Ergod. Th. & Dynam. Sys., (2023), **43**, 615–645 © The Author(s), 2021. Published by Cambridge University Press.
doi:10.1017/etds.2021.127

On C^0-genericity of distributional chaos

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(*Received* 13 *November* 2020 *and accepted in revised form* 4 *October* 2021)

Abstract. Let M be a compact smooth manifold without boundary. Based on results by Good and Meddaugh [*Invent. Math.* **220** (2020), 715–736], we prove that a strong distributional chaos is C^0-generic in the space of continuous self-maps (respectively, homeomorphisms) of M. The results contain answers to questions by Li, Li and Tu [*Chaos* **26** (2016), 093103] and Moothathu [*Topology Appl.* **158** (2011), 2232–2239] in the zero-dimensional case. A related counter-example on the chain components under shadowing is also given.

Key words: distributional chaos, generic, shadowing, zero dimension, Mycielski set
2020 Mathematics Subject Classification: 37D45, 37B65 (Primary)

1. *Introduction*

Throughout, let X denote a compact metric space endowed with a metric d. We denote by $C(X)$ (respectively, $\mathcal{H}(X)$) the set of continuous self-maps (respectively, homeomorphisms) of X. Let $d_{C^0}: C(X) \times C(X) \to [0, \infty)$ be the metric defined by

$$d_{C^0}(f, g) = \sup_{x \in X} d(f(x), g(x))$$

for $f, g \in C(X)$. A metric $\hat{d}_{C^0}: \mathcal{H}(X) \times \mathcal{H}(X) \to [0, \infty)$ is given by

$$\hat{d}_{C^0}(f, g) = \max\{d_{C^0}(f, g), d_{C^0}(f^{-1}, g^{-1})\}$$

for $f, g \in \mathcal{H}(X)$. With respect to these metrics, $C(X)$ and $\mathcal{H}(X)$ are complete metric spaces.

A subset S of X is called a *Mycielski set* if it is a countable union of Cantor sets. Define $C_{\mathrm{DC1}_*}(X)$ to be the set of $f \in C(X)$ such that there is a Mycielski subset S of X, which is distributionally n-δ_n-scrambled for all $n \geq 2$ for some $\delta_n > 0$. Let

$$\mathcal{H}_{\mathrm{DC1}_*}(X) = \mathcal{H}(X) \cap C_{\mathrm{DC1}_*}(X).$$

We say that a subset F of a complete metric space Z is *residual* if it contains a countable intersection of open and dense subsets of Z. The aim of this paper is to prove the following theorem.

THEOREM 1.1. *Given any compact smooth manifold M without boundary, $C_{\mathrm{DC1}_*}(M)$ is a residual subset of $C(M)$, and if* $\dim M > 1$, *then $\mathcal{H}_{\mathrm{DC1}_*}(M)$ is also a residual subset of $\mathcal{H}(M)$.*

We recall the definition of distributional n-chaos [22, 37].

Definition 1.1. For $f \in C(X)$, an n-tuple $(x_1, x_2, \ldots, x_n) \in X^n$, $n \geq 2$, is said to be *distributionally n-δ-scrambled* for $\delta > 0$ if

$$\limsup_{m\to\infty} \frac{1}{m}\left|\left\{0 \leq k \leq m-1 \colon \max_{1\leq i<j\leq n} d(f^k(x_i), f^k(x_j)) < \epsilon\right\}\right| = 1$$

for all $\epsilon > 0$, and

$$\limsup_{m\to\infty} \frac{1}{m}\left|\left\{0 \leq k \leq m-1 \colon \min_{1\leq i<j\leq n} d(f^k(x_i), f^k(x_j)) > \delta\right\}\right| = 1.$$

Let $\mathrm{DC1}_n^\delta(X, f)$ denote the set of distributionally n-δ-scrambled n-tuples and let $\mathrm{DC1}_n(X, f) = \bigcup_{\delta>0} \mathrm{DC1}_n^\delta(X, f)$. A subset S of X is said to be *distributionally n-scrambled* (respectively, *n-δ-scrambled*) if

$$(x_1, x_2, \ldots, x_n) \in \mathrm{DC1}_n(X, f) \quad (\text{respectively, } \mathrm{DC1}_n^\delta(X, f))$$

for any distinct $x_1, x_2, \ldots, x_n \in S$. We say that f exhibits the *distributional n*-chaos of type 1 ($\mathrm{DC1}_n$) if there is an uncountable distributionally n-scrambled subset of X.

For any map $f \in C(X)$, let $h_{\mathrm{top}}(f)$ denote the topological entropy of f (see, for example, [38] for its definition). Then, for every compact topological manifold M, possibly with boundary, Yano showed that generic $f \in C(M)$ (respectively, $f \in \mathcal{H}(M)$, if $\dim M > 1$) satisfies $h_{\mathrm{top}}(f) = \infty$ [40]. Since the positive topological entropy is a characteristic feature of chaos, generic dynamics on M is, roughly speaking, chaotic. Another definition of chaos is the so-called *Li–Yorke chaos* derived from [23]; and in [6], any map $f \in C(X)$ is proved to be Li–Yorke chaotic whenever $h_{\mathrm{top}}(f) > 0$ (see [6, Corollary 2.4]), so Li–Yorke chaos is topologically generic on manifolds. From a statistical viewpoint, the notion of *distributional chaos* was introduced by Schweizer and Smítal in [35] as three variants of Li–Yorke chaos for interval maps. They are numbered in order of decreasing strength ($\mathrm{DC}\beta_2$, $\beta \in \{1, 2, 3\}$), therefore $\mathrm{DC1}_2$ is the strongest by definition, and $\mathrm{DC2}_2$ (also called *mean Li–Yorke chaos*) is still stronger than Li–Yorke chaos. Then it is natural to ask if $\mathrm{DC1}_n$, $n \geq 2$, is generic or not. Note that $\mathrm{DC1}_n$, $n \geq 2$, does not necessarily imply $\mathrm{DC1}_{n+1}$ [22, 37].

For an interval map $f \in C([0, 1])$, all $\mathrm{DC}\beta_2$, $\beta \in \{1, 2, 3\}$, are equivalent to $h_{\mathrm{top}}(f) > 0$ [35] (see also [34]). Since there is a Li–Yorke chaotic map $f \in C([0, 1])$ with $h_{\mathrm{top}}(f) = 0$, $\mathrm{DC2}_2$ is strictly stronger than Li–Yorke chaos in general [36, 39] (see also [34]). In general, improving the result of [6], Downarowicz showed that any map $f \in C(X)$ with $h_{\mathrm{top}}(f) > 0$ exhibits $\mathrm{DC2}_2$ [8] (see also [13]). On the other hand, Pikuła constructed a subshift (X, f) such that $h_{top}(f) > 0$ and $DC1_2(X, f) = \emptyset$ [31]. Since $DC1_2(X, f) \neq \emptyset$ implies the existence of a distal pair for f (see [29, Corollary 15]), any proximal map $f \in C(X)$ with $h_{top}(f) > 0$, given in, for example, [12, 19, 30], does not exhibit $DC1_2$. By [4, Theorem 2], we also know that a minimal map $f \in C(X)$ with a regularly recurrent point satisfies $DC1_2(X, f) = \emptyset$, so every Toeplitz subshift with arbitrary topological entropy does not exhibit $DC1_2$ (see also [7] and [15, Remark 2.2]). Thus, some additional assumptions besides $h_{top}(f) > 0$ are needed to ensure $DC1_n$, $n \geq 2$, for a general map $f \in C(X)$.

