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Infinite interval exchange transformations from shifts

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Abstract. We show that minimal shifts with zero topological entropy are topologically conjugate to interval exchange transformations, which are generally infinite. When these shifts have linear factor complexity (linear block growth), the conjugate interval exchanges are proved to satisfy strong finiteness properties.

1. *Introduction*

Interval exchange transformations (IETs) are maps over $I = [0, 1)$ which can be defined as permutations of intervals partitioning I. In the finite case, these maps are just piecewise isometries of I onto itself, or equivalently, injective measure-preserving maps having only a finite number of discontinuities. They happen to be a fundamental notion in dynamical systems and ergodic theory [**CFS82**, **Mañ87**, **HK02**]. Another important notion in the same context is the shift σ on the set $A^{\mathbb{N}}$ of infinite words (symbolic sequences) over a finite alphabet A. This simple continuous map consists of erasing the first letter of its argument. The pair $(A^{\mathbb{N}}, \sigma)$ forms a basic topological dynamical system, where $A^{\mathbb{N}}$ is then called the *full shift* over A. If L is a closed σ-invariant subset of $A^{\mathbb{N}}$, the pair (L, σ) also induces such a system, where L is then just called a *shift* [**LM95**, **Kit98**]. The topological entropy of a shift L depends on the factor complexity (block growth) of L [**MH38**, **Par66**, **CN10**], i.e. the map p_L on $\mathbb{N}^*$ giving for each n the number of distinct length-n factors (subblocks) occurring in the words of L. Now, a known general relationship between all the above concepts is the following: the support I of a finite IET can be embedded as a subset into a measured compact space in such a way that the IET extends to a measure-preserving continuous map, whose natural symbolic conjugate is a shift L with a factor complexity bounded by an affine function (thus L has zero topological entropy) [**Kea75**]. The main idea of this paper is to study this relationship the other way around, that is,

starting from shifts to build topologically conjugate IETs, generally with infinitely many discontinuities, and using factor complexity to determine families of these IETs.

Thus, given a shift L, we first describe an ordered measured compact space X_L containing I and a continuous self-map f_L on X_L in such a way that (L, σ) is topologically conjugate to (X_L, f_L). Next, we show how f_L can be seen as an extension over X_L of an IET T_L over I. We shall mostly consider aperiodic minimal shifts, i.e. shifts with no periodic word and containing no proper shift. With this respect, the first main result we prove is the following.

THEOREM 1. *Let L be a measured minimal aperiodic shift with zero topological entropy. Then (L, σ) is topologically conjugate to (X_L, f_L), where f_L is the extension over X_L of an IET T_L on $I \subset X_L$.*

This result is to be put into perspective with the fact that aperiodic measure-preserving transformations of a Lebesgue space are known to be isomorphic to infinite IETs [**AOW85**]. Besides the fact it defines a tight relationship between shifts and transformations over $[0, 1)$, Theorem 1 is about topological isomorphisms—i.e. homeomorphisms— and not only about isomorphisms—i.e. measure-preserving maps. Another comparison point is that the conjugate IETs given by Theorem 1 can have up to a null measure infinite set of discontinuities, while being always piecewise increasing. Also, the construction behind these conjugate maps has the following consistency property: starting with a piecewise increasing IET T, and coding its dynamics after a finite monotonicity partition of I, a shift is obtained to which Theorem 1 applies, yielding an IET coinciding with T (see Proposition 3.3.3 further).

The paper focuses next on aperiodic minimal shifts L with linear complexity [**Fog02**, **CN10**], i.e. $p_L(n) = O(n)$, forming a family of shifts with zero topological entropy, which contains all the simplest non-trivial ones. In particular, these linear complexity shifts include not only the conjugate shifts obtained from the natural coding of finite IETs, but also the shifts associated with primitive substitutions [**Qué10**]. Now, the simplicity behind linear complexity is reflected in the conjugate IETs. In this case, a conjugate IET given by Theorem 1 has three properties: in addition to being piecewise increasing, its discontinuities may accumulate only at a finite set, and all these discontinuities belong to only finitely many distinct iterates (full orbits) of the IET. We call an IET satisfying these properties *almost finite*. The second main result we prove is the following variation of Theorem 1.

THEOREM 2. *Let L be a measured minimal aperiodic shift with linear complexity. Then (L, σ) is topologically conjugate to (X_L, f_L), where f_L is the extension over X_L of an almost finite IET T_L on $I \subset X_L$.*

The above theorems can be constructive. In particular, we describe a technique to exhibit σ-invariant measures on the shifts L for which the conjugate IETs can be explicitly built (see Proposition 5.1.3, and the examples illustrating it).

2. *Basic definitions*

2.1. *Interval exchange transformations.*

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Definition 1. An orientation-preserving *finite interval exchange transformation*† (IET) is a map $T : I \to I$, where $I = [0, 1)$, with:

- a finite set $\{x_i\}_{i=0,\ldots,m-1}$ of discontinuities, denoted by $\mathcal{D}$, with $x_0 = 0 < x_1 < \cdots < x_{m-1} < x_m = 1$, which determines an ordered partition $\bigsqcup_{i=0}^{m-1} I_i$ of I formed by the right-open intervals $I_i = [x_i, x_{i+1})$,
  such that:
- T is injective, and T is a translation on each interval I_i, i.e. for each $i = 0, \ldots, m-1$, there exists $k_i \in \mathbb{R}$ such that for all $x \in I_i$, $T(x) = x + k_i$.

Such a finite IET T is just a piecewise order-preserving isometry of I. It is right-continuous, and measure-preserving, being injective and with derivative 1 on $I \setminus \mathcal{D}$. A usual way of interpreting T—giving its name to the notion—is the following: the intervals I_i, as ordered components of the partition $\bigsqcup_{i=0}^{m-1} I_i$ of I, are permuted in the image of T so as to form another partition $\bigsqcup_{i=0}^{m-1} I_{\pi(i)}$ of I, where π is the permutation over $\{0, \ldots, m-1\}$ induced by the k_i determining T. Indeed an IET can also be defined by a pair (λ, π), where λ is the vector of the lengths of the I_i, and π is a permutation of the I_i. Note that, given an IET, the partition of I can be refined by using any finite set $B \supset \mathcal{D}$ of points, and such a refinement can also be used in Definition 1 by replacing $\mathcal{D}$ with B. The points determining a specific partition of an IET T are then usually called the *separation points* of T.

The above definition of an IET can be extended to include infinity as follows.

Definition 2. An orientation-preserving *infinite interval exchange transformation* is a map $T : I \to I$, where $I = [0, 1)$, with:

- an infinite set Y of discontinuities such that the closure of Y in I, denoted by $\mathcal{D}$, has null measure and determines an ordered partition of I formed by:
  (i) the right-open intervals $[x, x')$ with $x \in \mathcal{D}$, $x' \in \mathcal{D} \cup \{1\}$, $[x, x') \cap \mathcal{D} = \{x\}$;
  (ii) the set $\mathcal{D}_{\mathrm{acc},r}$ of the accumulation points of $\mathcal{D}$ from the right,
  such that:
- T is injective on $I \setminus \mathcal{D}_{\mathrm{acc},r}$, right-continuous on I, and T is a translation on each right-open interval of the above partition‡.

An infinite IET T is also measure-preserving, being injective with derivative 1 on $I \setminus \mathcal{D}$, and $\mathcal{D}$ being of null measure. Since T is right-continuous on I, its values on $\mathcal{D}_{\mathrm{acc},r}$ are determined by its values on $I \setminus \mathcal{D}_{\mathrm{acc},r}$. Like in the finite case, $\mathcal{D}$ can be replaced in the above definition by any null-measure set B of separation points such that $B \supset \mathcal{D}$, where $B \cup \{1\}$ is compact. Note that we can use a null-measure infinite set B for finite IETs too.

2.2. *Words.* Let A be a finite alphabet, let A^+ be the set of finite words over A, let A^* be A^+ with the empty word ϵ, and let $A^{\mathbb{N}}$ be the set of infinite words over A. The alphabet A can be endowed with the discrete topology, and the resulting product topology on $A^{\mathbb{N}}$ is metrizable and compact. The usual associated metric is the *Cantor*

† In this paper, it is sufficient to consider the orientation-preserving case only, i.e. interval exchange transformations with $(+1)$ slopes and no (-1) slopes as in the general case [**Mañ87**].

‡ Infinite IETs are considered in different ways in the literature depending on the authors, e.g. in [**AOW85**], the discontinuities accumulate only at 1, whereas in [**Hoo15**], the discontinuities are countable and the partitioned interval may have infinite length.

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metric: if $w = a_0a_1a_2\ldots$ and $w' = a'_0a'_1a'_2\ldots$ in $A^{\mathbb{N}}$, with $a_i, a'_i \in A$, their distance is 0 if they are equal and 2^{-k} if not, where k is the smallest non-negative integer for which $a_k \neq a'_k$. Let $v = v_0v_1\ldots v_n \in A^*$, with $v_i \in A$, then its *cylinder set* (for short, *cylinder*) is $\mathrm{Cyl}(v) = \{w = a_0a_1a_2\ldots \in A^{\mathbb{N}} \mid a_0 = v_0, a_1 = v_1, \ldots, a_n = v_n\}$. A cylinder is a clopen set, and the collection of all the cylinders forms a basis for the topology on $A^{\mathbb{N}}$. For $L \subseteq A^{\mathbb{N}}$ and $v \in A^*$, the *cylinder* in the subspace topology is $\mathrm{Cyl}_L(v) = \mathrm{Cyl}(v) \cap L$. Let $\leq$ be an order over A, propagated to all the words in A^+ and $A^{\mathbb{N}}$ as the lexicographic order. Then for every $w, w' \in L$ with $w \leq w'$, the *word interval* $[w, w']$ in L is $\{w'' \in L \mid w \leq w'' \leq w'\}$. Note that the induced order topology is the same as the product topology.

A *factor* (or *subblock*) of a word w is a finite word v such that $w = w'vw''$, where w', w'' are possibly empty, and where $w'vw''$ denotes the word concatenation of w', v and w''. The length of v is denoted by $|v|$. The set of all distinct factors of a word w is denoted by Fact_w, and for a set $L \subseteq A^{\mathbb{N}}$, by $\mathrm{Fact}_L = \bigcup_{w\in L} \mathrm{Fact}_w$. For each $n \in \mathbb{N}^*$, the set of all the distinct factors v of w such that $|v| = n$ is denoted by $\mathrm{Fact}_w(n)$, and for $L \subseteq A^{\mathbb{N}}$, by $\mathrm{Fact}_L(n) = \bigcup_{w\in L} \mathrm{Fact}_w(n)$. The *(factor) complexity* (or *block growth*) [**MH38**, **CN10**] of a word w is the function $p_w : \mathbb{N}^* \to \mathbb{N}^*$, with $p_w(n) = |\mathrm{Fact}_w(n)|$, where $|\cdot|$ for a set denotes its cardinality, and for $L \subseteq A^{\mathbb{N}}$, it is $p_L(n) = |\mathrm{Fact}_L(n)|$. When L is a shift, its *topological entropy* [**Par66**] is defined as $\lim_{n\to\infty} \log(p_L(n))/n$ (which exists, since $\log(p_L(n))$ is subadditive).

An infinite word w is said to be *minimal* when each factor in Fact_w occurs infinitely often in w with bounded gaps, i.e. for each factor v of $w = a_0a_1, \ldots$, the ordered sequence of distinct indexes $\{n_j\}_{j\in\mathbb{N}}$ such that $a_{n_j} \ldots a_{n_j+|v|-1} = v$ is infinite, and $n_{j+1} - n_j$ is bounded independently of j [**MH38**]. Such a word w is *minimal aperiodic* when it is not periodic, i.e. when there is no factor v such that $w = v^\omega$. A set of finite words is *prolongable* if all its words can always be concatenated with letters to the right and to the left so that the resulting words still belong to the set. When $L \subseteq A^{\mathbb{N}}$ is made of minimal words, Fact_L is prolongable.

2.3. *Dynamical systems and shifts.* Considering a self-map $f : X \to X$, where X is a measured space and where f is measurable, the pair (X, f) is called a *measured dynamical system*. When X is a topological space and f is continuous, (X, f) is called a *topological dynamical system*. The *shift map* σ over $A^{\mathbb{N}}$ sends $a_0a_1\ldots$ to $a'_0a'_1, \ldots$, where $a'_i = a_{i+1}$ for every $i \in \mathbb{N}$, and it is a continuous map over $A^{\mathbb{N}}$. A *shift space* (or simply a *shift*) [**LM95**] is a closed σ-invariant set of words L in $A^{\mathbb{N}}$, and accordingly, (L, σ) is an instance of a topological dynamical system. For a dynamical system (X, f), the *(positive) orbit* of a point $x \in X$ is $\{f^n(x)\}_{n\in\mathbb{N}}$. When f is invertible, the *full orbit* is $\{f^n(x)\}_{n\in\mathbb{Z}}$, and when f is not invertible, it is the $\sim$-class of points in X containing x, where $x' \sim x$ if there are $n_1, n_2 \geq 0$ such that $f^{n_1}(x') = f^{n_2}(x)$ [**KST01**]. A continuous map $f : X \to X$ is said to be *minimal* if there is no non-empty closed proper subset $X' \subsetneq X$ such that $f(X') = X'$, or equivalently if the orbit of each $x \in X$ is dense in X. The map f is *minimal aperiodic* if X does not consist of a single periodic orbit. Accordingly, a *minimal shift* is a shift containing no proper shift, and a *minimal aperiodic shift* is a shift which is not made of the orbit of one single periodic word. If a word w belongs to a minimal shift L, it is minimal as a word, and $\mathrm{Fact}_w = \mathrm{Fact}_L$.

For two measured dynamical systems (X, f) and (Y, g), if there exists a measurable isomorphism $\phi : X \to Y$ such that $\phi \circ f = g \circ \phi$, then (X, f) and (Y, g) are said to be *conjugate* by ϕ. When (X, f) and (Y, g) are topological dynamical systems and ϕ is a continuous onto map (respectively a homeomorphism), (X, f) and (Y, g) are said to be *topologically semi-conjugate* (respectively *topologically conjugate*) by ϕ.

3. *From shifts to interval exchange transformations*

3.1. *The conjugacies.* We start here by presenting basic results about how to embed ordered measured spaces coming from shifts into [0, 1].

For any dynamical system (X, f), where X is a compact space and f is a continuous self-map on X, there exist Borel probability measures on X which are f-invariant [**HK02**]. If f is minimal aperiodic, any such measure is non-atomic and takes positive values on open sets. Borel measures on compact sets behave well with respect to approximating measurable sets by open and/or closed sets since they are *regular*, i.e. for any measurable set E, $\mu(E) = \sup\{\mu(E') \mid E' \text{ compact}, E' \subseteq E\}$ and $\mu(E) = \inf\{\mu(E') \mid E' \text{ open}, E \subseteq E'\}$. For a regular measure, to be non-atomic is equivalent to every singleton having measure 0.

From now on, L denotes a shift lexicographically ordered by $\leq$, and endowed with a σ-invariant Borel probability measure μ which is non-atomic (so that the measure of any single word in L is zero), positive on cylinders, and regular. If L is a minimal aperiodic shift, as just said above, such measures exist.

Remark 3.1.1. There is no isolated word in L.

