 1, \pm 2, |\mu| \neq |\nu|]. \tag{6.0}$$

The proof for the case $\mu = 1, \nu = 2$ is typical. We begin with the inequality

$$\begin{aligned} |k^{(1,2)}(s,t) - k^{1,2}(s,t)| \leqq\ & |k^{(1,2)}(s,t) - k(s+u, t+v)| \qquad [u > 0, v > 0] \\ &+ |k(s+u, t+v) - k^1(s, t+v)| \\ &+ |k^1(s, t+v) - k^{1,2}(s,t)|. \end{aligned} \tag{6.1}$$

Let e be a positive constant. Since $k(s + u, t + v) \to k^{(1,2)}(s, t)$ as $u \to 0+$, $v \to 0+$, there exists a $\delta < 0$ such that the first term on the right of (6.1) is at most e, provided

$$0 < u < \delta, \quad 0 < v < \delta.$$

Since $k^1(s, t + v) \to k^{1,2}(s, t)$ as $v \to 0+$, there exists a δ_2 with $0 < \delta_2 < \delta$, such that when $0 < v < \delta_2$ the third term on the right of (6.1) is at most e. Fix v with $0 < v < \delta_2$; there will then exist a δ_1 with $0 < \delta_1 < \delta$ such that the second term on the right is at most e when $0 < u < \delta_1$. For this choice of v and u, (6.1) shows that

$$|k^{(1,2)}(s,t) - k^{1,2}(s,t)| < 3e. \tag{6.2}$$

Since the left member of (6.2) is independent of (u, v)

$$k^{(1,2)}(s,t) = k^{1,2}(s,t).$$

The lemma follows similarly for other values of μ and ν.

Since the relation (6.0) holds on the edges of Q by definition of $k^{(\mu,\nu)}$ in (2.2), the following theorem is established.

THEOREM 6.1. *At each point of Q*

$$k^{(\mu,\nu)}(s,t) = k^{\mu,\nu}(s,t) \qquad [\mu, \nu = \pm 1, \pm 2, |\mu| \neq |\nu|]. \tag{6.3}$$

LEMMA 6.2. *On the edges of Q*

$$k^{\mu,\nu}(s,t) = k^{\nu,\mu}(s,t) \qquad |\mu| \neq |\nu|. \tag{6.4}$$

The existence of the limit functions follows from Lemmas 3.1 and 3.3.

For points on the lower and left edges of Q

$$0 = k(s,t) = k^\mu(s,t) = k^\nu(s,t) = k^{\mu,\nu}(s,t) = k^{\nu,\mu}(s,t).$$

On the right and upper edges of Q respectively

$$\begin{aligned} k^\mu(a', t) &= k(a', t), \qquad & |\mu| = 1, \\ k^\mu(s, b') &= k(s, b'), \qquad & |\mu| = 2, \end{aligned}$$

by definition of k^μ in **2**. Applying these relations one finds for $|\mu| \neq |\nu|$, that

$$k^{\mu,\nu}(a', t) = k^\nu(a', t), \qquad |\nu| = 2,$$

since $|\mu|$ is then 1, and $k^\mu(a', t) = k(a', t)$.

Similarly

$$\begin{aligned} k^{\mu,\nu}(a', t) &= k^\mu(a', t), \qquad & |\mu| = 2, \\ k^{\mu,\nu}(s, b') &= k^\nu(s, b'), \qquad & |\nu| = 1, \\ k^{\mu,\nu}(s, b') &= k^\mu(s, b'), \qquad & |\mu| = 1. \end{aligned}$$

The last four relations establish (6.4) on the right and upper edges of Q.

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THEOREM 6.2. *At each point* (s, t) *of* Q

$$k^{\mu,\nu}(s, t) = k^{\nu,\mu}(s, t) \qquad [\mu, \nu = \pm 1, \pm 2, |\mu| \neq |\nu|]. \tag{6.5}$$

At each interior point (s, t) of Q

$$k^{(\mu,\nu)}(s, t) = k^{(\nu,\mu)}(s, t)$$

by definition of $k^{(\mu,\nu)}$ and $k^{(\nu,\mu)}$ in **2**. At such a point of Q, (6.5) accordingly follows from (6.3). On the edges of Q, (6.5) follows from Lemma 6.2.