Shadowing is a natural candidate for such an assumption. In [21], Li, Li and Tu proved that for any transitive map $f \in C(X)$ with the shadowing property, $f \in C_{DC1_*}(X)$ if one of the following properties holds: (1) f is non-periodic and has a periodic point; or (2) f is non-trivial weakly mixing. Here, we have $h_{top}(f) > 0$ in both cases. This result is extended in [15] by using a relation defined by Richeson and Wiseman [33] (see also [14]). Note that it was previously known that shadowing with (chain) mixing implies the specification and so $DC1_2$, except for the degenerate case (see [20, 25, 28]). In [11], for every compact topological manifold M with $\dim M > 1$, Guihéneuf and Lefeuvre proved that generic $f \in \mathcal{H}_\mu(M)$ satisfies the shadowing property, where $\mathcal{H}_\mu(M)$ is the set of $f \in \mathcal{H}(M)$ preserving a non-atomic Borel probability measure μ on M with the full support and $\mu(\partial M) = 0$. Since such $f \in \mathcal{H}_\mu(M)$ is also (chain) mixing, $DC1_n$, $n \geq 2$, is generic for the conservative homeomorphisms. As for $\mathcal{H}(M)$, the situation is very different because, in particular, generic $f \in \mathcal{H}(M)$ has no isolated chain component, at least for any smooth closed M [2]. Nevertheless, it is still useful to focus on the chain components. Since shadowing is shown to be generic in $C(M)$ and $\mathcal{H}(M)$ [24, 32], any consequence of shadowing is a generic property in $C(M)$ and $\mathcal{H}(M)$. When we consider the result of Li, Li and Tu, one of the obvious difficulties in proving $DC1_n$, $n \geq 2$, for generic $f \in C(M)$ (or $f \in \mathcal{H}(M)$, if $\dim M > 1$) is that even if f has shadowing, its restriction $f|_C$ to a chain component C for f does not necessarily have the shadowing property. Another difficulty arises from the additional assumption such as (1) or (2) above. The main results of this paper resolve these difficulties and establish the genericity of $DC1_n$, $n \geq 2$. Note that for generic $f \in C(M)$ (respectively, $f \in \mathcal{H}(M)$), the chain recurrent set $CR(f)$ is known to be zero-dimensional, or equivalently, totally disconnected [2, 18].

In outline, the proof of Theorem 1.1 goes as follows. In [10], Good and Meddaugh found and investigated a fundamental relationship between subshifts of finite type (SFTs) and shadowing. The following two lemmas are from [10].

LEMMA 1.1. *Let $\pi = (\pi_n^{n+1} \colon (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n \geq 1}$ be an inverse sequence of equivariant maps and let $(X, f) = \lim_\pi (X_n, f_n)$. If $f_n \colon X_n \to X_n$ has the shadowing property for each $n \geq 1$, and π satisfies the Mittag-Leffler condition* (MLC)*, then f has the shadowing property.*

LEMMA 1.2. *Let $f \colon X \to X$ be a continuous map with the shadowing property. If* $\dim X = 0$*, then there is an inverse sequence of equivariant maps*

$$\pi = (\pi_n^{n+1} \colon (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n \geq 1}$$

such that the following properties hold:

(1) π *satisfies the* MLC*;*

(2) (X_n, f_n) *is an SFT for each* $n \geq 1$*;*

(3) (X, f) *is topologically conjugate to* $\lim_\pi(X_n, f_n)$.

Note that these results concern the so-called *Mittag-Leffler condition* of an inverse sequence of equivariant maps. Most of this paper is devoted to a study of the MLC focusing on the structure of chain components. By using the above lemmas and a method in [**15**] with Mycielski's theorem, we prove the following lemma. Here, $\mathcal{D}(f)$ is the partition of X with respect to the equivalence relation $\sim_f$ defined by Richeson and Wiseman (see §2.2 for details).

LEMMA 1.3. *Let* $f: X \to X$ *be a transitive continuous map with the shadowing property such that* $h_{\rm top}(f) > 0$*. If there are a compact metric space* Y *such that* $\dim Y = 0$*, a transitive continuous map* $g: Y \to Y$ *with the shadowing property, and a factor map*

$$\pi: (Y, g) \to (X, f),$$

then there exists a sequence of positive numbers $(\delta_n)_{n\geq 2}$ *such that every* $D \in \mathcal{D}(f)$ *contains a dense Mycielski subset* S *which consists of transitive points for* f *and is distributionally* n-δ_n*-scrambled for all* $n \geq 2$.

A question in [**21**] asks whether or not, for any continuous map $f: X \to X$, the shadowing property, transitivity, and positive topological entropy imply $DC1_n$, $n \geq 2$. Lemma 1.3 gives a partial answer to this question. As a direct consequence, we obtain the following corollary, which answers the question in the zero-dimensional case.

COROLLARY 1.1. *Let* $f: X \to X$ *be a transitive continuous map with the shadowing property. If* $\dim X = 0$ *and* $h_{\rm top}(f) > 0$*, then there exists a sequence of positive numbers* $(\delta_n)_{n\geq 2}$ *such that every* $D \in \mathcal{D}(f)$ *contains a dense Mycielski subset* S *which consists of transitive points for* f *and is distributionally* n-δ_n*-scrambled for all* $n \geq 2$.

Then, by dropping the transitivity assumption through Lemma 4.1, we obtain the following theorem. Here, $\mathcal{C}(f)$ is the set of chain components for f (see §2.2 for the definition).

THEOREM 1.2. *Given any continuous map* $f: X \to X$ *with the shadowing property, if* $\dim X = 0$ *and* $h_{\rm top}(f) > 0$*, then there exist* $C \in \mathcal{C}(f)$ *and a sequence of positive numbers* $(\delta_n)_{n\geq 2}$ *such that every* $D \in \mathcal{D}(f|_C)$ *contains a dense Mycielski subset* S *which consists of transitive points for* $f|_C$ *and is distributionally* n-δ_n*-scrambled for all* $n \geq 2$.

Let

- $C_{\rm sh}(X) = \{f \in C(X): f \text{ has the shadowing property}\}$,
- $C_{{\rm cr}^0}(X) = \{f \in C(X): \dim {\rm CR}(f) = 0\}$,
- $C_{h>0}(X) = \{f \in C(X): h_{\rm top}(f) > 0\}$,

and let $\mathcal{H}_\sigma(X) = \mathcal{H}(X) \cap C_\sigma(X)$ for $\sigma \in \{{\rm sh}, {\rm cr}^0, h > 0\}$. Note that for any

$$f \in C_{\rm sh}(X) \cap C_{{\rm cr}^0}(X) \cap C_{h>0}(X),$$

the restriction $f|_{CR(f)}\colon\ CR(f) \to CR(f)$ has the following properties:

- the shadowing property;
- $\dim CR(f) = 0$;
- $h_{top}(f|_{CR(f)}) = h_{top}(f) > 0$.