Proof. If there was an isolated word w in L, there would exist a prefix v of w such that $\mathrm{Cyl}_L(v)$ contains only w, but then $\{w\}$ would have positive measure. □

Using μ and the order $\leq$ over L, and denoting the smallest word of L by $w_{L,\min}$, we define:

$$\begin{array}{rccl} \phi_\mu\colon & L & \to & \overline{I} = [0, 1] \\ & w & \mapsto & \mu([w_{L,\min}, w]). \end{array}$$

LEMMA 3.1.2. *ϕ_μ is a monotonic non-decreasing continuous map.*

Proof. Since μ is a measure which takes its values in [0, 1], ϕ_μ is monotonic non-decreasing. For continuity, the involved spaces being compact, we just check that the image by ϕ_μ of every sequence in L converging to some $w \in L$ is a sequence converging to $\phi_\mu(w)$. Let $\mathcal{S}$ be such a sequence, from which we extract two subsequences $\{w_i^+\}$ and $\{w_i^-\}$, formed by decreasing words greater than w and by increasing words smaller than w, respectively. At least one of them is infinite, say $\{w_i^+\}$, and $[w_{L,\min}, w] = \bigcap_i [w_{L,\min}, w_i^+)$. Since μ is regular, $\mu([w_{L,\min}, w]) = \phi_\mu(w) = \inf_i \mu([w_{L,\min}, w_i^+))$, and since μ is zero on single words for all $i > 0$, $\mu([w_{L,\min}, w_i^+]) = \phi_\mu(w_i^+)$, thus $\inf_i \phi_\mu(w_i^+) = \phi_\mu(w)$. Hence since ϕ_μ is non-decreasing, $\{\phi_\mu(w_i^+)\}$ converges to $\phi_\mu(w)$, whatever the choice of $\{w_i^+\}$. If $\{w_i^-\}$ is also infinite, similar arguments as for $\{w_i^+\}$ apply using suprema instead of infima, whence the result. □

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LEMMA 3.1.3. *ϕ_μ is a surjective map.*

Proof. Since L is compact and ϕ_μ is continuous, $\overline{I} \setminus \phi_\mu(L)$ is open, and if not empty it is a disjoint union of open intervals, since 0 and 1 belong to $\phi_\mu(L)$. Let $J = (x, x')$, with $x, x' \in \phi_\mu(L)$, be one of these intervals, and let $w_{x,\sup} = \sup\{w \in L \mid \phi_\mu(w) = x\}$ and $w_{x',\inf} = \inf\{w \in L \mid \phi_\mu(w) = x'\}$. Since $x < x'$ and since ϕ_μ is non-decreasing, $w_{x,\sup} < w_{x',\inf}$. Also, there is no word $w \in L$ with $w_{x,\sup} < w < w_{x',\inf}$, otherwise $\phi_\mu(w_{x,\sup}) \leq \phi_\mu(w) \leq \phi_\mu(w_{x',\inf})$, and by the definition of J we would have $\phi_\mu(w) = x$ or $\phi_\mu(w) = x'$, contradicting the definitions of $w_{x,\sup}$ or $w_{x',\inf}$. Now, $\phi_\mu(w_{x',\inf}) - \phi_\mu(w_{x,\sup}) = \mu([w_{L,\min}, w_{x',\inf}]) - \mu([w_{L,\min}, w_{x,\sup}]) = \mu((w_{x,\sup}, w_{x',\inf}])$, which is equal to the non-zero length of J. But we just have checked that $(w_{x,\sup}, w_{x',\inf}] = \{w_{x',\inf}\}$ and μ is zero on single words. Hence there is no interval such as J, so ϕ_μ is onto. □

We now characterize the points where ϕ_μ is non-injective. For each $n > 0$, the length-n factor set $\mathrm{Fact}_L(n)$ induces an ordered finite partition of L defined as

$$\mathrm{CYL}_L(n): \quad L = \bigsqcup_{v \in \mathrm{Fact}_L(n)} \mathrm{Cyl}_L(v),$$

where the $\mathrm{Cyl}_L(v)$ are ordered in L according to the lexicographic order of the v in $\mathrm{Fact}_L(n)$. By compactness, each $\mathrm{Cyl}_L(v)$ has two *endpoints*: its smallest and its greatest words. We say that $w, w' \in L$ (respectively $\mathrm{Cyl}(u), \mathrm{Cyl}(u') \in \mathrm{CYL}_L(n)$, $n > 0$) are *consecutive* if $w < w'$ (respectively $\mathrm{Cyl}(u) < \mathrm{Cyl}(u')$) and if there is no word $w'' \in L$ such that $w < w'' < w'$ (respectively $\mathrm{Cyl}(u) < w'' < \mathrm{Cyl}(u')$).

LEMMA 3.1.4. *Let $w, w' \in L$. Then w, w' are consecutive if and only if for some $n > 0$ they are endpoints, respectively the greatest and the smallest words, of two consecutive cylinders of* $\mathrm{CYL}_L(n)$.

Proof. ($\Leftarrow$) Trivial. ($\Rightarrow$) Assume $w < w'$, with w, w' consecutive. Let $u \in A^*$ be their longest common prefix, so that $w = ua_1a_2\ldots$ and $w' = ua'_1a'_2, \ldots,$ with $a_i, a'_i \in A$. Thus $w, w' \in \mathrm{Cyl}_L(u)$ and $a_1 < a'_1$. Since there is no word between w and w', the prefix $ua_1 \ldots a_{j-1}a_j$ of w for each $j > 1$ is also the greatest prefix for all the words in $\mathrm{Cyl}_L(ua_1)$. Otherwise, there would be a factor $ua_1v \in \mathrm{Fact}_L(|u| + j)$, $v \in A^+$, such that $ua_1 \ldots a_{j-1}a_j < ua_1v$ as a prefix of a word $w'' = ua_1v \ldots \in L$. But then $w < w'' < w'$, which is impossible. Thus w is the greatest word in $\mathrm{Cyl}_L(ua_1)$. By similar arguments, w' is the smallest word in $\mathrm{Cyl}_L(ua'_1)$. Finally, there is no other $a \in A$ with $a_1 < a < a'_1$, such that there is $w''' = ua \ldots \in L$, otherwise again $w < w''' < w'$. Hence $\mathrm{Cyl}_L(ua_1)$ and $\mathrm{Cyl}_L(ua'_1)$ are consecutive cylinders in $\mathrm{CYL}_L(|u| + 1)$. □

LEMMA 3.1.5. *Let $w, w' \in L$. Then ϕ_μ is non-injective on $\{w, w'\}$ if and only if w, w' are consecutive.*

Proof. ($\Leftarrow$) Assume $w < w'$, with w, w' consecutive. Then we have $\phi_\mu(w') - \phi_\mu(w) = \mu([w_{L,\min}, w']) - \mu([w_{L,\min}, w]) = \mu((w, w']) = \mu(\{w'\}) = 0$, since μ is zero on single words. ($\Rightarrow$) Assume $w < w'$, with w, w' non-consecutive. Thus there exists $w'' \in L$ with $w < w'' < w'$, so that there is a length-n prefix u of w'', with $n > 0$, distinct from the length-n prefixes of w and w'. But then, every $w''' \in \mathrm{Cyl}_L(u)$ is such that $w < w''' < w'$,

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and we have $\phi_\mu(w') - \phi_\mu(w) = \mu((w, w']) \geq \mu(\mathrm{Cyl}_L(u)) > 0$, since μ is positive on cylinders. □

Thus $\phi_\mu^{-1}(x)$, $x \in \overline{I}$, consists of either one or two words in L, and a consequence of Lemma 3.1.4 is that the set of points with two-word preimages is countable.

Let us then transform ϕ_μ into an injective map by embedding $\overline{I}$ as a subset of a larger compact space (using a similar construction to the classic one for transforming piecewise continuous self-maps into homeomorphisms, as in e.g. [**Kea75**] for IETs). We first define the following spaces and maps.

- $Z_0 \subset \overline{I}$ denotes the image by ϕ_μ of the set of points where ϕ_μ is not injective.
- Z_0^- denotes a copy of Z_0.
- X_L denotes $\overline{I} \sqcup Z_0^-$, ordered in such a way that each point in Z_0 lies to the right of its copy in Z_0^-, with no other point in between. We endow X_L with the order topology for this order relation.
- $\iota : \overline{I} \to X_L$ denotes the inclusion map. It is increasing and right-continuous on I.
- $\kappa : X_L \to \overline{I}$ denotes the canonical map associated with the equivalence relation in X_L which identifies each point in Z_0 with its copy in Z_0^-. It is a non-decreasing continuous map, and it is onto. Accordingly, $\kappa \circ \iota$ is the identity map on $\overline{I}$.

Having in mind Lemma 3.1.5 we then define the following map from ϕ_μ:

$$\begin{array}{rccl} \phi: & L & \to & X_L \\ & w & \mapsto & \begin{cases} \iota(\phi_\mu(w)) & \text{if } \phi_\mu(w) \notin Z_0, \text{ or} \\ & \text{if } \phi_\mu(w) \in Z_0, \text{ with } \phi_\mu^{-1}(\phi_\mu(w)) = \{w, w'\}, w > w', \\ \iota(\phi_\mu(w))^- & \text{if } \phi_\mu(w) \in Z_0, \text{ with } \phi_\mu^{-1}(\phi_\mu(w)) = \{w, w'\}, w < w', \\ & \text{where } \iota(\phi_\mu(w))^- \text{ is the copy of } \iota(\phi_\mu(w)) \text{ in } Z_0^-. \end{cases} \end{array}$$

Note that $\kappa \circ \phi = \phi_\mu$.

LEMMA 3.1.6. *ϕ is an increasing homeomorphism.*

Proof. By Lemma 3.1.2, ϕ is increasing where ϕ_μ is injective since $\phi = \phi_\mu$ on these points. Where ϕ_μ is not injective, that is, for each $x \in Z_0$ where $\phi_\mu^{-1}(x) = \{w, w'\}$ for some $w, w' \in L$ with $w < w'$, then by definition of X_L and ϕ we have $\phi(w) < \phi(w')$. Therefore ϕ is everywhere increasing, thus into. Since $X_L = \iota(\overline{I}) \sqcup Z_0^-$, ϕ is also onto. Hence ϕ is a monotonic bijection between two totally ordered sets endowed with their respective order topologies, so it is a homeomorphism. □

We define two more maps:

$$\begin{array}{rccl} f_L: & X_L & \to & X_L \\ & x & \mapsto & \phi(\sigma(\phi^{-1}(x))) \end{array} \qquad \begin{array}{rccl} T_L: & \overline{I} & \to & \overline{I} \\ & x & \mapsto & \kappa(f_L(\iota(x))). \end{array}$$

PROPOSITION 3.1.7. *Consider the following diagram:*

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$$\begin{tikzcd}
L \arrow[r, "\phi"] \arrow[rr, bend left, "\phi_\mu"] \arrow[d, "\sigma"'] & X_L \arrow[r, shift left, "\kappa"] \arrow[d, "f_L"'] & \overline{I} \arrow[l, shift left, "\iota"] \arrow[d, "T_L"] \\
L \arrow[r, "\phi"] \arrow[rr, bend right, "\phi_\mu"'] & X_L \arrow[r, shift left, "\kappa"] & \overline{I} \arrow[l, shift left, "\iota"]
\end{tikzcd}$$

(a) *for every* $x \in I$, $\iota(T_L(x)) = f_L(\iota(x))$;
(b) (L, σ) *is topologically conjugate to* (X_L, f_L) *by* ϕ;
(c) $((\iota(\overline{I}) \subset X_L), f_L)$ *is topologically semi-conjugate to* $(\overline{I}, T_L)$ *by* κ;
(d) $((\phi^{-1}(\iota(\overline{I})) \subset L), \sigma)$ *is topologically semi-conjugate to* $(\overline{I}, T_L)$ *by* ϕ_μ.

Proof. (a) Note that $0, 1 \notin Z_0$. Indeed, ϕ_μ being non-decreasing, if $\phi_\mu^{-1}(0)$ (respectively $\phi_\mu^{-1}(1)$) was made of two words, consecutive by Lemma 3.1.5, they would be the two smallest (respectively greatest) in L, and by Lemma 3.1.4 the smaller one (respectively greater one) would be isolated, contradicting Remark 3.1.1. Now, we put $I' = \overline{I} \setminus (Z_0 \cup \{0, 1\})$. By definition, κ is injective on $\iota(I)$, hence on $\iota(I')$, thus since κ is a left inverse of ι, it is also a right inverse of ι when restricted to this set. We compose $T_L = \kappa \circ f_L \circ \iota$ on both sides to the left by ι, so that $\iota \circ T_L = f_L \circ \iota$ is valid on I', provided that $f_L(\iota(I')) \subset \iota(I')$, which is proved as follows: by definition of I', for every $x \in I'$, $\phi_\mu^{-1}(x) = \phi^{-1}(\iota(x))$, that is, $\phi_\mu^{-1}(I') = \phi^{-1}(\iota(I'))$. Also, by Lemmas 3.1.4 and 3.1.5, a word w is in $\phi_\mu^{-1}(I')$ if and only if it is not a cylinder endpoint. Thus, there are words in L arbitrarily close to w from the right and the left, that is, words smaller and greater than w with arbitrarily long prefixes common with those of w. By continuity of σ, the same holds for $\sigma(w)$, so $\sigma(\phi^{-1}(\iota(I'))) \subset \phi^{-1}(\iota(I'))$. Composing on both sides to the left by ϕ yields $f_L(\iota(I')) \subset \iota(I')$, as required. Now, Z_0 being countable, I' is dense in $\overline{I}$. Since both $\iota \circ T_L$ and $f_L \circ \iota$ are compositions of right-continuous maps on I, they are right-continuous too, so $\iota \circ T_L = f_L \circ \iota$ extends to all of I.

(b) ϕ is a homeomorphism, thus by definition of f_L, the result follows.

(c) By definition of T_L, we have the conjugacy $T_L \circ \kappa = \kappa \circ f_L$ on $\iota(I')$. Both members of this identity are right-continuous maps, and by the same arguments as in (a), this conjugacy holds on all of $\iota(I)$. Next, again by definition of T_L, we have $T_L(1) = \kappa(f_L(\iota(1)))$, and 1 has only one preimage by κ which is $\iota(1)$, so the conjugacy holds on $\iota(\overline{I})$ too. Finally, κ is measure-preserving and onto.

(d) Using the fact that $\kappa \circ \phi = \phi_\mu$, and the commutativity properties given by (b) and (c) on κ and ϕ, the result follows. □

3.2. *Checking the infinite exchange transformation properties.* We now focus on when T_L is an IET. Here are first three general lemmas.

LEMMA 3.2.1. *The image of* μ *by* ϕ_μ *is the Lebesgue measure on* $\overline{I}$, *which is preserved by* T_L.

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Proof. We denote by $f_*\mu$ the image of the measure μ by the map f. By Lemmas 3.1.2 and 3.1.3, ϕ_μ is continuous and surjective, thus the preimage of any interval $[0, x]$, with $x \in \overline{I}$, is an interval $[w_{L,\min}, w]$ in L, where $\phi_\mu(w) = x = \mu([w_{L,\min}, w])$. Hence $(\phi_\mu)_*\mu$ is the Lebesgue measure on $\overline{I}$. Next, by Proposition 3.1.7(d), $\phi_\mu \circ \sigma = T_L \circ \phi_\mu$ on $\phi^{-1}(\iota(\overline{I}))$, a set of full measure in L. Thus $(\phi_\mu)_*\sigma_*\mu = T_{L*}(\phi_\mu)_*\mu$, and since σ preserves μ, $(\phi_\mu)_*\sigma_*\mu = (\phi_\mu)_*\mu$, that is, T_L preserves the Lebesgue measure on $\overline{I}$. □

From $\mathrm{CYL}_L(1)$, that is, the partition of L defined as $\bigsqcup_{a\in A} \mathrm{Cyl}_L(a)$, we define the induced partition of I as

$$P_A : I = \bigsqcup_{a\in A} I_{L,a}, \quad \text{where } I_{L,a} = \{x \in I \mid \phi^{-1}(\iota(x)) \in \mathrm{Cyl}_L(a)\}. \tag{1}$$

Note that each $I_{L,a}$ is a right-open interval in I. Indeed, if $x < x'$, with $x, x' \in I_{L,a}$, then every $x'' \in I$, with $x < x'' < x'$, is also in $I_{L,a}$ since ϕ^{-1} and ι both preserve the order. Moreover, if $a < a'$, where $a, a' \in A$ are consecutive, then if w is the greatest word in $\mathrm{Cyl}_L(a)$ and w' the smallest in $\mathrm{Cyl}_L(a')$, we have $\phi_\mu(w) = \phi_\mu(w') = y$, where $y \in Z_0$. By definition of Z_0 and ϕ and since $w < w'$, $\phi^{-1}(\iota(y)) = w'$, thus $y \in I_{L,a'}$, which means that $I_{L,a}$ is right-open.