THEOREM 6.3. *The points at which k fails to be continuous can be covered by a countable set of lines parallel to the coordinate axes.*

Given $e > 0$, let S_e be a set of points p of Q with the following properties. At p, k shall be discontinuous with a jump exceeding e, and no two points of S_e shall have the same s or t coordinate. The maximum number N_e of points in such sets S_e is finite. Otherwise there would exist a set S_e containing an infinite set of points p, and these points would have some point (σ, τ) of Q as a limit point. In one at least of the four open quadrants defined by the lines $s = \sigma$, $t = \tau$, there would be an infinite subset of points of S_e. This is impossible if the limits $k^{(\mu,\nu)}(\sigma, \tau)$ exist. Hence N_e is finite.

A countable set Ω of lines parallel to the coordinate axes on which all points of discontinuity of k lie can be enumerated as follows.

With $e = 1$, let S_1 be a set of points of Q defined as above with S_1 maximal in that S_1 contains just N_1 points. In Ω take first the lines $s = s_1$, $t = t_1$ determined by points (s_1, t_1) of S_1. Let $S_{.1}$ be a second maximal set. To Ω now add all lines $s = s_2$, $t = t_2$ corresponding to points of $S_{.1}$ at which the jump of k does not exceed 1. Let $S_{.01}$ be a third maximal set, and add to Ω all lines $s = s_3$, $t = t_3$ corresponding to points (s_3, t_3) of $S_{.01}$ at which the jump of k does not exceed .1, etc. The resulting set of lines is at most countable and will cover the points of discontinuity of k.

THEOREM 6.4. *The value of*

$$P[k^{(\mu,\nu)}] \qquad [\mu, \nu = \pm 1, \pm 2, |\mu| \neq |\nu|] \tag{6.6}$$

is independent of μ and ν, and at most $P(k)$.

We begin by proving Lemma 6.3.

LEMMA 6.3. *Let (μ, ν) be a pair admitted in* (6.6) *and (μ', ν') a second such pair. On setting $k^{(\mu',\nu')} = h$ one has the relation*

$$h^{(\mu,\nu)} = k^{(\mu,\nu)}. \tag{6.7}$$

In accordance with the preceding theorem the points at which h and k are continuous and equal are everywhere dense on G. Hence (6.7) holds at each interior point of G. On the lower and left edges of G both members of (6.7) vanish and so are equal. The proof that (6.7) holds on the edge of G on which $s = a'$ is as follows.

Of the members μ, ν one, say σ, is ± 2, and of the members μ', ν' one, say τ, is ± 2. In accordance with the equalities used in proving Lemma 6.2,

$$k^{\mu',\nu'}(a', t) = k^{\tau}(a', t) \quad h^{\mu,\nu}(a', t) = h^{\sigma}(a', t).$$

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Using these relations and Lemma 6.1

$$\begin{aligned} h^{\mu,\nu}(a', t) &= h^{\sigma}(a', t) = [k^{\mu',\nu'}(a', t)]^{\sigma} \\ &= [k^{\tau}(a', t)]^{\sigma} = k^{\sigma}(a', t) = k^{\mu,\nu}(a', t), \end{aligned}$$

from which (6.7) follows on the right edge of Q. The validity of (6.7) on the upper edge of Q is similarly proved.

Proof of Theorem 6.4. Set

$$k^{(\mu',\nu')} = h \qquad k^{(\mu,\nu)} = g.$$

In accordance with (6.7), $h^{(\mu,\nu)} = g$ so that it follows from Theorem 6.1 and Theorem 3.1 that

$$P(h) \leqq P(g) \leqq P(k).$$

Similarly $g^{(\mu',\nu')} = h$ so that

$$P(g) \leqq P(h) \leqq P(k).$$

Hence $$P(h) = P(g) \leqq P(k).$$

This completes the proof of the theorem.

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