By applying Theorem 1.2 to $f|_{CR(f)}$, we obtain $f \in C_{DC1_*}(X)$; therefore,

$$C_{sh}(X) \cap C_{cr^0}(X) \cap C_{h>0}(X) \subset C_{DC1_*}(X)$$

and so

$$\mathcal{H}_{sh}(X) \cap \mathcal{H}_{cr^0}(X) \cap \mathcal{H}_{h>0}(X) \subset \mathcal{H}_{DC1_*}(X).$$

Let M be a compact smooth manifold without boundary. Then Theorem 1.1 follows from the previous claims and the following results in the literature:

- shadowing
  - $C_{sh}(M)$ is a residual subset of $C(M)$ [**24**],
  - $\mathcal{H}_{sh}(M)$ is a residual subset of $\mathcal{H}(M)$ [**32**];
- chain recurrence
  - $C_{cr^0}(M)$ is a residual subset of $C(M)$ [**2**, **18**],
  - $\mathcal{H}_{cr^0}(M)$ is a residual subset of $\mathcal{H}(M)$ [**2**];
- topological entropy
  - $C_{h>0}(M)$ is a residual subset of $C(M)$ [**40**],
  - if $\dim M > 1$, then $\mathcal{H}_{h>0}(M)$ is a residual subset of $\mathcal{H}(M)$ [**40**].

The proof also gives an insight into how the distributionally scrambled sets exist in the chain recurrent set. As shown in the proof of Lemma 4.1 in §4, any chain component with positive topological entropy is approximated by one also with the shadowing property. The latter component is partitioned into the equivalence classes of the relation by Richeson and Wiseman. Then Corollary 1.1 implies that all equivalence classes densely contain distributionally scrambled Mycielski sets within them. It deepens the understanding about the chaotic aspect of C^0-generic dynamics on manifolds.

We remark that for generic $f \in \mathcal{H}(M)$ with $\dim M > 1$, the set of $x \in CR(f)$ which is contained in some $C \in \mathcal{C}(f)$, such that $(C, f|_C)$ has a non-trivial subshift of finite type as a factor, is dense in $CR(f)$ [**2**]; therefore,

$$\bigsqcup\{C \in \mathcal{C}(f)\colon h_{top}(f|_C) > 0\}$$

is a dense subset of $CR(f)$. We also recall from [**2**] that for generic $f \in \mathcal{H}(M)$ with $\dim M > 1$, the set of $x \in CR(f)$ which lies in some $C \in \mathcal{C}(f)$ such that C is *initial* or *terminal*, implying, if $\dim CR(f) = 0$, that C is a periodic orbit or $(C, f|_C)$ is topologically conjugate to an *odometer*, is a residual subset of $CR(f)$. Thus, for such $f \in \mathcal{H}(M)$ with shadowing, the distributionally scrambled Mycielski sets should be contained in intermediate chain components, and the distributional chaos occurs in a dense but meager subset of $CR(f)$.

Lastly, Theorem 1.2 also provides a method to prove the genericity of $DC1_n$, $n \geq 2$, for continuous self-maps or homeomorphisms of various underlying spaces which are not necessarily manifolds. We can find many results on the genericity of shadowing, zero-dimensionality of the chain recurrent set, and positive topological entropy in the context of topological dynamics (see, for example, [**9**, **17**, **18**]). Here, let us mention only the case where X is the Cantor set. In this case, it is shown that $\mathcal{H}(X)$ has a residual conjugacy class [**1**, **16**]. Then generic $f \in \mathcal{H}(X)$ has the shadowing property but satisfies $h_{\mathrm{top}}(f) = 0$ and has no Li–Yorke pair [**5**, **9**]; therefore, the generic homeomorphisms of X are *not* chaotic.

This paper consists of six sections. The basic notation, definitions, and facts are briefly collected in §2. In §3 we prove some preparatory lemmas. In §4 we prove Lemma 4.1 to reduce Theorem 1.2 to Lemma 1.3. In §5 we prove Lemma 1.3. In §6, as a bi-product of the proof of Lemma 4.1, we answer a question by Moothathu [**26**] in the zero-dimensional case, and give a related counter-example showing that the chain components with the shadowing property can be relatively few.

2. *Preliminaries*

In this section we collect some basic definitions, notations, facts, and prove some lemmas which will be used in what follows.

2.1. *Chains, cycles, pseudo-orbits and the shadowing property.* Given a continuous map $f: X \to X$, a finite sequence $(x_i)_{i=0}^k$ of points in X, where $k > 0$ is a positive integer, is called a δ-*chain* of f if $d(f(x_i), x_{i+1}) \le \delta$ for every $0 \le i \le k-1$. A δ-chain $(x_i)_{i=0}^k$ of f is said to be a δ-*cycle* of f if $x_0 = x_k$. Let $\xi = (x_i)_{i \ge 0}$ be a sequence of points in X. For $\delta > 0$, ξ is called a δ-pseudo-orbit of f if $d(f(x_i), x_{i+1}) \le \delta$ for all $i \ge 0$. For $\epsilon > 0$, ξ is said to be ϵ-shadowed by $x \in X$ if $d(f^i(x), x_i) \le \epsilon$ for all $i \ge 0$. We say that f has the *shadowing property* if, for any $\epsilon > 0$, there is $\delta > 0$ such that every δ-pseudo-orbit of f is ϵ-shadowed by some point of X.

2.2. *Chain components and a relation.*

2.2.1. *Chain recurrence and chain transitivity.* Given a continuous map $f: X \to X$, a point $x \in X$ is called a *chain recurrent point* for f if, for any $\delta > 0$, there is a δ-cycle $(x_i)_{i=0}^k$ of f with $x_0 = x_k = x$. We denote by $\mathrm{CR}(f)$ the set of chain recurrent points for f. It is a closed f-invariant subset of X, and the restriction $f|_{\mathrm{CR}(f)}: \mathrm{CR}(f) \to \mathrm{CR}(f)$ satisfies $\mathrm{CR}(f|_{\mathrm{CR}(f)}) = \mathrm{CR}(f)$. It is known that if f has the shadowing property, then so does $f|_{\mathrm{CR}(f)}$ [**26**]. We call f *chain recurrent* if $X = \mathrm{CR}(f)$. For any $x, y \in X$ and $\delta > 0$, the notation $x \to_{f,\delta} y$ means that there is a δ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x$ and $x_k = y$. Then f is said to be *chain transitive* if $x \to_{f,\delta} y$ for any $x, y \in X$ and $\delta > 0$. We say that f is *transitive* if, for any two non-empty open subsets U, $V$of X, there is $n > 0$ such that $f^n(U) \cap V \ne \emptyset$. If f is transitive, then f is chain transitive, and the converse holds when f has the shadowing property.

2.2.2. *Chain components.* For any continuous map $f: X \to X$, $\mathrm{CR}(f)$ admits a decomposition with respect to a relation $\leftrightarrow_f$ in $\mathrm{CR}(f)^2 = \mathrm{CR}(f) \times \mathrm{CR}(f)$ defined as follows: for any $x, y \in \mathrm{CR}(f)$, $x \leftrightarrow_f y$ if and only if $x \to_{f,\delta} y$ and $y \to_{f,\delta} x$ for every $\delta > 0$. Note that $\leftrightarrow_f$ is a closed $(f \times f)$-invariant equivalence relation in $\mathrm{CR}(f)^2$.

An equivalence class C of $\leftrightarrow_f$ is called a *chain component* for f. We denote by $\mathcal{C}(f)$ the set of chain components for f. Then the following properties hold.