LEMMA 3.2.2. *T_L is right-continuous on I, and piecewise increasing on P_A.*

Proof. T_L is right-continuous on I, being a composition of the right-continuous maps κ, f_L, and ι. Next, T_L is non-decreasing on each $I_{L,a}$ in P_A. Indeed, by Proposition 3.1.7(d), $T_L \circ \phi_\mu = \phi_\mu \circ \sigma$ on $\phi^{-1}(\iota(I_{L,a})) \subset \mathrm{Cyl}_L(a)$, and ϕ_μ is non-decreasing on I, while σ is increasing on $\mathrm{Cyl}_L(a)$. Now, assume there exist $x, x' \in I_{L,a}$, with $x < x'$, such that $T_L(x) = T_L(x')$. Since T_L is non-decreasing it must be constant on (x, x'), which is impossible since T_L is measure-preserving by Lemma 3.2.1, hence T_L is increasing on each $I_{L,a}$. □

LEMMA 3.2.3. *$T_L(I) \subset I$.*

Proof. By Lemma 3.2.2, T_L is piecewise increasing on the $I_{L,a}$. Let J be one of these intervals, and assume $T_L(x) = 1$ for some $x \in J$. But with T_L being increasing on J, and J being right-open, $T_L(y)$ would be > 1 for all $y \in J$ with $y > x$. □

As a next step, let us exhibit conditions so that there exists a partition of I on which T_L is a translation on its components (P_A is not such a partition in general). For that purpose, we first study the effect of σ over the word intervals in L. A factor $u \in \mathrm{Fact}_L$ is said to be *left special* (respectively *right special*) if u has at least two distinct left (respectively right) letter prolongations in Fact_L, i.e. if $u \in \mathrm{Fact}_L(n)$, then u is the suffix (respectively prefix) of at least two distinct factors in $\mathrm{Fact}_L(n+1)$.

LEMMA 3.2.4. *Let L be a shift such that Fact_L is prolongable. Let $w, w' \in \mathrm{Cyl}_L(au)$, with $a \in A$, $u \in \mathrm{Fact}_L$ not left special, and $w < w'$. Then*

$$\sigma([w, w']) = [\sigma(w), \sigma(w')]. \tag{2}$$

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Proof. ($\subseteq$) Let $w = auv$ and $w' = auv'$, with $v, v' \in A^{\mathbb{N}}$. Let $w'' = auv'' \in [w, w']$, with $v'' \in A^{\mathbb{N}}$. By lexicographic order, we have $uv \leq uv'' \leq uv'$. Hence $\sigma(w'') = uv''$ belongs to $[uv, uv'] = [\sigma(w), \sigma(w')]$.

($\supseteq$) Let again $w = auv$ and $w' = auv'$, with $v, v' \in A^{\mathbb{N}}$. Let $w'' = uv'' \in [\sigma(w), \sigma(w')]$, with $v'' \in A^{\mathbb{N}}$. Consider $\{uu_j\}_{j\in\mathbb{N}}$ the sequence of all the prefixes of w'' with $u_j \in A^*$ starting with u. By prolongability to the left, each uu_j has at least one left letter prolongation, but it can only be a since u is not left special. By prolongability to the right, each auu_j is the prefix of at least one word in L, which makes a sequence converging to the word aw'', belonging to L by compactness of L. Thus there is one left letter prolongation for w'' in L, unique since u is not left special, that is, $aw'' = \sigma^{-1}(w'') = \sigma^{-1}(uv'')$. Since $uv \leq uv'' \leq uv'$, we have $auv \leq aw'' \leq auv'$, that is, $aw'' \in [w, w']$. Hence $w'' \in \sigma([w, w'])$. □

Note that a specific case of the above is $\sigma(\mathrm{Cyl}_L(au)) = \mathrm{Cyl}_L(u)$, where $a \in A$ and u is not left special. Now, the idea is to build a partition of L essentially made of cylinders such that equality (2) holds in all of these cylinders. An *infinite left special word* (or *infinite left special branch*) with respect to Fact_L, where $L \subseteq A^{\mathbb{N}}$, is a one-way infinite word such that all its prefixes are left special factors in Fact_L. We denote by $\mathrm{SP}_L \subseteq A^{\mathbb{N}}$ the set of all the infinite left special words with respect to Fact_L. When L is a shift, SP_L is included in L.

LEMMA 3.2.5. *Let L be a shift having left special factors of arbitrary length, measured by μ, with $\mu(\mathrm{SP}_L) = 0$. Then, there is an infinite partition of L defined as*

$$\mathrm{PART}_L : L = \bigsqcup_{k>0} \mathrm{Cyl}_L(v^{(k)}) \sqcup W_{\mathrm{SP}_L}, \tag{3}$$

where:

- *for each $k > 0$, $v^{(k)} \in \mathrm{Fact}_L$, such that for all $w, w' \in \mathrm{Cyl}_L(v^{(k)})$ with $w < w'$, $\sigma([w, w']) = [\sigma(w), \sigma(w')]$ (i.e. equality (2) holds).*
- *$W_{\mathrm{SP}_L} = \sigma^{-1}(\mathrm{SP}_L)$ is a null measure non-empty set in L, which is formed by the accumulation of the endpoints of $\mathrm{Cyl}_L(v^{(k)})$ in L.*

Proof. Let us describe the cylinders of PART_L by an iterative process. As a first step, consider the partition $\mathrm{CYL}_L(2)$ of L, i.e. $\bigsqcup_{a_i a_j \in \mathrm{Fact}_L(2)} \mathrm{Cyl}_L(a_i a_j)$, with $a_i, a_j \in A$. By Lemma 3.2.4, for each non-left special a_j, we have $\sigma([w, w']) = [\sigma(w), \sigma(w')]$ for all $w, w' \in \mathrm{Cyl}_L(a_i a_j)$, so that $\mathrm{Cyl}_L(a_i a_j)$ is put in PART_L, and $a_i a_j$ is one of the $v^{(k)}$. As a second step, each of the remaining $\mathrm{Cyl}_L(a_i a_j)$ not put in PART_L during the first step, is partitioned with cylinders of the form $\mathrm{Cyl}_L(a_i a_j a_k)$ in $\mathrm{CYL}_L(3)$. Again, for each non-left special suffix $a_j a_k$, the word intervals in $\mathrm{Cyl}_L(a_i a_j a_k)$ satisfy equality (2), and $\mathrm{Cyl}_L(a_i a_j a_k)$ is put in PART_L, while $a_i a_j a_k$ is also one of the $v^{(k)}$. This refinement process is inductively applied as long as cylinders remain at step n by partitioning them with cylinders in $\mathrm{CYL}_L(n+1)$, defining then all the cylinders of PART_L.

Since the left special factors of L can have arbitrary lengths, and since every prefix of a left special word is left special, there is at least one left special factor in L for each length. Thus, given any $n > 0$ and a left special factor $u \in \mathrm{Fact}_L(n)$, the cylinder $\mathrm{Cyl}_L(au)$, $a \in A$, must still be refined during the nth step of the above refinement process. Therefore, this process is necessarily infinite. It determines infinite sequences of nested cylinders of the

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form $\{\mathrm{Cyl}_L(au_j)\}_{j\in\mathbb{N}^*}$, where $u_j \in \mathrm{Fact}_L(j)$ is left special while being a prefix of u_{j+1}, for all j. Such a sequence $\{au_j\}_{j\in\mathbb{N}^*}$ gives a limit infinite word, which belongs to W_{SP_L} since $\{u_j\}_{j\in\mathbb{N}^*}$ gives a limit infinite word in SP_L. As a result, if a word $w \in L$ does not belong to some $\mathrm{Cyl}_L(v^{(k)})$, it belongs to W_{SP_L}. Conversely, if $w \in \mathrm{SP}_L$, there exist more than one left letter prolongation to each of its prefixes. Consider any of these letters, say $a \in A$, so that $aw \in \sigma^{-1}(\mathrm{SP}_L) = W_{\mathrm{SP}_L}$. According to the same above argument, each prefix u_j of w determines a cylinder $\mathrm{Cyl}_L(au_j)$ which has to be refined further. We have $\bigcap_{u_j \text{ prefix of } w} \mathrm{Cyl}_L(au_j) = aw$, where aw does not belong to any cylinder $\mathrm{Cyl}_L(v^{(k)})$, and induces an infinite refinement. Thus PART_L is a partition since a word w in L belongs either to one of the cylinders $\mathrm{Cyl}_L(v^{(k)})$ obtained through a finite number of steps in the refinement process, or to W_{SP_L} if w induces an infinite number of steps.

The set W_{SP_L} is not empty since SP_L is not empty when there are infinitely many left special factors. Also, W_{SP_L} is of null measure since SP_L has been assumed of null measure and σ is measure-preserving. Finally, W_{SP_L} occurs as the set of the accumulation of the $\mathrm{Cyl}_L(v^{(k)})$ endpoints, since again SP_L is of null measure, thus there is no cylinder having all its words in SP_L. □

Note that if L is minimal aperiodic, it has left special factors of arbitrary length.

In order to exhibit examples of aperiodic minimal shifts, we use the following classic technique. A *substitution* is a map $\theta : A \to A'^*$, where A and A' are alphabets, which is extended to words by sending $w = \ldots a_i a_{i+1} a_{i+2} \ldots$ to $\theta(w) = \ldots \theta(a_i)\theta(a_{i+1})\theta(a_{i+2}) \ldots$, that is, on finite words, θ is just a monoid morphism from A^* to A'^*. For instance, the *Thue–Morse substitution* θ_{tm} over $A = A' = \{a, b\}$ is defined by $\theta_{tm}(a) = ab$ and $\theta_{tm}(b) = ba$. When $A = A'$, a substitution can be iterated. The set of factors of such a substitution is defined as $\mathrm{Fact}_\theta = \{v \in A^* \mid v \in \mathrm{Fact}_{\theta^n(a)}, a \in A, n \in \mathbb{N}\}$, and its *associated shift* as $L_\theta = \{w \in A^{\mathbb{N}} \mid \mathrm{Fact}_w \subseteq \mathrm{Fact}_\theta\}$. When θ is *non-erasing*, i.e. there is no $a \in A$ whose image is the empty word, and when a is a strict prefix of $\theta(a)$, then for all $n \geq 0$, $\theta^n(a)$ is a strict prefix of $\theta^{(n+1)}(a)$. Thus $\{\theta^n(a)\}_{n\in\mathbb{N}}$ gives rise to a limit word in $A^{\mathbb{N}}$, denoted by $\theta^\omega(a)$, which is a fixed point of θ. For instance, the Thue–Morse substitution has two fixed points: $\theta_{tm}^\omega(a) = w_1 = abbabaabbaababba \ldots$ and $\theta_{tm}^\omega(b) = w_2 = baababbaabbabaab \ldots$. Considering a fixed point w of a substitution θ, its *associated shift* $L_{\theta,w}$ is then defined as the closure of $\{\sigma^n(w) \mid n \in \mathbb{N}\}$. A substitution θ is said to be *primitive* if there exists $n > 0$ such that for every $a, b \in A$, the word $\theta^n(a)$ contains b. For a primitive substitution θ, it is known that [**Qué10**]: (1) if w is any of its fixed points, $L_\theta = L_{\theta,w}$; (2) for all $n > 0$, $L_{\theta^n} = L_\theta$, so that periodic points can also be considered; (3) L_θ is minimal. Thus for instance the Thue–Morse substitution θ_{tm} generates a minimal shift L_{tm}, equal to $L_{\theta_{tm}} = L_{\theta_{tm},w_1} = L_{\theta_{tm},w_2}$. A primitive substitution is said to be *aperiodic* if its associated shift is minimal aperiodic, e.g. θ_{tm} is known to be aperiodic [**Thu12**]†.

Example 3.2.6. (Step-by-step construction of a partition PART_L) Considering the above Thue–Morse shift L_{tm} and its fixed points w_1, w_2, we build here the first components of

† More generally, aperiodicity for a substitution is decidable using [**ER83**].

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$\mathrm{PART}_{L_{tm}}$ as in the proof of Lemma 3.2.5. The first factor sets are:

$$\begin{aligned}
&\mathrm{Fact}_{L_{tm}}(1) = \{a, b\}\\
&\mathrm{Fact}_{L_{tm}}(2) = \{aa, ab, ba, bb\}\\
&\mathrm{Fact}_{L_{tm}}(3) = \{aab, aba, abb, baa, bab, bba\}\\
&\mathrm{Fact}_{L_{tm}}(4) = \{aaba, aabb, abaa, abab, abba, baab, baba, babb, bbaa\}\\
&\mathrm{Fact}_{L_{tm}}(5) = \{aabab, aabba, abaab, ababb, abbaa, abbab, baaba, baabb,\\
&\qquad\qquad babaa, babba, bbaab, bbaba\}.
\end{aligned}$$

Thus its first left special factors are: $a, b, ab, ba, aba, abb, baa, bab, abba, baab$. Applying the refinement process, the first step, based on $\mathrm{CYL}_L(2)$, gives no cylinders of $\mathrm{PART}_{L_{tm}}$ since a and b are both left special factors. The second step, based on $\mathrm{CYL}_L(3)$, gives $\mathrm{Cyl}_{L_{tm}}(abb)$ and $\mathrm{Cyl}_{L_{tm}}(baa)$, since aa and bb are not left special factors, thus $v^{(1)} = abb$ and $v^{(2)} = baa$. The third step gives no new cylinders since $abba$ and $baab$ are the only factors whose right suffix is not left special, already contained in the cylinders of the preceding step. The fourth step gives $\mathrm{Cyl}_{L_{tm}}(aabab)$, $\mathrm{Cyl}_{L_{tm}}(ababb)$, $\mathrm{Cyl}_{L_{tm}}(babaa)$, and $\mathrm{Cyl}_{L_{tm}}(bbaba)$, thus $v^{(3)} = aabab$, $v^{(4)} = ababb$, $v^{(5)} = babaa$, and $v^{(6)} = bbaba$. And so on. ◊

Now, given a shift L and PART_L as obtained from Lemma 3.2.5, we define the induced partition of I as follows:

$$\mathrm{PART}_{L,I} : I = \bigsqcup_{k>0} I_L^{(k)} \sqcup B_{\mathrm{SP}_L}, \tag{4}$$

where $I_L^{(k)} = \{x \in I \mid \phi^{-1}(\iota(x)) \in \mathrm{Cyl}_L(v^{(k)})\}$ and $B_{\mathrm{SP}_L} = \{x \in I \mid \phi^{-1}(\iota(x)) \in W_{\mathrm{SP}_L}\}$. Each $I_L^{(k)}$ is a right-open interval by the same arguments as for the intervals of the partition P_A (see p. 1943).

Let B_L be the closure in $\overline{I}$ of the left ends of the $I_L^{(k)}$, intersected with I (so that $1 \notin B_L$). Let $B_{L,\mathrm{acc}}$ (respectively $B_{L,\mathrm{acc},r}$) be the set of accumulation points (respectively accumulation points from the right) of B_L in $\overline{I}$. Note that $B_{\mathrm{SP}_L} \subset B_{L,\mathrm{acc}}$ and even that $B_{\mathrm{SP}_L} = B_{L,\mathrm{acc},r}$ since an accumulation point x only to the left is the left endpoint of some $I_L^{(k)}$ and does not belong to B_{SP_L} (in this case, $\phi^{-1}(\iota(x)) \notin W_{\mathrm{SP}_L}$ because of the differences between $\iota(I)$ and X_L—see also Definition 2).

LEMMA 3.2.7. *$B_L \sqcup \{1\}$ is a null-measure infinite compact set of points in $\overline{I}$.*

Proof. The set $B_L \sqcup \{1\}$ is compact by definition. Since the set of the left endpoints of the $I_L^{(k)}$ is countable, the measure of B_L is the same as that of $B_{L,\mathrm{acc}}$, and also as that of $B_{L,\mathrm{acc},r}$ which is equal to B_{SP_L}. Now, recall from Lemma 3.2.1 that the Lebesgue measure is the image of μ by ϕ_μ. Since $\phi_\mu^{-1}(B_{\mathrm{SP}_L}) = W_{\mathrm{SP}_L}$ and $\mu(W_{\mathrm{SP}_L}) = 0$, the measure of B_L is also zero in $\overline{I}$. □

LEMMA 3.2.8. *T_L is a translation on each interval component $I_L^{(k)}$ of* $\mathrm{PART}_{L,I}$.