(1) $\mathrm{CR}(f) = \bigsqcup_{C \in \mathcal{C}(f)} C$, here $\bigsqcup$ denotes the disjoint union.
(2) Every $C \in \mathcal{C}(f)$ is a closed f-invariant subset of $\mathrm{CR}(f)$.
(3) $f|_C : C \to C$ is chain transitive for all $C \in \mathcal{C}(f)$.

Note that f is chain transitive if and only if f is chain recurrent and satisfies $\mathcal{C}(f) = \{X\}$.

2.2.3. *A relation.* Let $f : X \to X$ be a chain transitive map. For $\delta > 0$ and a δ-cycle $\gamma = (x_i)_{i=0}^k$ of f, k is called the *length* of γ. Let $m = m(f, \delta) > 0$ be the greatest common divisor of all the lengths of δ-cycles of f. We define a relation $\sim_{f,\delta}$ in X^2 as follows: for any $x, y \in X$, $x \sim_{f,\delta} y$ if and only if there is a δ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x$, $x_k = y$, and $m|k$. Then the following properties hold.

(1) $\sim_{f,\delta}$ is an open and closed $(f \times f)$-invariant equivalence relation in X^2.
(2) For any $x \in X$ and $n \geq 0$, $x \sim_{f,\delta} f^{mn}(x)$.
(3) There exists $N > 0$ such that, for any $x, y \in X$ with $x \sim_{f,\delta} y$ and $n \geq N$, there is a δ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x$, $x_k = y$ and $k = mn$.

Following [33], define a relation $\sim_f$ in X^2 as follows: for any $x, y \in X$, $x \sim_f y$ if and only if $x \sim_{f,\delta} y$ for every $\delta > 0$. This is a closed $(f \times f)$-invariant equivalence relation in X^2. We denote by $\mathcal{D}(f)$ the set of equivalence classes of $\sim_f$. This gives a closed partition of X. A pair $(x, y) \in X^2$ is said to be *chain proximal* if, for any $\delta > 0$, there is a pair $((x_i)_{i=0}^k, (y_i)_{i=0}^k)$ of δ-chains of f such that $(x_0, y_0) = (x, y)$ and $x_k = y_k$. As claimed in [33, Remark 8], for any $(x, y) \in X^2$, (x, y) is chain proximal if and only if $x \sim_f y$.

2.3. *Inverse limit.*

2.3.1. *Inverse limit spaces.* Given an inverse sequence of continuous maps

$$\pi = (\pi_n^{n+1} : X_{n+1} \to X_n)_{n \geq 1},$$

where $(X_n)_{n \geq 1}$ is a sequence of compact metric spaces, define $\pi_n^m : X_m \to X_n$ by

$$\pi_n^m = \begin{cases} \mathrm{id}_{X_n} & \text{if } m = n, \\ \pi_n^{n+1} \circ \pi_{n+1}^{n+2} \circ \cdots \circ \pi_{m-1}^m & \text{if } m > n, \end{cases}$$

for all $m \geq n \geq 1$. Note that $\pi_n^l = \pi_n^m \circ \pi_m^l$ for any $l \geq m \geq n \geq 1$. The *inverse limit space* $X = \lim_\pi X_n$ is defined by

$$X = \left\{ x = (x_n)_{n \geq 1} \in \prod_{n \geq 1} X_n : \pi_n^{n+1}(x_{n+1}) = x_n, \text{ for all } n \geq 1 \right\},$$

which is a compact metric space.

For any $n \geq 1$, note that $\pi_n^m(X_m) \supset \pi_n^{m+1}(X_{m+1})$ for every $m \geq n$, and let

$$\hat{X}_n = \bigcap_{m \geq n} \pi_n^m(X_m).$$

https://doi.org/10.1017/etds.2021.127 Published online by Cambridge University Press

By compactness, we easily see that, for any $n \geq 1$ and $x \in X_n$, $x \in \hat{X}_n$ if and only if there is a sequence

$$(x_m)_{m \geq n} \in \prod_{m \geq n} X_m$$

with $\pi_m^{m+1}(x_{m+1}) = x_m$ for every $m \geq n$. For each $n \geq 1$, $\pi_n^{n+1}(\hat{X}_{n+1}) = \hat{X}_n$, that is, $\hat{\pi}_n^{n+1} = (\pi_n^{n+1})|_{\hat{X}_{n+1}} \colon \hat{X}_{n+1} \to \hat{X}_n$ is surjective. Let

$$\hat{\pi} = (\hat{\pi}_n^{n+1} \colon \hat{X}_{n+1} \to \hat{X}_n)_{n \geq 1}$$

and $\hat{X} = \lim_{\hat{\pi}} \hat{X}_n$. Since any $x = (x_n)_{n \geq 1} \in X$ satisfies $x_n \in \hat{X}_n$ for every $n \geq 1$, we see that the inclusion $i \colon \hat{X} \to X$ is a homeomorphism.

2.3.2. *The Mittag-Leffler condition.* Given $\pi = (\pi_n^{n+1} \colon X_{n+1} \to X_n)_{n \geq 1}$, an inverse sequence of continuous maps, π is said to satisfy the *Mittag-Leffler Condition* if, for any $n \geq 1$, there is $N \geq n$ such that $\pi_n^N(X_N) = \pi_n^m(X_m)$ for all $m \geq N$. We say that π satisfies MLC(1) if

$$\pi_n^{n+1}(X_{n+1}) = \pi_n^{n+2}(X_{n+2})$$

for any $n \geq 1$.

LEMMA 2.1. *Let $\pi = (\pi_n^{n+1} \colon X_{n+1} \to X_n)_{n \geq 1}$ be an inverse sequence of continuous maps. Then the following properties are equivalent.*

(1) *π satisfies* MLC(1).

(2) *For any $n \geq 1$ and $m \geq n+1$, $\pi_n^{n+1}(X_{n+1}) = \pi_n^m(X_m)$.*

(3) *For every $n \geq 1$, $\hat{X}_n = \pi_n^{n+1}(X_{n+1})$.*

Proof. (1) $\Rightarrow$ (2): We use an induction on m. For $m = n+1$, $\pi_n^{n+1}(X_{n+1}) = \pi_n^m(X_m)$ is trivially true. Assume $\pi_n^{n+1}(X_{n+1}) = \pi_n^m(X_m)$ for some $m \geq n+1$. Then we have

$$\pi_n^{m+1}(X_{m+1}) = \pi_n^{m-1}(\pi_{m-1}^{m+1}(X_{m+1})) = \pi_n^{m-1}(\pi_{m-1}^m(X_m)) = \pi_n^m(X_m) = \pi_n^{n+1}(X_{n+1}),$$

completing the induction.

(2) $\Rightarrow$ (1): Put $m = n+2$ in (2).

(2) $\Rightarrow$ (3): Property (2) implies

$$\hat{X}_n = \pi_n^n(X_n) \cap \bigcap_{m \geq n+1} \pi_n^m(X_m) = X_n \cap \pi_n^{n+1}(X_{n+1}) = \pi_n^{n+1}(X_{n+1})$$

for every $n \geq 1$.

(3) $\Rightarrow$ (2): Since $\pi_n^{n+1}(X_{n+1}) \supset \pi_n^{n+2}(X_{n+2}) \supset \cdots \supset \hat{X}_n$, $\hat{X}_n = \pi_n^{n+1}(X_{n+1})$ implies $\pi_n^m(X_m) = \hat{X}_n = \pi_n^{n+1}(X_{n+1})$ for any $m \geq n+1$, completing the proof. □

Remark 2.1. Property (2) in the above lemma implies that π satisfies the MLC.

LEMMA 2.2. *Let $\pi = (\pi_n^{n+1} \colon X_{n+1} \to X_n)_{n \geq 1}$ be an inverse sequence of continuous maps. If π satisfies the* MLC, *then there is a sequence $1 \leq n(1) < n(2) < \cdots$ such that, letting $\pi' = (\pi_{n(j)}^{n(j+1)} \colon X_{n(j+1)} \to X_{n(j)})_{j \geq 1}$, π' satisfies* MLC(1).

Proof. Put $n(0) = 1$. Inductively, define a sequence $1 = n(0) < n(1) < n(2) < \cdots$ as follows: given $j \geq 0$ and $n(j)$, take $n(j+1) > n(j)$ such that $\pi_{n(j)}^{n(j+1)}(X_{n(j+1)}) = \pi_{n(j)}^{m}(X_m)$ for every $m \geq n(j+1)$. Then, for each $j \geq 1$, $\pi_{n(j)}^{n(j+1)}(X_{n(j+1)}) = \pi_{n(j)}^{n(j+2)}(X_{n(j+2)})$ since $n(j+2) > n(j+1)$, implying that π' satisfies MLC(1). □

2.3.3. *Equivariance, factor and the topological conjugacy.* Given two continuous maps $f : X \to X$ and $g : Y \to Y$, where X and Y are compact metric spaces, a continuous map $\pi : X \to Y$ is said to be *equivariant* if $g \circ \pi = \pi \circ f$, and such π is also denoted by $\pi : (X, f) \to (Y, g)$. An equivariant map $\pi : (X, f) \to (Y, g)$ is called a *factor map* (respectively, *topological conjugacy*) if it is surjective (respectively, a homeomorphism). Two systems (X, f) and (Y, g) are said to be *topologically conjugate* if there is a topological conjugacy $h : (X, f) \to (Y, g)$.

2.3.4. *Inverse limit systems.* For an inverse sequence of equivariant maps

$$\pi = (\pi_n^{n+1} : (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n\geq 1},$$

the *inverse limit system* $(X, f) = \lim_\pi (X_n, f_n)$ is well defined by $X = \lim_\pi X_n$, and $f(x) = (f_n(x_n))_{n\geq 1}$ for all $x = (x_n)_{n\geq 1} \in X$.

For every $n \geq 1$, note that $\hat{X}_n$ is a closed f_n-invariant subset of X_n, and let $\hat{f}_n = (f_n)|_{\hat{X}_n} : \hat{X}_n \to \hat{X}_n$. For all $n \geq 1$, $\hat{\pi}_n^{n+1} = (\pi_n^{n+1})|_{\hat{X}_{n+1}} : \hat{X}_{n+1} \to \hat{X}_n$ gives a factor map

$$\hat{\pi}_n^{n+1} : (\hat{X}_{n+1}, \hat{f}_{n+1}) \to (\hat{X}_n, \hat{f}_n).$$

Let

$$\hat{\pi} = (\hat{\pi}_n^{n+1} : (\hat{X}_{n+1}, \hat{f}_{n+1}) \to (\hat{X}_n, \hat{f}_n))_{n\geq 1}$$

and $(\hat{X}, \hat{f}) = \lim_{\hat{\pi}} (\hat{X}_n, \hat{f}_n)$. Then the inclusion $i : \hat{X} \to X$ is a topological conjugacy $i : (\hat{X}, \hat{f}) \to (X, f)$.

LEMMA 2.3. *Let $\pi = (\pi_n^{n+1} : (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n\geq 1}$ be an inverse sequence of equivariant maps and let $(X, f) = \lim_\pi (X_n, f_n)$. If $f_n : X_n \to X_n$ is chain recurrent (respectively, chain transitive) for each $n \geq 1$, then $f : X \to X$ is chain recurrent (respectively, chain transitive).*

Proof. Let $n \geq 1$. Then, for any $m \geq n$, since $f_m : X_m \to X_m$ is chain recurrent (respectively, chain transitive), and $\pi_n^m : (X_m, f_m) \to (X_n, f_n)$ is an equivariant map,

$$(f_n)|_{\pi_n^m(X_m)} : \pi_n^m(X_m) \to \pi_n^m(X_m)$$

is chain recurrent (respectively, chain transitive). Because $\hat{X}_n = \bigcap_{m\geq n} \pi_n^m(X_m)$,

$$\hat{f}_n : \hat{X}_n \to \hat{X}_n$$

is chain recurrent (respectively, chain transitive). Since

$$\hat{\pi} = (\hat{\pi}_n^{n+1} : (\hat{X}_{n+1}, \hat{f}_{n+1}) \to (\hat{X}_n, \hat{f}_n))_{n\geq 1}$$

is a sequence of factor maps, we see that $\hat{f}\colon \hat{X} \to \hat{X}$ is chain recurrent (respectively, chain transitive). Thus, $f\colon X \to X$ is chain recurrent (respectively, chain transitive), because (X, f) and $(\hat{X}, \hat{f})$ are topologically conjugate, completing the proof. □

For an inverse sequence of equivariant maps

$$\pi = (\pi_n^{n+1}\colon (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n\geq 1}$$

and a sequence $1 \leq n(1) < n(2) < \cdots$, $\pi_{n(j)}^{n(j+1)}\colon X_{n(j+1)} \to X_{n(j)}$ is an equivariant map for each $j \geq 1$. Letting

$$\pi' = (\pi_{n(j)}^{n(j+1)}\colon (X_{n(j+1)}, f_{n(j+1)}) \to (X_{n(j)}, f_{n(j)}))_{j\geq 1}$$

and $(Y, g) = \lim_{\pi'}(X_{n(j)}, f_{n(j)})$, we have a topological conjugacy $h\colon (X, f) \to (Y, g)$ given by $h(x) = (x_{n(j)})_{j\geq 1}$ for all $x = (x_n)_{n\geq 1} \in X$. By this and Lemma 2.2, we obtain the following lemma.

LEMMA 2.4. *Let $\pi = (\pi_n^{n+1}\colon (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n\geq 1}$ be an inverse sequence of equivariant maps. If π satisfies the* MLC, *then there is a sequence* $1 \leq n(1) < n(2) < \cdots$ *such that, letting $\pi' = (\pi_{n(j)}^{n(j+1)}\colon (X_{n(j+1)}, f_{n(j+1)}) \to (X_{n(j)}, f_{n(j)}))_{j\geq 1}$, π' satisfies* MLC(1), *and* $\lim_{\pi}(X_n, f_n)$ *is topologically conjugate to* $\lim_{\pi'}(X_{n(j)}, f_{n(j)})$.