Proof. Let the interior of $I_L^{(k)}$ be denoted by $I_L^{(k)^o}$. By definition of T_L, for all $x \in I_L^{(k)^o}$,

$$T_L(x) - x = \kappa(\phi(\sigma(\phi^{-1}(\iota(x))))) - x. \tag{5}$$

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We first put $\phi^{-1}(\iota(x)) = w$. Next, recalling that $\phi_\mu = \kappa \circ \phi$ and that $\kappa \circ \iota$ is the identity, we compose $\phi^{-1}(\iota(x)) = w$ on both sides to the left by $\kappa \circ \phi$ and get $x = \kappa(\phi(w)) = \phi_\mu(w)$, so that (5) becomes for all $x \in I_L^{(k)^o}$,

$$T_L(x) - x = \kappa(\phi(\sigma(w))) - \kappa(\phi(w)) = \phi_\mu(\sigma(w)) - \phi_\mu(w). \tag{6}$$

By definition of $\mathrm{PART}_{L,I}$, the inverse image of $I_L^{(k)^o}$ by ϕ_μ is the interior $C^{(k)^o}$, where $C^{(k)}$ denotes $\mathrm{Cyl}_L(v^{(k)})$ of PART_L. As a closed ordered set, $C^{(k)}$ has a smallest word $w_{C^{(k)},\min}$, and recall that $w_{L,\min}$ denotes the smallest word of L, so that for all $w \in C^{(k)^o}$, $[w_{L,\min}, w] = [w_{L,\min}, w_{C^{(k)},\min}] \sqcup (w_{C^{(k)},\min}, w]$, then

$$\phi_\mu(w) = \mu([w_{L,\min}, w_{C^{(k)},\min}]) + \mu([w_{C^{(k)},\min}, w]). \tag{7}$$

Since $w_{C^{(k)},\min}$ and w belong to the same cylinder, and $w_{C^{(k)},\min} < w$, we have $\sigma(w_{C^{(k)},\min}) < \sigma(w)$, hence

$$\phi_\mu(\sigma(w)) = \mu([w_{L,\min}, \sigma(w_{C^{(k)},\min})]) + \mu([\sigma(w_{C^{(k)},\min}), \sigma(w)]).$$

Next, using the properties over the components of PART_L,

$$\begin{aligned}
\phi_\mu(\sigma(w)) &= \mu([w_{L,\min}, \sigma(w_{C^{(k)},\min})]) + \mu(\sigma([w_{C^{(k)},\min}, w])) \quad \text{(by equality (2))}\\
&= \mu([w_{L,\min}, \sigma(w_{C^{(k)},\min})]) + \mu(\sigma^{-1}\sigma([w_{C^{(k)},\min}, w])) \quad (\mu\text{-preservation})\\
&= \mu([w_{L,\min}, \sigma(w_{C^{(k)},\min})]) + \mu([w_{C^{(k)},\min}, w]) \quad (\sigma \text{ injectivity on } C^{(k)^o}).
\end{aligned}$$

Therefore, from the above last equation and (7), we get for all $w \in C^{(k)^o}$,

$$\phi_\mu(\sigma(w)) - \phi_\mu(w) = \mu([w_{L,\min}, \sigma(w_{C^{(k)},\min})]) - \mu([w_{L,\min}, w_{C^{(k)},\min}]) = K_{C^{(k)}}.$$

Thus $K_{C^{(k)}}$ depends only on the $C^{(k)^o}$ to which w belongs, that is, it depends only on the open interval $I_L^{(k)^o}$ to which x belongs. Hence, for every $I_L^{(k)}$ of $\mathrm{PART}_{L,I}$, and its corresponding cylinder $C^{(k)} = \mathrm{Cyl}_L(v^{(k)})$ of PART_L, there is a constant $K_{C^{(k)}}$ such that for all $x \in I_L^{(k)^o}$, $T_L(x) - x = K_{C^{(k)}}$. By right-continuity of T_L (see Lemma 3.2.2), this equality extends to all of $I_L^{(k)}$. □

Following Definition 2, let $\mathcal{D}_L$ be the closure of Y_L in I, where Y_L is the set of discontinuities of T_L, which by Lemma 3.2.8 is necessarily included into B_L. Let $\mathcal{D}_{L,\mathrm{acc},r}$ be the set of accumulation points of $\mathcal{D}_L$ from the right.

LEMMA 3.2.9. *T_L is injective on $I \setminus \mathcal{D}_{L,\mathrm{acc},r}$.*

Proof. By Lemma 3.2.8, T_L is a translation on each of the right-open interval components $I_L^{(k)}$ of $\mathrm{PART}_{L,I}$. The set $\mathcal{D}_L$ can be obtained by dropping all the points from B_L having a neighborhood where T_L extends as a translation map, so that T_L is still a translation map on each right-open interval component of $I \setminus \mathcal{D}_L$. By Lemma 3.2.1, the Lebesgue measure is preserved by T_L, and by Lemma 3.2.7, B_L has null measure, thus $\mathcal{D}_L \subseteq B_L$ has too. Now, if the open interval components of $I \setminus \mathcal{D}_L$ had overlaps in their translated images by T_L, the global image would be a set I' with $\mu(I') < 1$, such that $\mu(T^{-1}(I')) = \mu(I \setminus \mathcal{D}_L) = 1$, contradicting measure preservation. Moreover, no image by T_L of a left endpoint of these open intervals can lie in the image of another interval, thus T is injective on all the corresponding right-open intervals. Since $\mathcal{D}_L \setminus \mathcal{D}_{L,\mathrm{acc},r}$ consists of all these left endpoints, the result follows. □

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LEMMA 3.2.10. *Let L be a shift measured by μ. If L has zero topological entropy, then $\mu(\mathrm{SP}_L) = 0$.*

Proof. The topological entropy dominates the (measure-theoretic) entropies on L with respect to all the invariant probability measures [**HK02**, Proposition 4.4.1]. Thus with respect to μ, the shift L has necessarily zero entropy. But then, a dynamical system with zero entropy with respect to an invariant measure is known to be invertible [**HK02**, Section 3.7k], that is, it has an inverse on a subset of full measure. In the case of a shift L, the set of words on which the shift map is not invertible is exactly SP_L, so that $\mu(\mathrm{SP}_L) = 0$. □

We are now ready to prove Theorem 1 as stated in the introduction.

Proof. Since L is assumed aperiodic minimal, the results of §3.1 apply, so that (L, σ) is a topological conjugate by ϕ of (X_L, f_L), where I is embedded as a subset of X_L in such a way that f_L is an extension of T_L. By Lemma 3.2.2, T_L is right-continuous, and by Lemma 3.2.10, $\mu(\mathrm{SP}_L) = 0$ for any σ-invariant Borel probability measure μ chosen for L. Minimal aperiodicity means L has left special factors of arbitrary length, thus Lemma 3.2.5 applies. Hence, by Lemmas 3.2.7–3.2.9, and since the set $\mathcal{D}_L$ of T_L is included into B_L, T_L is an IET. □

Note that Theorem 1 remains valid under the assumptions on L in Lemma 3.2.5, in particular $\mu(\mathrm{SP}_L) = 0$ can be used instead of zero topological entropy. Note also that the above construction relies on B_L, a set determined from $\mathrm{PART}_{L,I}$, determined in turn from PART_L. This set of separation points for T_L could have been different, as it is just a set with the property of including $\mathcal{D}_L$ (see §2.1). However, following the construction of PART_L in the proof of Lemma 3.2.5, this set has some word-combinatorics properties, and we shall use them henceforth.

As a complement, in view of Definition 2 of an infinite IET and of Lemma 3.2.9, we give the following example.

Example 3.2.11. (T_L can be non-injective on $\mathcal{D}_{L,\mathrm{acc},r}$) Let us reconsider the Thue–Morse substitution θ_{tm} over $A = \{a, b\}$ (see Example 3.2.6), its associated shift L_{tm}, and its fixed points $w_1 = \theta_{tm}^{\omega}(a)$ and $w_2 = \theta_{tm}^{\omega}(b)$. A *bispecial factor* is a factor which is both right and left special. According to [**Cas96**], the bispecial factors in L_{tm} are $\theta_{tm}^n(a)$, $\theta_{tm}^n(b)$, $\theta_{tm}^n(aba)$, $\theta_{tm}^n(bab)$, with $n \geq 0$. Thus we have $\{w_1, w_2\} = \mathrm{SP}_{L_{tm}}$. In fact, in an aperiodic minimal shift, every infinite special word has always infinitely many prefixes which are bispecial factors.

Let the lexicographic order over L_{tm} be determined by $a < b$. We claim that w_1, w_2 are consecutive in L_{tm}, with w_1 being the greatest word in $\mathrm{Cyl}_{L_{tm}}(a)$, and w_2 the smallest word in $\mathrm{Cyl}_{L_{tm}}(b)$. Assume on the contrary there is $w_1' \in \mathrm{Cyl}_{L_{tm}}(a)$ with $w_1 < w_1'$. Let u be the maximal common prefix so that $w_1 = uv_1$ and $w_1' = uv_1'$. Then u is bispecial, since at least three of the factors aua, aub, bua, bub are in $\mathrm{Fact}_{L_{tm}}$ (both ua and ub are prolongable to the left, and at least one of them is left special, being a prefix of w_1). By [**Cas96**], such a bispecial factor in L_{tm} with at least three continuation factors and starting by a must be of the form $\theta_{tm}^n(a)$ for some $n \geq 0$. Now, since $w_1 < w_1'$, the word v_1

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must also start with a, and v_1' with b. However, since there is $n > 0$ such that $u = \theta_{tm}^n(a)$ and since $\theta_{tm}^{n+1}(a) = \theta_{tm}^n(a)\theta_{tm}^n(b)$ is also a prefix of w_1, then u must be followed by b in w_1. Hence there is no such word as w_1' and w_1 is the greatest word in $\mathrm{Cyl}_{L_{tm}}(a)$. Similar arguments apply to w_2 showing it is the smallest word in $\mathrm{Cyl}_{L_{tm}}(b)$. Thus by Lemma 3.1.4, w_1, w_2 are consecutive.

As a result, $\{aw_1, aw_2\}$ and $\{bw_1, bw_2\}$ are two pairs of consecutive words in L_{tm}. Moreover, $\mathrm{Cyl}_{L_{tm}}(abb)$ is a non-empty cylinder between aw_2 and bw_1, then by Lemmas 3.1.2 and 3.1.5, $\phi_\mu(aw_1) = \phi_\mu(aw_2) < \phi_\mu(bw_1) = \phi_\mu(bw_2)$. Now, since L_{tm} has linear complexity [**Cas96**], it has zero topological entropy, and Theorem 1 applies. We then obtain a conjugate IET $T_{L_{tm}}$ with $B_{L_{tm}}$ as set of separation points. By Proposition 3.1.7(d) and since $\{aw_2, bw_2\} \subset \phi^{-1}(\iota(\overline{I}))$ by definition of Z_0, we have $T_{L_{tm}}(\phi_\mu(aw_2)) = \phi_\mu(\sigma(aw_2)) = \phi_\mu(w_2) = \phi_\mu(\sigma(bw_2)) = T_{L_{tm}}(\phi_\mu(bw_2))$. Hence $T_{L_{tm}}$ is not injective.

Now, by Lemma 3.2.9, $\phi_\mu(aw_2)$ and $\phi_\mu(bw_2)$ must belong to the set of discontinuities $\mathcal{D}_{L_{tm},\mathrm{acc},r}$. Note that this is coherent with the fact that by Lemma 3.2.5 $\{aw_1, bw_1, aw_2, bw_2\} = W_{\mathrm{SP}_{L_{tm}}}$, so that $\{\phi_\mu(aw_2), \phi_\mu(bw_2)\} = B_{\mathrm{SP}_{L_{tm}}}$, and that by definition, $B_{\mathrm{SP}_{L_{tm}}} = B_{L_{tm},\mathrm{acc},r} \supseteq \mathcal{D}_{L_{tm},\mathrm{acc},r}$. ◊

Also, according to the construction behind Lemma 3.2.5, the set of separation points B_L is always infinite while including the set $\mathcal{D}_L$ which can be in fact finite (see e.g. Corollary 3.3.4 further).

Example 3.2.12. (The set $\mathcal{D}_L$ of T_L can be infinite) Let us consider again the Thue–Morse associated shift L_{tm} and its associated definitions. If there were only finitely many discontinuities for $T_{L_{tm}}$, we could assume there is an open interval $J \subset I$, having $\phi_\mu(aw_2)$ as the left endpoint, on which $T_{L_{tm}}$ is continuous. We could also assume J avoids the finite set $\phi_\mu(W_{\mathrm{SP}_{L_{tm}}})$, hence J would be contained in a subset of I where $T_{L_{tm}}$ is injective. But $T_{L_{tm}}(\phi_\mu(aw_2)) = T_{L_{tm}}(\phi_\mu(bw_2))$, so by right-continuity, $T_{L_{tm}}$ cannot be injective on J and there is no such interval. ◊

3.3. *Codings and reconstructions.* In this section, we describe some more relationships between L and T_L.

First, an IET T is said to be *minimal* if all its orbits are dense in I. Next, a *monotonicity partition* P of I with respect to T is a finite partition made of right-open intervals on each of which T is increasing, but not necessarily continuous. Recall then that by Lemma 3.2.2, the conjugate IETs given by Theorem 1 are piecewise increasing, thus admitting monotonicity partitions. The associated *coding* of the orbits of T with a partition P is based on a map α assigning a distinct letter of an alphabet A to each interval of P, so that

$$\begin{array}{rccl} \mathrm{cod}_{0,P}: & I & \to & A \\ & x & \mapsto & \alpha(J) \quad \text{if } x \in J, \text{ with } J \in P, \end{array}$$

which is extended as

$$\begin{array}{rccl} \mathrm{cod}_P: & I & \to & A^{\mathbb{N}} \\ & x & \mapsto & \mathrm{cod}_{0,P}(x)\,\mathrm{cod}_{0,P}(T(x))\,\mathrm{cod}_{0,P}(T^2(x))\ldots \end{array}$$

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The word $\mathrm{cod}_P(x)$ is called the *symbolic orbit* of x by cod_P, and the closure $\overline{\mathrm{cod}_P(I)}$, denoted by L_P, is called the *associated shift* of cod_P. A lexicographic order on L_P is induced by the order over A determined by the order of the component intervals of P on I, that is, $\mathrm{cod}_{0,P}(x) < \mathrm{cod}_{0,P}(x')$, if $x \in J$, $x' \in J'$ with $J < J'$, i.e. J occurs before J' on I.

The next two technical lemmas make it possible to apply Theorem 1 to L_P.

LEMMA 3.3.1. *Let T be a piecewise increasing IET, and let P be a monotonicity partition of I. Then* cod_P *is non-decreasing, Borel measurable, and right-continuous. Moreover, if T is minimal,* cod_P *is increasing.*

Proof. Let $x, x' \in I$, with $x < x'$. If $x \in J$ and $x' \in J$, where J, J' are intervals of P with $J < J'$, we have $\mathrm{cod}_{0,P}(x) < \mathrm{cod}_{0,P}(x')$, hence $\mathrm{cod}_P(x) < \mathrm{cod}_P(x')$. Otherwise, $J = J'$, so that $\mathrm{cod}_{0,P}(x) = \mathrm{cod}_{0,P}(x')$. Since on the intervals of P, we have $T(x) < T(x')$, we can inductively apply the same argument until for some $n > 0$, $\mathrm{cod}_{0,P}(T^n(x)) < \mathrm{cod}_{0,P}(T^n(x'))$, in which case $\mathrm{cod}_P(x) < \mathrm{cod}_P(x')$, and if there is no such n, then $\mathrm{cod}_P(x) = \mathrm{cod}_P(x')$. Hence, cod_P is non-decreasing. If T is minimal, the above n always exists since $T^n(x)$ can be arbitrarily close to the left end of an interval in P, and as long as both $T^n(x)$ and $T^n(x')$ are in the same interval of P, we have $T^n(x') - T^n(x) \geq x' - x$. Hence, $\mathrm{cod}_P(x) < \mathrm{cod}_P(x')$.