2.4. *Subshifts.*

2.4.1. *Subshifts of finite type.* Let S be a finite set with the discrete topology. The *shift map* $\sigma\colon S^{\mathbb{N}} \to S^{\mathbb{N}}$ is defined by $\sigma(x) = (x_{n+1})_{n\geq 1}$ for all $x = (x_n)_{n\geq 1} \in S^{\mathbb{N}}$. Note that σ is continuous with respect to the product topology of $S^{\mathbb{N}}$. The product space $S^{\mathbb{N}}$ (and also $(S^{\mathbb{N}}, \sigma)$) is called the (one-sided) *full-shift* over S. A closed σ-invariant subset X of $S^{\mathbb{N}}$ (and also the subsystem $(X, \sigma|_X)$ of $(S^{\mathbb{N}}, \sigma)$) is called a *subshift*. A subshift X of $S^{\mathbb{N}}$ (and also $(X, \sigma|_X)$ of $(S^{\mathbb{N}}, \sigma)$) is called a *subshift of finite type* if there are $N > 0$ and $F \subset S^{N+1}$ such that, for any $x = (x_n)_{n\geq 1} \in S^{\mathbb{N}}$, $x \in X$ if and only if $(x_i, x_{i+1}, \ldots, x_{i+N}) \in F$ for all $i \geq 1$. The shift map $\sigma\colon S^{\mathbb{N}} \to S^{\mathbb{N}}$ is positively expansive and has the shadowing property. We know that a subshift X of $S^{\mathbb{N}}$ is of finite type if and only if $\sigma|_X\colon X \to X$ has the shadowing property [3].

2.4.2. *Some properties of SFTs.* Let $(X, \sigma|_X)$ be an SFT (of some full-shift over S) and put $f = \sigma|_X$. Then f has the following properties.

(1) $\mathrm{CR}(f) = \overline{\mathrm{Per}(f)}$, where $\mathrm{Per}(f)$ denotes the set of periodic points for f.

(2) For any $x \in X$, there is $y \in \mathrm{CR}(f)$ such that $\lim_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

In fact, these two properties are consequences of the positive expansiveness and the shadowing property of $f\colon X \to X$. Since the restriction $f|_{\mathrm{CR}(f)}\colon \mathrm{CR}(f) \to \mathrm{CR}(f)$ is surjective and positively expansive, it is c-expansive. Also, it has the shadowing property. Applying [3, Theorem 3.4.4] to $f|_{\mathrm{CR}(f)}$, we obtain the following property.

(3) There is a finite set $\mathcal{C}$ of clopen f-invariant subsets of $\mathrm{CR}(f)$ such that

$$\mathrm{CR}(f) = \bigsqcup_{C \in \mathcal{C}} C,$$

and $f|_C : C \to C$ is transitive for every $C \in \mathcal{C}$.

An element of $\mathcal{C}$ is called a *basic set*. We easily see that $\mathcal{C} = \mathcal{C}(f)$, that is, the basic sets coincide with the chain components for f. For every $C \in \mathcal{C}$, $f|_C$ has the shadowing property, so C (or $(C, f|_C)$) is a transitive SFT.

Consider the case where f is transitive (or (X, f) is a transitive SFT). Then we have $X = \mathrm{CR}(f)$ and $\mathcal{C} = \mathcal{C}(f) = \{X\}$. Again by [3, Theorem 3.4.4], X admits a decomposition

$$X = \bigsqcup_{i=0}^{m-1} f^i(D),$$

where $m > 0$ is a positive integer, such that $f^i(D)$, $0 \le i \le m-1$, are clopen f^m-invariant subsets of X, and

$$f^m|_{f^i(D)} : f^i(D) \to f^i(D)$$

is mixing for every $0 \le i \le m-1$. Here, a continuous map $g : Y \to Y$ is said to be *mixing* if, for any two non-empty open subsets U, V of Y, there is $N > 0$ such that $g^n(U) \cap V \neq \emptyset$ for all $n \ge N$. In this case, we easily see that $\mathcal{D}(f) = \{f^i(D) : 0 \le i \le m-1\}$.

3. *Preparatory lemmas*

In this section we prove some preparatory lemmas needed for the proofs of the main results. The first two lemmas give an expression of the chain recurrent set (respectively, chain components) for the inverse limit system.

LEMMA 3.1. *Let $\pi = (\pi_n^{n+1} : (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n\ge1}$ be an inverse sequence of equivariant maps and let $(X, f) = \lim_\pi (X_n, f_n)$. Then*

$$\mathrm{CR}(f) = \{x = (x_n)_{n\ge1} \in X : x_n \in \mathrm{CR}(f_n), \text{ for all } n \ge 1\}.$$

Proof. Let R denote the right-hand side of the equation. $\mathrm{CR}(f) \subset R$ is clearly true. Let us prove $R \subset \mathrm{CR}(f)$. Note that $\pi_n^{n+1}(\mathrm{CR}(f_{n+1})) \subset \mathrm{CR}(f_n)$ for every $n \ge 1$. Let $Y_n = \mathrm{CR}(f_n)$, $g_n = (f_n)|_{Y_n} : Y_n \to Y_n$, and $\tilde{\pi}_n^{n+1} = (\pi_n^{n+1})|_{Y_{n+1}} : Y_{n+1} \to Y_n$ for each $n \ge 1$. Consider the inverse sequence of equivariant maps

$$\tilde{\pi} = (\tilde{\pi}_n^{n+1} : (Y_{n+1}, g_{n+1}) \to (Y_n, g_n))_{n\ge1}$$

and let $(Y, g) = \lim_{\tilde{\pi}} (Y_n, g_n)$. Since g_n is chain recurrent for all $n \ge 1$, by Lemma 2.3, g is chain recurrent. On the other hand, R is a closed f-invariant subset of X and satisfies $Y = R$. The inclusion $i : Y \to R$ gives a topological conjugacy $i : (Y, g) \to (R, f|_R)$, and so $f|_R : R \to R$ is chain recurrent, which clearly implies $R \subset \mathrm{CR}(f)$; therefore, the lemma has been proved. □

Let $\pi = (\pi_n^{n+1} : (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n\ge1}$ be an inverse sequence of equivariant maps and let $(X, f) = \lim_\pi (X_n, f_n)$. Note that for any $n \ge 1$ and $C_{n+1} \in \mathcal{C}(f_{n+1})$, there

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is $C_n \in \mathcal{C}(f_n)$ such that $\pi_n^{n+1}(C_{n+1}) \subset C_n$. Let

$$\mathcal{C}_\pi = \left\{C_* = (C_n)_{n\geq 1} \in \prod_{n\geq 1} \mathcal{C}(f_n) : \pi_n^{n+1}(C_{n+1}) \subset C_n, \text{ for all } n \geq 1\right\}.$$

Also, for any $C_* = (C_n)_{n\geq 1} \in \mathcal{C}_\pi$, let

$$[C_*] = \{x = (x_n)_{n\geq 1} \in X : x_n \in C_n, \text{ for all } n \geq 1\}.$$

The next lemma gives an expression of $\mathcal{C}(f)$.

LEMMA 3.2. *It holds that $\mathcal{C}(f) = \{[C_*] : C_* \in \mathcal{C}_\pi\}$, where $\mathcal{C}(f)$ is the set of chain components for f.*

Proof. For any $C_* = (C_n)_{n\geq 1} \in \mathcal{C}_\pi$, $[C_*]$ is a closed f-invariant subset of X. We prove that $f|_{[C_*]} : [C_*] \to [C_*]$ is chain transitive. For each $n \geq 1$, let $g_n = f|_{C_n} : C_n \to C_n$ and let

$$\tilde{\pi}_n^{n+1} = (\pi_n^{n+1})|_{C_{n+1}} : C_{n+1} \to C_n.$$

Consider the inverse sequence of equivariant maps

$$\tilde{\pi} = (\tilde{\pi}_n^{n+1} : (C_{n+1}, g_{n+1}) \to (C_n, g_n))_{n\geq 1}$$

and let $(Y, g) = \lim_{\tilde{\pi}}(C_n, g_n)$. Since g_n is chain transitive for all $n \geq 1$, by Lemma 2.3, g is chain transitive. On the other hand, we have $Y = [C_*]$. The inclusion $i : Y \to [C_*]$ gives a topological conjugacy $i : (Y, g) \to ([C_*], f|_{[C_*]})$, which implies that $f|_{[C_*]}$ is chain transitive.

Given any $C_* = (C_n)_{n\geq 1} \in \mathcal{C}_\pi$, from what is shown above, there is $C \in \mathcal{C}(f)$ such that $[C_*] \subset C$. Fix $x = (x_n)_{n\geq 1} \in [C_*]$. Then, for every $y = (y_n)_{n\geq 1} \in C$, we easily see that $\{x_n, y_n\} \in C_n$ for all $n \geq 1$; therefore, $y \in [C_*]$. This implies $C \subset [C_*]$ and so $[C_*] = C$, proving

$$\{[C_*] : C_* \in \mathcal{C}_\pi\} \subset \mathcal{C}(f).$$

To prove

$$\mathcal{C}(f) \subset \{[C_*] : C_* \in \mathcal{C}_\pi\},$$

for any $C \in \mathcal{C}(f)$, fix $x = (x_n)_{n\geq 1} \in C$, and take $C_n \in \mathcal{C}(f_n)$ with $x_n \in C_n$ for each $n \geq 1$. Then $C_* = (C_n)_{n\geq 1} \in \mathcal{C}_\pi$ and $x \in [C_*] \subset C$. Similarly to the argument above, we obtain $C = [C_*]$, completing the proof. □

The next lemma gives an expression for $\mathcal{D}(f)$, which was introduced in §2.2, for the inverse limit system under MLC(1). Given

$$\pi = (\pi_n^{n+1} : (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n\geq 1},$$

an inverse sequence of equivariant maps, let $(X, f) = \lim_\pi(X_n, f_n)$ and suppose that $f_n : X_n \to X_n$ is chain transitive for all $n \geq 1$. Then, by Lemma 2.3, $f : X \to X$ is chain transitive. Note that, for any $n \geq 1$ and $D_{n+1} \in \mathcal{D}(f_{n+1})$, there is $D_n \in \mathcal{D}(f_n)$ such that

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$\pi_n^{n+1}(D_{n+1}) \subset D_n$. Let

$$\mathcal{D}_\pi = \left\{ D_* = (D_n)_{n\geq 1} \in \prod_{n\geq 1} \mathcal{D}(f_n) \colon \pi_n^{n+1}(D_{n+1}) \subset D_n, \text{ for all } n \geq 1 \right\}.$$

Also, for any $D_* = (D_n)_{n\geq 1} \in \mathcal{D}_\pi$, let

$$[D_*] = \{x = (x_n)_{n\geq 1} \in X \colon x_n \in D_n, \text{ for all } n \geq 1\}.$$

LEMMA 3.3. *If π satisfies* MLC(1)*, then* $\mathcal{D}(f) = \{[D_*] \colon D_* \in \mathcal{D}_\pi\}$.

Proof. Let $D_* = (D_n)_{n\geq 1} \in \mathcal{D}_\pi$ and let $x = (x_n)_{n\geq 1}$, $y = (y_n)_{n\geq 1} \in [D_*]$. We prove that $(x, y) \in X^2$ is chain proximal for *f*. Fix any $N > 0$ and $\delta > 0$. Since $\{x_{N+1}, y_{N+1}\} \subset D_{N+1} \in \mathcal{D}(f_{N+1})$, $(x_{N+1}, y_{N+1}) \in X_{N+1}^2$ is chain proximal for f_{N+1}, implying that there is a pair

$$((x_{N+1}^{(i)})_{i=0}^k, (y_{N+1}^{(i)})_{i=0}^k)$$

of δ-chains of f_{N+1} such that $(x_{N+1}^{(0)}, y_{N+1}^{(0)}) = (x_{N+1}, y_{N+1})$ and $x_{N+1}^{(k)} = y_{N+1}^{(k)}$. Let $(z^{(0)}, w^{(0)}) = (x, y) \in X^2$ and note that

$$(z_n^{(0)}, w_n^{(0)}) = (x_n, y_n) = (\pi_n^{N+1}(x_{N+1}), \pi_n^{N+1}(y_{N+1})) = (\pi_n^{N+1}(x_{N+1}^{(0)}), \pi_n^{N+1}(y_{N+1}^{(0)}))$$

for every $1 \leq n \leq N$. For each $0 < i \leq k$, since

$$\{\pi_N^{N+1}(x_{N+1}^{(i)}), \pi_N^{N+1}(y_{N+1}^{(i)})\} \subset \pi_N^{N+1}(X_{N+1}) = \hat{X}_N$$

by MLC(1) (see Lemma 2.1), there are $(z^{(i)}, w^{(i)}) \in X^2$, $0 < i \leq k-1$, and $z^{(k)} = w^{(k)} \in X$ such that

$$(z_n^{(i)}, w_n^{(i)}) = (\pi_n^{N+1}(x_{N+1}^{(i)}), \pi_n^{N+1}(y_{N+1}^{(i)}))$$

and also

$$z_n^{(k)} = w_n^{(k)} = \pi_n^{N+1}(x_{N+1}^{(k)}) = \pi_n^{N+1}(y_{N+1}^{(k)})$$

for every $1 \leq n \leq N$. Let d_n, $n \geq 1$, be the metric on X_n. For any $0 \leq i \leq k-1$ and $1 \leq n \leq N$, we have

$$\begin{aligned} d_n(f(z^{(i)})_n, z_n^{(i+1)}) &= d_n(f_n(z_n^{(i)}), z_n^{(i+1)}) \\ &= d_n(f_n(\pi_n^{N+1}(x_{N+1}^{(i)})), \pi_n^{N+1}(x_{N+1}^{(i+1)})) \\ &= d_n(\pi_n^{N+1}(f_{N+1}(x_{N+1}^{(i)})), \pi_n^{N+1}(x_{N+1}^{(i+1)})) \end{aligned}$$

with $d_{N+1}(f_{N+1}(x_{N+1}^{(i)}), x_{N+1}^{(i+1)}) \leq \delta$, and similarly,

$$d_n(f(w^{(i)})_n, w_n^{(i+1)}) = d_n(\pi_n^{N+1}(f_{N+1}(y_{N+1}^{(i)})), \pi_n^{N+1}(y_{N+1}^{(i+1)}))$$

with $d_{N+1}(f_{N+1}(y_{N+1}^{(i)}), y_{N+1}^{(i+1)}) \leq \delta$. Therefore, for every $\epsilon > 0$, if *N* is large enough, and then δ is sufficiently small,

$$((z^{(i)})_{i=0}^k, (w^{(i)})_{i=0}^k)$$

https://doi.org/10.1017/etds.2021.127 Published online by Cambridge University Press

is a pair of ϵ-chains of f with $(z^{(0)}, w^{(0)}) = (x, y)$ and $z^{(k)} = w^{(k)}$, proving that $(x, y) \in X^2$ is chain proximal for f.

Given any $D_* = (D_n)_{n\geq 1} \in \mathcal{D}_\pi$, from what is shown above, we have $[D_*] \subset D$ for some $D \in \mathcal{D}(f)$. The rest of the proof is identical to that of Lemma 3.2. □

The final lemma gives a sufficient condition for an inverse sequence of subsystems to continue satisfying condition MLC(1).