The map cod_P is the product of the maps $\mathrm{cod}_{0,P} \circ T^i$, $i \geq 0$, each being continuous on the interiors of the intervals of a finite partition of I, since T^i is piecewise increasing. Hence cod_P is Borel measurable. Likewise, each $\mathrm{cod}_{0,P} \circ T^i$ is right-continuous since cod_P and T^i are right-continuous, hence cod_P is too. □

From the above lemma, the image by cod_P of the Lebesgue measure on I is a Borel measure on L_P that we denote by μ_P. Since cod_P conjugates σ and T, μ_P is invariant by σ. Also, as cod_P is right-continuous, when T is minimal, L_P is aperiodic minimal, and μ_P is non-atomic and positive on the cylinder sets of L_P.

LEMMA 3.3.2. *Let T be a piecewise increasing and minimal IET. Let P be a monotonicity partition of I, and let L_P be the associated shift. Then* $\mu_P(\mathrm{SP}_{L_P}) = 0$.

Proof. By definition of μ_P for all $x \in I$, $\mu_P([w_{L_P,\min}, \mathrm{cod}_P(x)])$ is equal to the Lebesgue measure of $\mathrm{cod}_P^{-1}([w_{L_P,\min}, \mathrm{cod}_P(x)))$, that is, of $[0, x)$, since by Lemma 3.3.1 cod_P is increasing. Thus, using the map ϕ_{μ_P} on L_P for all $i \in I$,

$$\phi_{\mu_P}(\mathrm{cod}_P(x)) = x. \tag{8}$$

By Lemma 3.1.5 and by definition of Z_0, recall that if $x \notin Z_0$, $\phi_{\mu_P}^{-1}(x)$ is well-defined, and by equality (8) is equal to $\mathrm{cod}_P(x)$. If $x \in Z_0$, $\phi_{\mu_P}^{-1}(x)$ consists of two words, one of them being $\mathrm{cod}_P(x)$. Accordingly, $\mathrm{cod}_P(I)$ misses some words of L_P, but $L_P \setminus \mathrm{cod}_P(I)$ is countable, since Z_0 is countable.

Let us now estimate the measure of $\sigma^{-1}(\mathrm{SP}_{L_P})$. We define the subset $W = \sigma^{-1}(\mathrm{SP}_{L_P}) \cap (L_P \setminus \mathrm{cod}_P(I))$, which is countable too. Let $w_0 \in \sigma^{-1}(\mathrm{SP}_{L_P})$. By definition of SP_{L_P}, $\{w_0\} \subsetneq \sigma^{-1}(\sigma(w_0))$, and there are three cases.

- $w_0 \in W$: thus $w_0 \in \sigma^{-1}(\sigma(W))$.

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- $w_0 \notin W$ and there is $w_1 \in \sigma^{-1}(\sigma(w_0))$ such that $w_1 \in W$. Thus $w_0 \in \sigma^{-1}(\sigma(W))$ too, since $\sigma(w_0) = \sigma(w_1)$.
- $w_0 \notin W$ and there is no $w_1 \in \sigma^{-1}(\sigma(w_0))$ such that $w_1 \in W$. Thus there are $x_0, x_1 \in I$ such that $\{w_0, w_1\} \subset \sigma^{-1}(\sigma(w_0))$ with $w_0 = \mathrm{cod}_P(x_0)$, $w_1 = \mathrm{cod}_P(x_1)$, and $x_0 \neq x_1$ since $w_0 \neq w_1$. By definition of cod_P, for all $x \in I$,

$$\mathrm{cod}_P(T(x)) = \sigma(\mathrm{cod}_P(x)). \tag{9}$$

  Since $\sigma(\mathrm{cod}_P(x_0)) = \sigma(\mathrm{cod}_P(x_1))$, then $\mathrm{cod}_P(T(x_0)) = \mathrm{cod}_P(T(x_1))$, that is, since cod_P is injective, $T(x_0) = T(x_1)$. Hence according to Definition 2, $x_0, x_1 \in \mathcal{D}_{\mathrm{acc},r}$, and $w_0 \in \sigma^{-1}(\mathrm{cod}_P(\mathcal{D}_{\mathrm{acc},r}))$.

Thus $\sigma^{-1}(\mathrm{SP}_{L_P}) \subseteq \sigma^{-1}(\sigma(W)) \cup \sigma^{-1}(\mathrm{cod}_P(\mathcal{D}_{\mathrm{acc},r}))$. Then, $\mu_P(\sigma^{-1}(\sigma(W)) = 0$, since $\sigma^{-1}(\sigma(W))$ is countable and μ_P is non-atomic. Next, $\mu_P(\sigma^{-1}(\mathrm{cod}_P(\mathcal{D}_{\mathrm{acc},r}))) = \mu_P(\mathrm{cod}_P(\mathcal{D}_{\mathrm{acc},r}))$ since μ_P is preserved by σ. Now, since μ_P is the image of the Lebesgue measure on I by cod_P and $\mathcal{D}_{\mathrm{acc},r}$ has measure 0, T being an IET, $\mu_P(\mathrm{cod}_P(\mathcal{D}_{\mathrm{acc},r})) = 0$, so $\mu_P(\sigma^{-1}(\mathrm{cod}_P(\mathcal{D}_{\mathrm{acc},r}))) = 0$ too. Hence $\mu_P(\sigma^{-1}(\mathrm{SP}_{L_P})) = 0$, and again by measure preservation, $\mu_P(\mathrm{SP}_{L_P}) = 0$. □

Then, given a monotonicity partition P, Lemma 3.2.5 applies to L_P measured by μ_P, so that Theorem 1 applies to L_P too, and we can get a conjugate IET T_{L_P}.

PROPOSITION 3.3.3. *Let T be a piecewise increasing and minimal IET. Let P be a monotonicity partition of I, and let L_P be the associated shift with its conjugate IET T_{L_P}. Then $T_{L_P} = T$, i.e.*

$$\textit{for every monotonicity partition } P : T \xrightarrow{\mathrm{cod}_P} L_P \xrightarrow{\text{Theorem 1}} T_{L_P} = T.$$

Proof. In equality (8), $T(x)$ can replace x, and by using (9), for all $x \in I$, we get

$$T(x) = \phi_{\mu_P}(\mathrm{cod}_P(T(x))) = \phi_{\mu_P}(\sigma(\mathrm{cod}_P(x))).$$

Also, right-continuity of cod_P (see Lemma 3.3.1) means that for every $x \in I$, $\mathrm{cod}_P(x)$ is the infimum of $\{\mathrm{cod}_P(y) \mid y > x,\ y \in I\}$. Thus $\mathrm{cod}_P(x)$ cannot be the right endpoint of a cylinder in L_P, hence $\mathrm{cod}_P(x) \in \phi^{-1}(\iota(I))$. In fact, since $\phi_{\mu_P} = \kappa \circ \phi$, and since $\iota \circ \kappa$ is the identity on $\phi(\mathrm{cod}_P(I))$, we get $\mathrm{cod}_P(x) = \phi^{-1}(\iota(x))$ by composing equality (8) to the left with $\phi^{-1} \circ \iota$. By Proposition 3.1.7(d), $T_{L_P}(\phi_{\mu_P}(w)) = \phi_{\mu_P}(\sigma(w))$ on $\phi^{-1}(\iota(I))$, thus with $w = \phi^{-1}(\iota(x))$, for all $x \in I$,

$$T_{L_P}(x) = \phi_{\mu_P}(\sigma(\mathrm{cod}_P(x))),$$

whence T and T_{L_P} coincide on I. □

Of course the above result applies to minimal *finite* IETs, as these are piecewise increasing by definition. In this case, the partition P can be the one induced by the set $\mathcal{D}$ of discontinuities, or by any finite $B \supseteq \mathcal{D}$. Then, cod_P becomes the usual *natural coding* of the orbits of T with B as set of separation points [**Kea75**], that is, a coding based on the assignment of a distinct letter to each component of a partition on which T is continuous in the interior of these components.

COROLLARY 3.3.4. *Let T be a finite minimal IET. Let L be the associated shift given by a natural coding of T, with its conjugate IET T_L. Then $T = T_L$.*

Note that in this finite IET case, T is reobtained as T_L from L, but according to the construction behind Lemma 3.2.5, the set of separation points B_L is infinite, whereas $\mathcal{D} \subset B_L$ is finite (see Example 5.2.4 (iv) further).

A shift L can also be recovered from T_L. Recall that P_A is the monotonicity partition of I for T_L relying on the letters of A (see Lemma 3.2.2).

PROPOSITION 3.3.5. *Let L be any shift with a conjugate IET T_L (Theorem 1 applies to L). Then for every $x \in I$, $\mathrm{cod}_{P_A}(x) = \phi^{-1}(\iota(x))$, so that $L_{P_A} = L$, i.e.*

$$\textit{for the monotonicity partition } P_A : L \xrightarrow{\text{Theorem 1}} T_{L_{P_A}} \xrightarrow{\mathrm{cod}_{P_A}} L_{P_A} = L.$$

Proof. For $x \in I$, by definition of cod_{0,P_A} and P_A, $\mathrm{cod}_{P_A}(x) = a$ when $x \in I_{L,a}$, since $\alpha(I_{L,a}) = a$. Next, by definition of $I_{L,a}$, $\phi^{-1}(\iota(x)) \in \mathrm{Cyl}(a)$, that is, the first letter of $\mathrm{cod}_{P_A}(x)$ and of $\phi^{-1}(\iota(x))$ are equal. To check equality on their second letters, that is, the first letter of $\sigma(\mathrm{cod}_{P_A}(x))$ and of $\sigma(\phi^{-1}(\iota(x)))$, consider that since x above can be arbitrary, the first letter of $\mathrm{cod}_{P_A}(T_L(x))$ and of $\phi^{-1}(\iota(T_L(x)))$ are equal too. Then by equality (9) $\mathrm{cod}_{P_A}(T_L(x)) = \sigma(\mathrm{cod}_{P_A}(x))$, and by Proposition 3.1.7 (a,b) $\phi^{-1}(\iota(T_L(x))) = \sigma(\phi^{-1}(\iota(x)))$. By inductively applying the same arguments, the equality holds for the next letters. Hence $\mathrm{cod}_{P_A}(I) = \phi^{-1}(\iota(I))$. □

If an IET is given with an infinite set $B \supseteq \mathcal{D}$ of separation points, its natural coding is such that the coding alphabet must be then infinite too. However, in the case of T_L with B_L as set of separation points, this coding is related to L, that is, to a language over a finite alphabet. Indeed, let A_∞ be an infinite alphabet and let α_∞ be a map assigning a distinct letter of A_∞ to each component of the partition $\mathrm{PART}_{L,I}$ (cf. p. 1946) so that its natural coding is defined as

$$\begin{array}{lcl} \mathrm{cod}_{0,\infty}: & I & \to & A_\infty \\ & x & \mapsto & \begin{cases} \alpha_\infty(I_L^{(k)}) & \text{if } x \in I_L^{(k)}, \\ \alpha_\infty(x) & \text{if } x \in B_{\mathrm{SP}_L}. \end{cases} \\ \mathrm{cod}_\infty: & I & \to & A_\infty^{\mathbb{N}} \\ & x & \mapsto & \mathrm{cod}_{0,\infty}(x)\,\mathrm{cod}_{0,\infty}(T_L(x))\,\mathrm{cod}_{0,\infty}(T_L^2(x))\ldots \end{array}$$

Recall now that the partition PART_L of L (cf. p. 1944) is made of the cylinders determined by the finite words $v^{(k)}$ in Fact_L and by the infinite words in W_{SP_L}. We consider then the following injective recoding map of the letters of A_∞:

$$\begin{array}{lcl} \zeta_\infty: & A_\infty & \to & \mathrm{Fact}_L \cup W_{\mathrm{SP}_L} \\ & a & \mapsto & \begin{cases} v^{(k)} & \text{if } a = \alpha_\infty(I_L^{(k)}), \\ w & \text{if } a = \alpha_\infty(x), \text{ where } x \in B_{\mathrm{SP}_L} \text{ and } \phi^{-1}(\iota(x)) = w. \end{cases} \end{array}$$

Next, we define

$$\delta_{0,\infty} : L \to A_\infty,$$

where for each $w \in L$:

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- $\delta_{0,\infty}(w) = (\zeta_\infty)^{-1}(au)$, if au occurs as a prefix of w, with $a \in A$, and $u \in \mathrm{Fact}_L$ is non-left special, that is, by the construction of PART_L in Lemma 3.2.5, au is a factor $v^{(k)}$ such that $\mathrm{Cyl}_L(v^{(k)}) \in \mathrm{PART}_L$;
- $\delta_{0,\infty}(w) = (\zeta_\infty)^{-1}(w)$, if w is of the form $w = aw'$, with $a \in A$, and w' is an infinite left special word in SP_L, that is, $w \in W_{\mathrm{SP}_L}$.

The map $\delta_{0,\infty}$ is well-defined firstly because the two above cases cover L and are independent, reflecting how PART_L is built, and secondly because, if au exists, it is unique since the $v^{(k)}$ determining the cylinders of PART_L can never be prefixes to each other (these cylinders do not overlap). Then, $\delta_{0,\infty}$ is extended as follows:

$$\begin{array}{rccl} \delta_\infty : & L & \to & A_\infty^{\mathbb{N}} \\ & w & \mapsto & \delta_{0,\infty}(w)\,\delta_{0,\infty}(\sigma(w))\,\delta_{0,\infty}(\sigma^2(w))\ldots \end{array}$$

We can now map the words in L to symbolic orbits of T_L obtained by cod_∞.

PROPOSITION 3.3.6. *Let L be any shift with a conjugate IET T_L (Theorem 1 applies to L). Then for every $x \in I$, $\mathrm{cod}_\infty(x) = \delta_\infty(\phi^{-1}(\iota(x)))$, i.e.*

$$L \overset{\text{Theorem 1}}{\longrightarrow} T_L \overset{\mathrm{cod}_\infty}{\longrightarrow} \mathrm{cod}_\infty(I) = \delta_\infty(\mathrm{cod}_{P_A}(I)).$$

Proof. For $x \in I$, the first letter of $\mathrm{cod}_\infty(x)$ and of $\delta_\infty(w)$, where $w = \phi^{-1}(\iota(x))$, are equal. Indeed, since w belongs to some $\mathrm{Cyl}(v^{(k)})$ or to W_{SP_L}, $\delta_{0,\infty}(w)$ recovers the letter assigned by α_∞ to the component of $\mathrm{PART}_{L,I}$ containing x, i.e. $\mathrm{cod}_\infty(x)$. Next, $\sigma(\mathrm{cod}_\infty(x)) = \mathrm{cod}_\infty(T_L(x))$ and $\sigma(\delta_\infty(\phi^{-1}(\iota(x)))) = \delta_\infty(\sigma(\phi^{-1}(\iota(x)))) = \delta_\infty(\phi^{-1}(\iota(T_L(x))))$ so that the second letters are also equal. By inductively applying the same arguments, we obtain $\mathrm{cod}_\infty(x) = \delta_\infty(\phi^{-1}(\iota(x)))$. Hence $\mathrm{cod}_\infty(I) = \delta_\infty(\phi^{-1}(\iota(I)))$, and by Proposition 3.3.5, $\delta_\infty(\phi^{-1}(\iota(I))) = \delta_\infty(\mathrm{cod}_{P_A}(I))$. □

This mapping of L to the natural coding of T_L will be in use in the next section.

4. *The linear complexity shift case*

We now focus on shifts L with *linear complexity*, i.e. those such that $p_L(n) = O(n)$, and prove Theorem 2. For instance, aperiodic minimal shifts associated with primitive substitutions have linear complexity [**Pan84**, **Qué10**], e.g. the shift associated with the Thue–Morse substitution. Since these shifts form a specific family of shifts with zero topological entropy, the idea behind Theorem 2 is to investigate the properties of their conjugate IETs as given by Theorem 1.

4.1. *Almost finite interval exchange transformations.*

We begin by defining a specific kind of IETs.

Definition 3. An IET T with a set $\mathcal{D}$ of discontinuities is *almost finite* if:
(1) T is piecewise increasing on I;
(2) the set $\mathcal{D}_{\mathrm{acc}}$ of accumulation points of $\mathcal{D}$ is finite;
(3) the points in $\mathcal{D}$ belong to a finite number of full orbits of T†.

† A sufficient condition for minimality of a finite IET is that the full orbits of the discontinuity points are infinite and distinct [**Kea75**] (cf. the *infinite distinct orbit condition* (IDOC)). An almost finite IET with an infinite number of discontinuities is thus far from satisfying the IDOC.

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In the above definition—like in Definitions 1 and 2—$\mathcal{D}$ can be replaced by a null-measure set of separation points B such that $B \supset \mathcal{D}$. Note also that a finite IET is almost finite. Now, the first property of almost finiteness is always satisfied by a conjugate IET T_L (see Lemma 3.2.2). About the second property, when L has linear complexity, we first show the following technical result.

LEMMA 4.1.1. *Let L be a minimal shift with linear complexity. Then* SP_L *is finite.*

Proof. For each finite $r \leq |\mathrm{SP}_L|$, there is $n_0 > 0$ such that for all $n \geq n_0$, the number of distinct length-n prefixes of the words in SP_L is at least r. If $\mathrm{sp}_{L,l}(n)$ denotes the number of left special factors in $\mathrm{Fact}_L(n)$, then $\mathrm{sp}_{L,l}(n) \geq r$. Since L is minimal, Fact_L is prolongable, and thus for all $n \geq 1$, $p_L(n+1) - p_L(n) \geq \mathrm{sp}_{L,l}(n)$. Therefore, if SP_L was infinite, for every $r \in \mathbb{N}$, there would exist $n_0 > 0$ such that for all $n \geq n_0$, $p_L(n+1) - p_L(n) \geq r$, whence $p_L(n) \neq O(n)$. □

Then we have the following.

LEMMA 4.1.2. *Let L be an aperiodic minimal shift with linear complexity, and let T_L be its conjugate IET. Then the set $\mathcal{D}_{L,\mathrm{acc}}$ of the accumulation points of the set $\mathcal{D}_L$ of the discontinuities of T_L is finite.*

Proof. By definition, $\mathcal{D}_L \subseteq B_L$. Next, $B_{L,\mathrm{acc}}$ is determined by W_{SP_L} since $\phi_\mu(W_{\mathrm{SP}_L}) = B_{L,\mathrm{acc}}$, and we also have $W_{\mathrm{SP}_L} = \sigma^{-1}(\mathrm{SP}_L)$. The alphabet A is finite, and by Lemma 4.1.1 SP_L is too, whence the result. □

In order to prove the third property of almost finiteness for T_L, the idea is to study similar relationships to known ones for finite IETs, between natural codings, complexity of the associated shifts, and distinctiveness of the orbits of the separation points [**Kea75**]. However, as seen in §3.3, a natural coding cod_∞ of an infinite IET T uses an infinite alphabet A_∞. To deal with this situation, we shall code the orbits of T using finer and finer *finite* partitions of I, together with finite alphabets with more and more letters, as is developed in the next section.

4.2. *A finite number of full orbits.* For an IET T with $B \supset \mathcal{D}$ as set of separation points, let B_{acc} denote the set of accumulation points of B in $\overline{I}$, and let $B_{\mathrm{acc}} = B^1_{\mathrm{acc}} \sqcup B^2_{\mathrm{acc}}$, where B^1_{acc} denotes the one-sided accumulation points, and B^2_{acc} the two-sided ones. An ordered finite subset $B_m = \{0 = b_0, b_1, \ldots, b_{m-1}\}$, $m > 0$, of B is said to be *admissible* if for $B_m \cup \{b_m = 1\}$, each induced interval $I_{m,i} = [b_i, b_{i+1})$ has one of the following types.

(1) $I_{m,i}$ is an interval of T with respect to B (b_i is the only point of B in $I_{m,i}$).

(2) $I_{m,i}$ covers infinitely many consecutive intervals of T, and either:

  (a) $[b_i, b_{i+1}]$ has exactly one point of B_{acc}, belonging to B^1_{acc}, either b_i or b_{i+1};

  (b) (b_i, b_{i+1}) has exactly one point of B_{acc}, belonging to B^2_{acc}, with $b_i, b_{i+1} \notin B_{\mathrm{acc}}$.

Thus, an admissible B_m induces a finite partition $\bigsqcup_i I_{m,i}$ of I, where for each point $x \in B_{L,\mathrm{acc}}$, there is exactly one I_{m,i_x} such that almost every point of any sequence of points of B converging to x lies in I_{m,i_x}. For instance, here is a possible subset B_4 of a set B with two accumulation points $x_1 \in B^1_{\mathrm{acc}}$ and $x_2 \in B^2_{\mathrm{acc}}$, such that the induced intervals $I_{4,0}$, $I_{4,2}$ are of type 1, $I_{4,1}$ is of type 2a, and $I_{4,3}$ is of type 2b.

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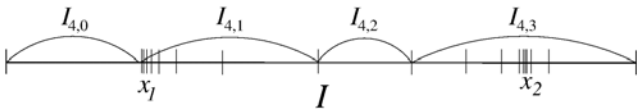

LEMMA 4.2.1. *Let B be a set of separation points. Then B_{acc} is finite if and only if for some $m > 0$, there is an admissible $B_m \subset B$.*

Proof. ($\Leftarrow$) Trivial. ($\Rightarrow$) First, for each point $x \in B^2_{\mathrm{acc}}$ there is an interval of type 2b containing it, because such an x is the intersection of a sequence of closed nested intervals with endpoints in B, and if every interval in the sequence contained a point in B_{acc}, then B_{acc} would be infinite. Next, by the same argument, each $x \in B^1_{\mathrm{acc}}$, which is the intersection of a sequence of nested closed intervals to its right, is the left endpoint of an interval of type 2a, while in its other side, x is the right endpoint of an interval of type 1, or else $x = 0$. The same occurs the other way around when the accumulation is on the left. Then there are finitely many points of B in the complement of the union of the above intervals and together with the endpoints of those intervals, they form an admissible B_m. □

Given an admissible set B_m, let A_m be an alphabet with m letters, and let α_m be a map assigning a distinct letter in A_m to each $I_{m,i}$ induced by B_m, so that

$$\begin{array}{rccl} \mathrm{cod}_{0,m}: & I & \to & A_m \\ & x & \mapsto & \alpha_m(I_{m,i}), \quad \text{if } x \in I_{m,i}. \end{array}$$

$$\begin{array}{rccl} \mathrm{cod}_m: & I & \to & A_m^{\mathbb{N}} \\ & x & \mapsto & \mathrm{cod}_{0,m}(x)\,\mathrm{cod}_{0,m}(T(x))\,\mathrm{cod}_{0,m}(T^2(x))\ldots \end{array}$$

The *associated shift* L_m of cod_m is then defined as $\overline{\mathrm{cod}_m(I)}$, as usual.

Now, let L be a shift over A, with a conjugate IET T_L having B_L as set of separation points, and $B_{L,\mathrm{acc}} = B^1_{L,\mathrm{acc}} \sqcup B^2_{L,\mathrm{acc}}$ as accumulation point sets. Let $B_{L,m}$ be an admissible subset of B_L, inducing a finite partition $\bigsqcup_i I_{L,m,i}$ of I. Then, we define a recoding map similar to and based on ζ_∞ (see p. 1952), sending each letter of A_m to words over A, which are either $v^{(k)}$ or common prefixes of sets of $v^{(k)}$ (the $v^{(k)}$ determining the cylinder components of PART_L):

$$\zeta_m : A_m \to \mathrm{Fact}_L \cup (\mathrm{Fact}_L \times \mathrm{Fact}_L),$$

where for each $a \in A_m$:

(1) $\zeta_m(a) = \zeta_\infty(\alpha_\infty(I_{L,m,i}))$, if $a = \alpha_m(I_{L,m,i})$ where $I_{L,m,i}$ is of type 1, that is, ζ_m yields the same $v^{(k)}$ as ζ_∞ if a is assigned to a type 1 interval;

(2) (a) $\zeta_m(a) = v_{\mathrm{pref}}$, if $a = \alpha_m(I_{L,m,i})$ where $I_{L,m,i}$ is of type 2a, and v_{pref} is the longest common prefix of $\{\zeta_\infty(\alpha_\infty(J)) | J \in \mathrm{PART}_{L,I},\ J \subset I_{L,m,i}\}$, that is, ζ_m yields the longest common prefix of the $v^{(k)}$ given by ζ_∞ applied to all the intervals covered by $I_{L,m,i}$;

(b) $\zeta_m(a) = (v_{\mathrm{pref},l}, v_{\mathrm{pref},r})$, if $a = \alpha_m(I_{L,m,i})$ where $I_{L,m,i}$ of type 2b, and $v_{\mathrm{pref},l}$, $v_{\mathrm{pref},r}$ are defined as follows: let $I_{L,m,i} = [b_i, b_{i+1})$ be divided into $J_l = [b_i, x)$ and $J_r = [x, b_{i+1})$, where $x \in B^2_{L,\mathrm{acc}}$ is the accumulation point in $I_{L,m,i}$; then $v_{\mathrm{pref},l}$ (respectively $v_{\mathrm{pref},r}$) is the longest common prefix of $\{\zeta_\infty(\alpha_\infty(J)) \mid J \in \mathrm{PART}_{L,I},\ J \in J_l\}$ (respectively $J \in J_r$).

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The case 2(b) above deals with the possible existence of points $x \in B^2_{L,\mathrm{acc}}$ such that $\phi_\mu^{-1}(x)$ is made of two consecutive distinct words in W_{SP_L}.

We show then that if m is sufficiently large, the map ζ_m can be used like ζ_∞ to define a map δ_m similar to δ_∞. First of all, we say a sequence $\{B_m\}_{m\geq m_0}$, with $m_0 > 0$, is *admissible* if for all $m \geq m_0$: (i) B_m is an admissible subset of B; (ii) $|B_{m+1}| = |B_m| + 1$; (iii) $B = \bigcup_{m\geq m_0} B_m \cup (B_{\mathrm{acc}} \setminus \{1\})$.

LEMMA 4.2.2. *Let B be a set of separation points. Then B_{acc} is finite if and only if for some $m_0 > 0$, there is an admissible sequence $\{B_m\}_{m\geq m_0}$.*

Proof. ($\Leftarrow$) A consequence of Lemma 4.2.1. ($\Rightarrow$) By Lemma 4.2.1 also, for some $m_0 > 0$ there is an admissible B_{m_0} of B. If the first point x of B_{acc} in I lies in an interval $I_{m_0,i}$ of type 2b, we add to B_{m_0} the smallest point of B in the interior of $I_{m_0,i}$ to obtain B_{m_0+1}, then the greatest one to obtain B_{m_0+2}. If x lies in $I_{m_0,i}$ of type 2a we do the same, but using the smallest or the greatest point of B in the interior of $I_{m_0,i}$, according to whether the point is an accumulation point on the left or the right. Then we proceed to the next point of B_{acc}, and so on, going back to the first one after having processed the last one. □

We denote by $F_{m,1}$ (respectively $F_{m,2}$) the set of the $v^{(k)}$ (respectively the prefixes of the $v^{(k)}$) given by $\zeta_m(\alpha_m(I_{L,m,i}))$ when $I_{L,m,i}$ is of type 1 (respectively type 2, including $v_{\mathrm{pref},l}$ and $v_{\mathrm{pref},r}$ as distinct words when $I_{L,m,i}$ is of type 2b). The technicalities in the proof of the following lemma come from the topological differences between L and I.

LEMMA 4.2.3. *Let L be a shift with a conjugate T_L and an admissible sequence $\{B_{L,m}\}_{m\geq m_0}$, $m_0 > 0$. Then there is $m_1 \geq m_0$, such that for all $m \geq m_1$, the map ζ_m is injective, such that for $i = 1, 2$, $F_{m,i}$ does not include the empty word, nor words prefix to each other, and no word in $F_{m,1}$ is a prefix of a word in $F_{m,2}$.*

Proof. By the construction of PART_L in Lemma 3.2.5, the set $F_{m,1}$, being made of $v^{(k)}$, has the first two required properties. Next, a word v in $F_{m,1}$ cannot be a prefix of any word in $F_{m,2}$, since v would be then a common prefix of words of type $v^{(k)}$, and again the $v^{(k)}$ are not prefixes to each other.

Now, from the point-of-view of L, each word $w \in W_{\mathrm{SP}_L}$ is an accumulation of endpoints of cylinder components of PART_L, and can be seen as the intersection of a sequence of nested cylinders of L (see again the proof of Lemma 3.2.5). Also, by Lemma 4.2.2, the set $B_{L,\mathrm{acc}}$ must be finite, so that W_{SP_L} is too. Therefore, there exists $n > 0$ such that at the nth step of the construction of PART_L, the nested cylinders to be refined at each further step are such that they contain only one word in W_{SP_L}, and such that they are all disjoint. Thus for each $w \in W_{\mathrm{SP}_L}$, there is a sequence of nested cylinders $\{C_{w,j}\}_{j\in\mathbb{N}^*}$ such that $\bigcap_{j\in\mathbb{N}^*} C_{w,j} = \{w\}$, disjoint from the other sequences, and having one of these two forms:

(a_L) $\{C_{w,j}\}_{j\in\mathbb{N}^*}$ is such that w is the smallest or the greatest element of every $C_{w,j}$, so that w is a one-sided accumulation word in L;

(b_L) $\{C_{w,j}\}_{j\in\mathbb{N}^*}$ is such that w belongs to the interior of every cylinder $C_{w,j}$, so that w is a two-sided accumulation word in L.

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From the point-of-view of I, the corresponding sequences $\{\phi_\mu(C_{w,j})\}_{j\in\mathbb{N}^*}$ are sequences of nested closed intervals with pairwise disjoint interiors, such that $\bigcap_{j\in\mathbb{N}^*}\phi_\mu(C_{w,j})=\{\phi_\mu(w)\}$, with $\phi_\mu(w)\in B_{L,\mathrm{acc}}$, taking one of these two forms:

(a_I) $\{\phi_\mu(C_{w,j})\}_{j\in\mathbb{N}^*}$ is such that $\phi_\mu(w)$ is equal to the right or the left endpoint of every interval $\phi_\mu(C_{w,j})$ so that $\phi_\mu(w)\in B^1_{L,\mathrm{acc}}$ or $B^2_{L,\mathrm{acc}}$;

(b_I) $\{\phi_\mu(C_{w,j})\}_{j\in\mathbb{N}^*}$ is such that $\phi_\mu(w)$ belongs to the interior of $\phi_\mu(C_{w,j})$, so that $\phi_\mu(w)\in B^2_{L,\mathrm{acc}}$.

From the point-of-view of $\{B_{L,m}\}_{m\geq m_0}$, for each $m\geq m_0$, every $x\in B_{L,\mathrm{acc}}$ is such that there is a $I_{L,m,i}$ of type 2 containing almost every point of any sequence in B_L converging to x, so that putting $J_{x,m}=\overline{I_{L,m,i}}$, we get a sequence of nested intervals $\{J_{x,m}\}_{m\geq m_0}$, with $\bigcap_{m\geq m_0}J_{x,m}=\{x\}$, taking one of these three forms:

(a_B) $\{J_{x,m}\}_{m\geq m_0}$ is made of type 2a intervals, i.e. $x\in B^1_{L,\mathrm{acc}}$, where $\phi_\mu^{-1}(x)=w$, $w\in W_{\mathrm{SP}_L}$. There is then a corresponding sequence $\{C_{w,j}\}_{j\in\mathbb{N}^*}$ of form a_L with intersection $\{w\}$, and $\{\phi_\mu(C_{w,j})\}_{j\in\mathbb{N}^*}$ of form a_I with intersection $\{x\}$;

(b_B) $\{J_{x,m}\}_{m\geq m_0}$ is made of type 2b intervals, i.e. $x\in B^2_{L,\mathrm{acc}}$, where either:

(i) $\phi_\mu^{-1}(x)=w$, with $w\in W_{\mathrm{SP}_L}$. There is then a corresponding sequence $\{C_{w,j}\}_{j\in\mathbb{N}^*}$ of form b_L with intersection $\{w\}$, and $\{\phi_\mu(C_{w,j})\}_{j\in\mathbb{N}^*}$ of form b_I with intersection $\{x\}$;

(ii) $\phi_\mu^{-1}(x)=\{w,w'\}$, with $w,w'\in W_{\mathrm{SP}_L}$. There are then corresponding sequences $\{C_{w,j}\}_{j\in\mathbb{N}^*}$, $\{C_{w',j}\}_{j\in\mathbb{N}^*}$ of form a_L, respectively with intersection $\{w\}$ and $\{w'\}$, and $\{\phi_\mu(C_{w,j})\}_{j\in\mathbb{N}^*}$, $\{\phi_\mu(C_{w',j})\}_{j\in\mathbb{N}^*}$ of form a_I, both with intersection $\{x\}$.

Now, since the sequences $\{\phi_\mu(C_{w,j})\}_{j\in\mathbb{N}^*}$ and $\{J_{x,m}\}_{m\geq m_0}$ have the same intersection point in $B_{L,\mathrm{acc}}$, and because of the properties of the $C_{w,1}$, there exists $m_1>0$, such that one of the following cases occur.

- $\{J_{x,m}\}_{m\geq m_0}$ is of form a_B or $\mathrm{b}_B\mathrm{i}$, and for all $m>m_1$, $J_{x,m}\subset\phi_\mu(C_{w,1})$, that is, the word determining the cylinder $C_{w,1}$ is a common prefix of the words in $\{\zeta_\infty(\alpha_\infty(J_{x,m}))\mid m>m_1\}$, i.e. a prefix of the corresponding $v_{\mathrm{pref}}\in F_{m,2}$.
- $\{J_{x,m}\}_{m\geq m_0}$ is of form $\mathrm{b}_B\mathrm{ii}$, and for all $m>m_1$, $J_{x,m}\subset\phi_\mu(C_{w,1})\cup\phi_\mu(C_{w',1})$, that is, the two words determining $C_{w,1}$ and $C_{w',1}$ are respective common prefixes of the factors in $\{\zeta_\infty(\alpha_\infty(J_l))\mid m>m_1\}$ and $\zeta_\infty(\alpha_\infty(J_r))\mid m>m_1\}$, where J_l and J_r are the two components of $J_{x,m}\setminus x$, i.e. prefixes of the corresponding $(v_{\mathrm{pref},l},v_{\mathrm{pref},r})$, with $v_{\mathrm{pref},l},v_{\mathrm{pref},r}\in F_{m,2}$.

Thus, for all $m\geq m_1$, since all the $C_{w,1}$ are disjoint, the set $F_{m,2}$ does not contain the empty word, nor words which are prefix to each other. □

Similarly to $\delta_{0,\infty}$, we can now define $\delta_{0,m}$ for each $m>m_1$, using the properties of ζ_m, so as to obtain letters of A_m out of the prefixes of the words in L:

$$\delta_{0,m}:L\to A_m,$$

where for each $w\in L$:

(1) $\delta_{0,m}(w)=(\zeta_m)^{-1}(au)$, if au occurs as a prefix of w in $F_{m,1}$, that is, au is a factor $v^{(k)}$ which belongs to $\zeta_m(A_m)$, and such that $\mathrm{Cyl}_L(v^{(k)})\in\mathrm{PART}_L$;

(2) or, one of the two following cases occurs:

- $\delta_{0,m}(w)=(\zeta_m)^{-1}(v_{\mathrm{pref}})$, if v_{pref} occurs as a prefix of w in $F_{m,2}$, which belongs as a single word to $\zeta_m(A_m)$;

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- $\delta_{0,m}(w) = (\zeta_m)^{-1}(v_{\text{pref},l}, v_{\text{pref},r})$, if $v_{\text{pref},l}$ or $v_{\text{pref},r}$ occurs as a prefix of w in $F_{m,2}$, which belongs to a pair $(v_{\text{pref},l}, v_{\text{pref},r})$ in $\zeta_m(A_m)$.

This map is well-defined because, firstly, all the above cases cover L by definition of $F_{m,1}$ and $F_{m,2}$, and secondly, the properties of $F_{m,1}$ and $F_{m,2}$ proved by Lemma 4.2.3 imply that taken in order, only one case applies for each $w \in L$. Accordingly, $\delta_{0,m}$ is extended as follows to obtain words in $A_m^{\mathbb{N}}$ from words in L:

$$\begin{array}{rccl} \delta_m : & L & \to & A_m^{\mathbb{N}} \\ & w & \mapsto & \delta_{0,m}(w)\,\delta_{0,m}(\sigma(w))\,\delta_{0,m}(\sigma^2(w))\ldots \end{array}$$

We then have a similar result to Proposition 3.3.6 (m_1 comes from Lemma 4.2.3).

LEMMA 4.2.4. *Let L be any shift with a conjugate T_L and an admissible sequence $\{B_{L,m}\}_{m\geq m_1}$. Then, for all $m \geq m_1$ and for all $x \in I$, $\text{cod}_m(x) = \delta_m(\phi^{-1}(\iota(x)))$, i.e.*

$$L \xrightarrow{\text{Theorem 1}} T_L \xrightarrow{\text{cod}_m} L_m = \delta_m(L).$$

Proof. For $x \in I$, the first letter of $\text{cod}_m(x)$ and of $\delta_m(w)$, where $w = \phi^{-1}(\iota(x))$, are equal. Indeed, w belongs to some cylinder determined by a word in $F_{m,1} \cup F_{m,2}$. Then, Case (1) of the $\delta_{0,m}(w)$ definition applies if and only if x belongs to a type 1 interval and $\delta_{0,m}(w)$ recovers the letter assigned by α_m to this interval, i.e. $\text{cod}_m(x)$. Otherwise, x belongs to a type 2 interval, and Case (2) of the $\delta_{0,m}(w)$ definition necessarily applies, with the same effect. This is well-defined since by Lemma 4.2.3 only words in $F_{m,2}$ can be prefixes of words in $F_{m,1}$. Next, similarly to Proposition 3.3.6, using the conjugacies given by Proposition 3.1.7, the same is true for the other letters of $\text{cod}_m(x)$ and $\delta_m(w)$. Hence $\text{cod}_m(I) = \delta_m(\phi^{-1}(\iota(I)))$. □

LEMMA 4.2.5. *Let L be any shift with a conjugate T_L and an admissible sequence $\{B_{L,m}\}_{m\geq m_1}$. Then, for all $m \geq m_1$, there exists $h_m > 0$ such that for all $n > 0$, $p_{L_m}(n) \leq p_L(n + h_m)$,*

Proof. Fix some $m \geq m_1$, and let $w \in L$ and $w' = \delta_m(w) \in L_m$. Let h_m be the maximum length of the factors in $F_{m,1} \cup F_{m,2}$. According to the definition of $\delta_{0,m}$ and to Lemma 4.2.4, for each $n > 0$, the length-$(n + h_m)$ prefix of w determines by δ_m at least one length-n prefix of w', hence, $p_{w'}(n) \leq p_w(n + h_m)$. Now, L being minimal, L_m is too, and w, w' are minimal words. Thus each of the factors in Fact_w, i.e. Fact_L, occurs as a prefix in the words in $\{\sigma^i(w)\}_{i\in\mathbb{N}}$, and these factors determine $\text{Fact}_{w'}$, i.e. Fact_{L_m}. Hence for all $n > 0$, $p_{L_m}(n) \leq p_L(n + h_m)$. □

LEMMA 4.2.6. *Let L be an aperiodic minimal shift with linear complexity. Let T_L be its conjugate IET. Then B_L belongs to a finite number of full orbits of T_L.*

Proof. According to the proof of Lemma 4.1.2, $B_{L,\text{acc}}$ is finite, thus according to Lemma 4.2.2 there is an admissible sequence $\{B_{L,m}\}_{m\geq m_1}$ of B_L, with $m_1 > 0$ given by Lemma 4.2.3. Fix some $m \geq m_1$ and consider the shift L_m associated with cod_m. For each $v \in \text{Fact}_{L_m}$, let $E_v = \{x \in I \mid \text{cod}_{0,m}(x)\text{cod}_{0,m}(T_L(x))\ldots\text{cod}_{0,m}(T_L^{|v|-1}(x)) = v\}$, i.e. the set of the points in I having symbolic orbits by cod_m with v as the prefix (note that E_v is not necessarily connected). For each $n > 0$, $\bigsqcup_{v\in\text{Fact}_{L_m}(n)} E_v$ is a partition of I, and the set

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$\{T_L^{-j}(B_{L,m}) \mid j = 0, \ldots, n-1\}$ defines the endpoints of the interval components of this partition. Then, for any point $x \in B_{L,m}$, if the set $T_L^{-n}(x)$ has a point y in the interior of some E_v with $v \in \mathrm{Fact}_{L_m}(n)$, it means that to each side of y there are components of E_{v_1} and E_{v_2}, with $v_1, v_2 \in \mathrm{Fact}_{L_m}(n+1)$ with v as the prefix, and $v_1 \neq v_2$. Thus v is a right special factor and it induces an increase in the complexity p_{L_m} of L_m.

Now, let $k_m \leq |B_{L,m}| = m$ be the number of distinct full orbits of $B_{L,m}$. Since L_m is aperiodic minimal, each of these orbits may contain at most one occurrence of each point in $B_{L,m}$. And since $B_{L,m}$ is finite, in each of these orbits there is at least $-j \in \mathbb{Z}^-$ for which a point of $B_{L,m}$ occurs in $\{T_L^{-j}(B_{L,m}) \mid j = 0, \ldots\}$. Thus there are at least k_m points in $B_{L,m}$ such that for all $n > 0$, their preimages by T_L^{-n} avoid $B_{L,m}$, and being in the interior of some E_v, they induce an increase of the number of the length-n factors. Hence, for all $m \geq m_1$ and for all $n > 0$, $k_m n \leq p_{L_m}(n)$. By Lemma 4.2.5, and since L has linear complexity, there exists $a > 1$ such that

$$p_{L_m}(n) \leq p_L(n + h_m) \leq a(n + h_m),$$

that is,

$$k_m \leq a + \frac{ah_m}{n}.$$

Since n is a free variable, we have $k_m \leq a$ for all $m \geq m_1$, and thus letting m go to infinity, this inequality holds also for B_L of T_L. □

We can then prove Theorem 2 as stated in the introduction.

Proof. Since L has linear complexity, Theorem 1 applies, and we get a conjugate IET T_L. By Lemmas 3.2.2 and 4.1.2, T_L has the two first properties of almost finiteness. Next, applying Lemma 4.2.6, and using the fact that $\mathcal{D}_L \subseteq B_L$, then T_L has also the third property of almost finiteness. □

5. *Building infinite interval exchange transformations*

In this section, we present results in the context of zero entropy about how to build effective approximations of the conjugate IETs T_L given by Theorem 1 for a set of corresponding σ-invariant measures on the shifts L.

5.1. *An approximation scheme.* Given a shift L over an ordered alphabet A, and its complexity p_L, we define a sequence of maps $\{T_n\}_{n>1}$, where each T_n is a map $I \to I$, $x \mapsto y = T_n(x)$, such that the source on the x-axis is divided into $p_L(n)$ right-open intervals of equal length, and the range on the y-axis is divided into $p_L(n-1)$ ones. The intervals of the source are then put into correspondence with the cylinders of the factors in $\mathrm{Fact}_L(n)$, using the lexicographic order between these factors, and the same is done for the intervals of the range with cylinders of the factors in $\mathrm{Fact}_L(n-1)$. Next, T_n is defined as the piecewise affine map which sends for each $v \in \mathrm{Fact}_L(n)$ the interval corresponding to $\mathrm{Cyl}_L(v)$ to the interval corresponding to $\mathrm{Cyl}_L(\sigma(v))$, where $\sigma(v)$ denotes v minus its first letter, by using a slope $p_L(n)/p_L(n-1)$.

First, here is a technical result about the above slopes converging to 1. Recall that $\mathrm{sp}_{L,l}(n)$ denotes the number of left special factors in $\mathrm{Fact}_L(n)$, and let $\mathrm{sp}_{L,r}(n)$ denote the same for the right special factors.

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LEMMA 5.1.1. *Let L be a shift such that* Fact_L *is prolongable. Let* $\{n_s\}_{s\in\mathbb{N}}$ *be a subsequence of* $\mathbb{N}\setminus\{0,1\}$. *Then* $\lim_{s\to\infty}(p_L(n_s)/p_L(n_s-1))=1$ *if and only if* $\lim_{s\to\infty}(\mathrm{sp}_{L,l}(n_s-1)/p_L(n_s-1))=0$. *The same holds for* $\mathrm{sp}_{L,r}(n_s-1)$.

Proof. We prove the result for $\mathrm{sp}_{L,l}$ (the proof for $\mathrm{sp}_{L,r}$ is similar): ($\Rightarrow$) $(p_L(n_s)/p_L(n_s-1))-1=(p_L(n_s)-p_L(n_s-1))/p_L(n_s-1)\geq \mathrm{sp}_{L,l}(n_s-1)/p_L(n_s-1)\geq 0$, ($\Leftarrow$) $|A|\,\mathrm{sp}_{L,l}(n_s-1)/p_L(n_s-1)\geq (p_L(n_s)-p_L(n_s-1))/p_L(n_s-1)=(p_L(n_s)/p_L(n_s-1))-1\geq 0$. □

For zero entropy, using subsequences is indeed useful.

LEMMA 5.1.2. *Let L be a minimal aperiodic shift with zero topological entropy. Then there is a subsequence* $\{n_s\}_{s\in\mathbb{N}}$ *of* $\mathbb{N}\setminus\{0,1\}$ *such that* $\lim_{s\to\infty}(p_L(n_s)/p_L(n_s-1))=1$.

Proof. Since L is minimal aperiodic, then for all $n>1$, $p_L(n)/p_L(n-1)>1$. Next, $p_L(n)/p_L(n-1)$ is not bounded away from 1. Assume on the contrary there is $c>0$ such that for all $n>1$, $p_L(n)\geq(1+c)p_L(n-1)$. Hence $p_L(n)\geq(1+c)^n$, and the topological entropy of L would not be less than $\log(1+c)$, a contradiction. □

Let $\mathrm{Fact}_L(u,n)=\{v\in\mathrm{Fact}_L(n)\mid v=uv',\ v'\in A^*\}$, i.e. the set of length-n factors with u as prefix, and let $p_L(u,n)=|\mathrm{Fact}_L(u,n)|$. In the next result we simultaneously build measures on L and approximations of the corresponding T_L.

PROPOSITION 5.1.3. *Let L be a minimal aperiodic shift with zero topological entropy. Let* $\{T_n\}_{n>1}$ *be the corresponding sequence of maps as defined above, and let* $\mathcal{T}=\{T_{n_s}\}_{s\in\mathbb{N}}$, *where* $\{n_s\}_{s\in\mathbb{N}}$ *is a subsequence of* $\mathbb{N}\setminus\{0,1\}$ *such that* $\lim_{s\to\infty}(p_L(n_s)/p_L(n_s-1))=1$. *Then:*

(1) *there is at least one shift-invariant measure μ on L, induced by the T_{n_s}'s interval lengths;*
(2) *there is a subsequence of $\mathcal{T}$ converging to a map T, equal almost everywhere to the conjugate IET T_L obtained by Theorem 1 from L measured by μ.*

Proof. (1) For every $s\in\mathbb{N}$ and for every $u\in\mathrm{Fact}_L$, we consider $p_L(u,n_s)/p_L(n_s)$, that is, the length of the interval corresponding to $\mathrm{Cyl}_L(u)$ on the x-axis of T_{n_s}. By a diagonal process on $\{p_L(u,n_s)/p_L(n_s)\}_{s\in\mathbb{N}}$, we obtain a sequence $Q=\{m_i\}_{i\in\mathbb{N}}$ such that, for every u, $\lim_{i\to\infty}(p_L(u,m_i)/p_L(m_i))$ exists, and we denote it by M_u. We then define μ on the cylinders of L by $\mu(\mathrm{Cyl}_L(u))=M_u$. Note that $\mu(L)=M_\epsilon=1$. The finite unions of the cylinders of $A^{\mathbb{N}}$ together with the empty cylinder form a semiring of sets, thus the intersection of this semiring with the cylinders of L too, and μ is a premeasure on this semiring. By Caratheodory's extension theorem, μ extends as a Borel probability measure on L.

Note that $\lim_{i\to\infty}(p_L(u,m_i)/p_L(m_i))=M_u$ implies that $\lim_{i\to\infty}(p_L(u,m_i-1)/p_L(m_i-1))=M_u$ too, that is, putting $d_i=(p_L(u,m_i)/p_L(m_i))-(p_L(u,m_i-1)/p_L(m_i-1))$, we have that $\lim_{i\to\infty}d_i=0$. Indeed, first, since L is aperiodic minimal, $p_L(m_i)>p_L(m_i-1)$ for every i, thus

$$d_i<\frac{p_L(u,m_i)}{p_L(m_i-1)}-\frac{p_L(u,m_i-1)}{p_L(m_i-1)}\leq|A|\frac{\mathrm{sp}_{L,l}(m_i-1)}{p_L(m_i-1)},$$

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which, by Lemma 5.1.1, converges to 0. Next, since $p_L(u, m_i) \geq p_L(u, m_i - 1)$,

$$d_i \geq \frac{p_L(u, m_i - 1)}{p_L(m_i)} - \frac{p_L(u, m_i - 1)}{p_L(m_i - 1)} = p_L(u, m_i - 1)\frac{p_L(m_i - 1) - p_L(m_i)}{p_L(m_i)p_L(m_i - 1)}$$
$$\geq -p_L(u, m_i - 1)|A|\frac{\mathrm{sp}_{L,l}(m_i - 1)}{p_L(m_i)p_L(m_i - 1)} = -|A|\frac{p_L(u, m_i - 1)}{p_L(m_i)}\frac{\mathrm{sp}_{L,l}(m_i - 1)}{p_L(m_i - 1)},$$

which also converges to 0, again by Lemma 5.1.1.

We now check the preservation of μ by σ. Assume that the letter prolongations in Fact_L of u to the left are $a_1, \ldots, a_h \in A$, where $h > 0$ since Fact_L is prolongable, L being minimal. Thus $\sigma^{-1}(\mathrm{Cyl}_L(u)) = \bigcup_{j=1}^{h} \mathrm{Cyl}_L(a_j u)$. For all $m_i \in Q$, we also have $\sum_{j=1}^{h} p_L(a_j u, m_i) \geq p_L(u, m_i - 1)$, and the difference between these two terms depends on the number of left special factors in $\mathrm{Fact}_L(u, m_i - 1)$ and on the number of their prolongations:

$$0 \leq \sum_{j=1}^{h} p_L(a_j u, m_i) - p_L(u, m_i - 1) \leq |A|\,\mathrm{sp}_{L,l}(m_i - 1). \tag{10}$$

Similarly, for each a_j, $p_L(a_j u, m_i) \geq p_L(a_j u, m_i - 1)$, and the difference between these terms depends on the number of right special factors in $\mathrm{Fact}_L(a_j u, m_i - 1)$:

$$0 \leq \sum_{j=1}^{h} p_L(a_j u, m_i) - \sum_{j=1}^{h} p_L(a_j u, m_i - 1) \leq |A|\,\mathrm{sp}_{L,r}(m_i - 1). \tag{11}$$

Reversing (11) and adding it to (10), for all $m_i \in Q$, we get

$$-|A|\,\mathrm{sp}_{L,r}(m_i - 1) \leq \sum_{j=1}^{h} p_L(a_j u, m_i - 1) - p_L(u, m_i - 1) \leq |A|\,\mathrm{sp}_{L,l}(m_i - 1).$$

We then divide the above terms by $p_L(m_i - 1)$, let i go to infinity, and using Lemma 5.1.1 again, we obtain $\mu(\mathrm{Cyl}_L(u)) = \sum_{j=1}^{h} \mu(\mathrm{Cyl}_L(a_j u))$. Hence μ is an invariant measure for L. Moreover, since L is aperiodic minimal, μ is non-atomic.

(2) On the x-axis for each $u \in \mathrm{Fact}_L$, the length of the interval corresponding to $\mathrm{Cyl}_L(u)$ converges to M_u as $\lim_{i\to\infty} (p_L(u, m_i)/p_L(m_i))$, and the same is true on the y-axis for $\sigma(u)$ with $M_{\sigma(u)}$ as $\lim_{i\to\infty} (p_L(\sigma(u), m_i - 1)/p_L(m_i - 1))$. Now, since L has zero topological entropy, and μ has all the requested properties, Theorem 1 applies to L measured by μ, so as to obtain T_L, with its associated partitions PART_L and $\mathrm{PART}_{L,I}$. The main components of PART_L are the cylinders $\mathrm{Cyl}_L(v^{(k)})$ (see Lemma 3.2.5), and following the construction of T_{m_i}, on the x-axis for each k, the associated intervals $I_{m_i}^{(k)}$ are the union of the intervals corresponding to $\mathrm{Cyl}_L(v^{(k)}v)$, $v^{(k)}v \in \mathrm{Fact}_L(m_i)$, then converging to an interval $I^{(k)}$ of length $M_{v^{(k)}}$. The same holds for the intervals $J_{m_i}^{(k)}$ in the y-axis with $\mathrm{Cyl}_L(\sigma(v^{(k)}v))$, respectively converging to intervals $J^{(k)}$ of length $M_{\sigma(v^{(k)})}$. Note then that for every i, T_{m_i} is continuous, hence is an affine map on each $I_{m_i}^{(k)}$. Indeed, σ is increasing on $\mathrm{Cyl}_L(\sigma(v^{(k)}))$, thus T_{m_i} on $I_{m_i}^{(k)}$ too. But if T_{m_i} had a discontinuity, it would mean that σ^{-1} is not well-defined in this cylinder, contradicting the definition of PART_L.

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Now, the left endpoint of $I^{(k)}$ is $o_k = \sum_{v<v^{(k)}, v\in \mathrm{Fact}_L(|v^{(k)}|)} M_v$, and the lower endpoint of $J^{(k)}$ is $o'_k = \sum_{v<\sigma(v^{(k)}), v\in \mathrm{Fact}_L(|v^{(k)}|-1)} M_v$. By convergence, for every k and $\varepsilon > 0$, there is $j > 0$ such that for all $i \geq j$, the endpoints of $I^{(k)}_{m_i}$ are at distances $< \varepsilon$ respectively from o_k and $o_k + M_{v^{(k)}}$. Thus let $T_{i,k,\varepsilon}$ be T_{m_i} restricted to $I^{(k)}_\epsilon = [o_k + \varepsilon, o_k + M_{v^{(k)}} - \varepsilon]$ for $i \geq j$, on which it is affine with slope $p_L(m_i)/p_L(m_i - 1)$. These slopes converge to 1, and the endpoints of $J^{(k)}_{m_i}$ converge respectively to o'_k and $o'_k + M_{\sigma(\mathrm{Cyl}_L(v^{(k)}))}$, hence $T_{i,k,\varepsilon}$ converges on $I^{(k)}_\epsilon$. Since ε is arbitrary, $\{T_{m_i}\}_{i\in\mathbb{N}}$ converges on all the $I^{(k)^o}$, i.e. the $I^{(k)}$ interiors. We put T as the limit map, so that for each $x \in I^{(k)^o}$, $T(x) = o'_k + (x - o_k)$. Now, in the proof of Lemma 3.2.8, $T_L(x) - x = K_{C^{(k)}}$, where $C^{(k)}$ denoted $\mathrm{Cyl}_L(v^{(k)})$, and where $K_{C^{(k)}} = \mu([w_{L,\min}, \sigma(w_{C^{(k)},\min})]) - \mu([w_{L,\min}, w_{C^{(k)},\min}])$. By the definition of T, the first term of this difference is o'_k and the second is o_k, that is, $T_L(x) - x = T(x) - x$ on $I^{(k)^o}$, thus by comparison with Lemma 3.2.8, $I^{(k)^o} = I^{(k)^o}_L$. Since by Lemma 3.2.10, $\mu(\mathrm{SP}_L) = 0$, the union of the $I^{(k)^o}$ has measure 1, whence $T = T_L$ almost everywhere. □

Note that the map T above can be extended to all of I by right-continuity, so that T can be made equal to T_L.

5.2. *Explicit graph examples.* A dynamical system (X, f) is said to be *uniquely ergodic* if there exists only one f-invariant Borel probability measure on X.

PROPOSITION 5.2.1. *Let L be an aperiodic minimal shift with linear complexity which is uniquely ergodic. Then $\mathcal{T}$ in Proposition 5.1.3, defined as $\{T_n\}_{n>1}$, converges almost everywhere to T_L.*

Proof. Since L has linear complexity, its topological entropy is zero. For the same reason, there exists $r > 0$ such that for all $n > 1$, $p_L(n) - p_L(n-1) \leq r$ [**Cas96**], hence $\lim_{n\to\infty} (p_L(n)/p_L(n-1)) = 1$, and Proposition 5.1.3 can be applied to L. Now, in the proof of this proposition, unique ergodicity means there can be only one accumulation point for each $\{p_L(u, n)/p_L(n)\}_{n>1}$, $u \in \mathrm{Fact}_L$. Thus these sequences converge without any subsequence extraction, as then does $\mathcal{T}$. □

In order to illustrate the above proposition, the simplest non-trivial examples are given by the primitive substitutive case (see p. 1945).

COROLLARY 5.2.2. *Let L be a shift associated with an aperiodic primitive substitution. Then Proposition 5.2.1 applies to L.*

Proof. Aperiodic primitive substitutions have associated shifts which are aperiodic minimal with linear complexity [**Pan84**], and uniquely ergodic [**Qué10**]. □

Note that in this primitive substitutive case, the limit measure of each cylinder can also be computed from the substitution itself [**Qué10**, § 5].

Example 5.2.3. Reconsider the Thue–Morse substitution θ_{tm} (see Examples 3.2.6, 3.2.11, and 3.2.12). Since θ_{tm} is aperiodic primitive, Corollary 5.2.2 applies to its associated shift L_{tm}. In the following picture we show the graph of T_{100} from the converging sequence $\mathcal{T} = \{T_n\}_{n>1}$ approximating the IET $T_{L_{tm}}$ with $B_{L_{tm}}$ as set of separation points.

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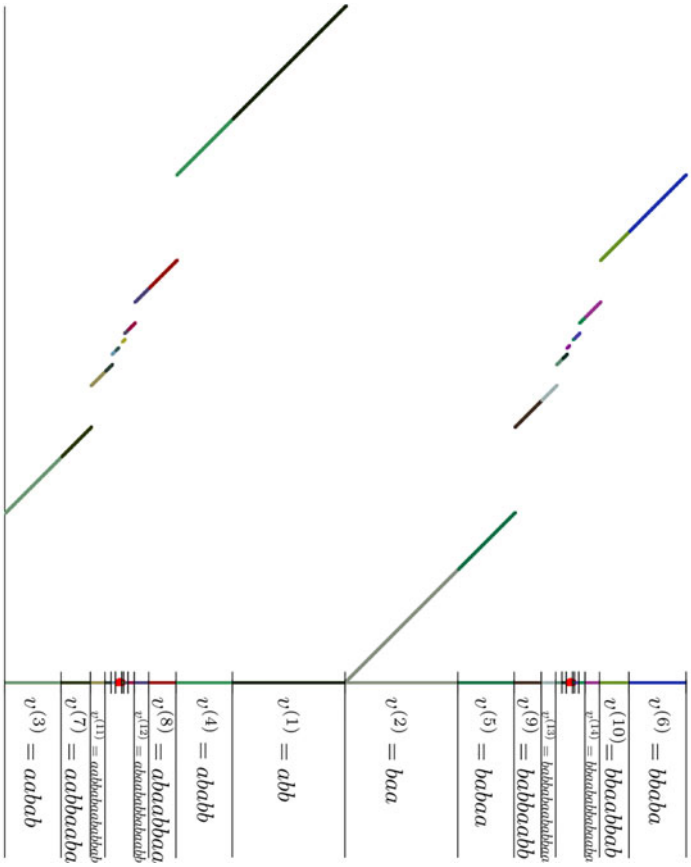
The intervals converging to their respective $I^{(k)}_{L_{tm}}$ of $\mathrm{PART}_{L_{tm},I}$ are indicated for the first k, by the associated $v^{(k)}$ determining the corresponding cylinders of $\mathrm{PART}_{L_{tm}}$ (see Lemma 3.2.5 and Example 3.2.6). As expected from the Examples 3.2.11 and 3.2.12, two accumulations of discontinuity points can be observed, where non-injectivity holds. Note first that, in this Thue–Morse case, the limit measure of each involved cylinder can be directly obtained from [**Dek92**]. Second, this example suggests existing connections with constructions of *infinite permutations* obtained from shifts associated with aperiodic primitive substitutions [**Mak09**, **AFP15**].



Example 5.2.4. Here are other examples of approximated infinite IETs from linear complexity shifts associated with aperiodic primitive substitutions. The pictures below are the T_{100} graphs from the convergent sequences $\mathcal{T} = \{T_n\}_{n>1}$ given by Corollary 5.2.2 for: (i) the *Tribonacci substitution*, i.e. $\theta(a) = ab$, $\theta(b) = ac$, $\theta(c) = a$; (ii) the *Tetranacci substitution*, i.e. $\theta(a) = ab$, $\theta(b) = ac$, $\theta(c) = ad$, $\theta(d) = a$; (iii) the *Rudin–Shapiro substitution*, i.e. $\theta(a) = ab$, $\theta(b) = ac$, $\theta(c) = db$, $\theta(d) = dc$; (iv) the *Fibonacci substitution*, i.e. $\theta(a) = ab$, $\theta(b) = a$. In this last example, L_θ is known to be the shift given by the natural coding of the minimal *finite* IET T over two intervals, where $B = \mathcal{D} = \{0, 1/\rho\}$, and ρ is the golden ratio [**Fog02**]. Note then that in accordance with

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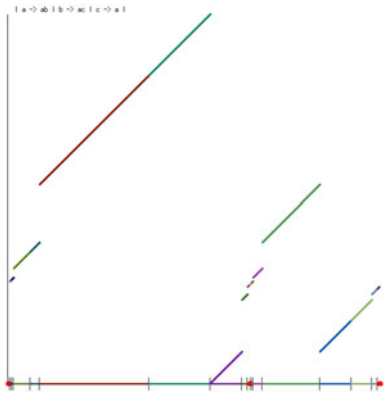
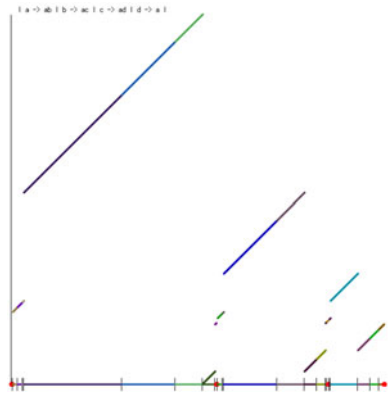
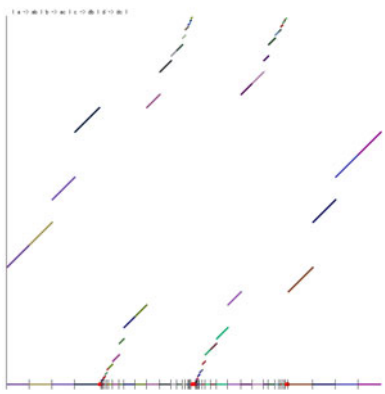
Corollary 3.3.4, the drawn graph is an approximation of the graph of T and of the graph of T_{L_θ} too, while the construction of T_{L_θ} describes it with an infinite set B_{L_θ} of separation points.



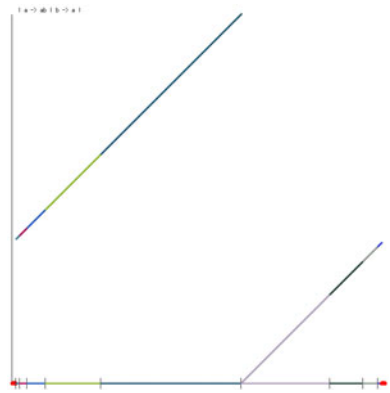
(i) Tribonacci



(ii) Tetranacci



(iii) Rudin-Shapiro



(iv) Fibonacci

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