LEMMA 3.4. *Let $\pi = (\pi_n^{n+1} \colon (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n\geq 1}$ be an inverse sequence of equivariant maps with* MLC(1). *Let $(X, f) = \lim_\pi (X_n, f_n)$ and suppose that a sequence of closed f_n-invariant subsets Y_n of X_n, $n \geq 1$, has the following properties:*

(1) $\pi_n^{n+1}(Y_{n+1}) \subset Y_n$ *for every* $n \geq 1$;

(2) *any* $x = (x_n)_{n\geq 1} \in X$ *satisfies* $x_n \in Y_n$ *for all* $n \geq 1$.

For each $n \geq 1$, let $g_n = (f_n)|_{Y_n} \colon Y_n \to Y_n$ and let $\tilde{\pi}_n^{n+1} = (\pi_n^{n+1})|_{Y_{n+1}} \colon Y_{n+1} \to Y_n$. Then the inverse sequence of equivariant maps

$$\tilde{\pi} = (\tilde{\pi}_n^{n+1} \colon (Y_{n+1}, g_{n+1}) \to (Y_n, g_n))_{n\geq 1}$$

satisfies MLC(1).

Proof. For any $n \geq 1$ and $q \in X_{n+1}$, since $\pi_n^{n+1}(q) \in \pi_n^{n+1}(X_{n+1}) = \hat{X}_n$ by MLC(1) of π (see Lemma 2.1), we have $x_n = \pi_n^{n+1}(q)$ for some $x = (x_j)_{j\geq 1} \in X$. Since $\pi_n^{n+1}(q) = x_n = \pi_n^{n+2}(x_{n+2})$, by property (2), we obtain $\pi_n^{n+1}(q) \in \pi_n^{n+2}(Y_{n+2})$, implying $\pi_n^{n+1}(X_{n+1}) \subset \pi_n^{n+2}(Y_{n+2})$. Then

$$\pi_n^{n+1}(Y_{n+1}) \subset \pi_n^{n+1}(X_{n+1}) \subset \pi_n^{n+2}(Y_{n+2}) \subset \pi_n^{n+1}(Y_{n+1}),$$

therefore $\pi_n^{n+1}(Y_{n+1}) = \pi_n^{n+2}(Y_{n+2})$. Since $n \geq 1$ is arbitrary, $\tilde{\pi}$ satisfies MLC(1). □

4. *Reduction of Theorem 1.2 to Lemma 1.3*

The aim of this section is to prove the following lemma to reduce Theorem 1.2 to Lemma 1.3.

LEMMA 4.1. *Let $f \colon X \to X$ be a continuous map with the shadowing property. If* $\dim X = 0$ *and $h_{\mathrm{top}}(f) > 0$, then there is $C \in C(f)$ such that $f|_C \colon C \to C$ has the shadowing property and satisfies $h_{\mathrm{top}}(f|_C) > 0$.*

A lemma is needed for the proof. It states that, for an inverse sequence of SFTs, we can consider the inverse sequence of chain recurrent sets without losing MLC(1).

LEMMA 4.2. *Let $\pi = (\pi_n^{n+1} \colon (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n\geq 1}$ be an inverse sequence of equivariant maps with* MLC(1). *Let $(X, f) = \lim_\pi (X_n, f_n)$ and suppose that (X_n, f_n) is an SFT for each $n \geq 1$. Let $Y_n = \mathrm{CR}(f_n)$, $g_n = (f_n)|_{Y_n} \colon Y_n \to Y_n$, and $\tilde{\pi}_n^{n+1} = (\pi_n^{n+1})|_{Y_{n+1}} \colon Y_{n+1} \to Y_n$ for every $n \geq 1$. Then the inverse sequence of equivariant maps*

$$\tilde{\pi} = (\tilde{\pi}_n^{n+1} \colon (Y_{n+1}, g_{n+1}) \to (Y_n, g_n))_{n\geq 1}$$

satisfies MLC(1).

Proof. Let $n \geq 1$. Since $Y_{n+1} = \overline{\mathrm{Per}(f_{n+1})}$ and $\pi_n^{n+1}(\mathrm{Per}(f_{n+1})) \subset \mathrm{Per}(f_n)$,

$$\pi_n^{n+1}(Y_{n+1}) = \pi_n^{n+1}(\overline{\mathrm{Per}(f_{n+1})}) \subset \overline{\pi_n^{n+1}(\mathrm{Per}(f_{n+1}))} \subset \overline{\pi_n^{n+1}(Y_{n+1}) \cap \mathrm{Per}(f_n)}.$$

By MLC(1) of π, $\pi_n^{n+1}(Y_{n+1}) \subset \pi_n^{n+1}(X_{n+1}) = \pi_n^{n+2}(X_{n+2})$; therefore, for any $p \in \pi_n^{n+1}(Y_{n+1}) \cap \mathrm{Per}(f_n)$, there is $q \in X_{n+2}$ such that $p = \pi_n^{n+2}(q)$. Then there is $r \in Y_{n+2}$ such that

$$\lim_{k\to\infty} d_{n+2}(f_{n+2}^k(q), f_{n+2}^k(r)) = 0,$$

implying

$$\begin{aligned}\lim_{k\to\infty} d_n(f_n^k(p), f_n^k(\pi_n^{n+2}(r))) &= \lim_{k\to\infty} d_n(f_n^k(\pi_n^{n+2}(q)), f_n^k(\pi_n^{n+2}(r)))\\ &= \lim_{k\to\infty} d_n(\pi_n^{n+2}(f_{n+2}^k(q)), \pi_n^{n+2}(f_{n+2}^k(r)))\\ &= 0,\end{aligned}$$

where d_n, d_{n+2} are the metrics on X_n, X_{n+2}. Note that $\pi_n^{n+2}(r) \in \pi_n^{n+2}(Y_{n+2})$. From $p \in \mathrm{Per}(f_n)$ and the f_n-invariance of $\pi_n^{n+2}(Y_{n+2})$, it follows that $p \in \overline{\pi_n^{n+2}(Y_{n+2})} = \pi_n^{n+2}(Y_{n+2})$. Since $p \in \pi_n^{n+1}(Y_{n+1}) \cap \mathrm{Per}(f_n)$ is arbitrary, we obtain

$$\pi_n^{n+1}(Y_{n+1}) \cap \mathrm{Per}(f_n) \subset \pi_n^{n+2}(Y_{n+2})$$

and so

$$\pi_n^{n+1}(Y_{n+1}) \subset \overline{\pi_n^{n+1}(Y_{n+1}) \cap \mathrm{Per}(f_n)} \subset \overline{\pi_n^{n+2}(Y_{n+2})} = \pi_n^{n+2}(Y_{n+2}).$$

Thus, $\pi_n^{n+1}(Y_{n+1}) = \pi_n^{n+2}(Y_{n+2})$, proving the lemma. □

We now prove Lemma 4.1. The proof is based on Lemma 1.2 and by carefully choosing an inverse sequence of chain components with MLC(1).

Proof of Lemma 4.1. By Lemmas 1.2 and 2.4, we may assume $(X, f) = \lim_\pi (X_n, f_n)$, where (X_n, f_n), $n \geq 1$, are SFTs, and

$$\pi = (\pi_n^{n+1} : (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n\geq 1}$$

is an inverse sequence of equivariant maps with MLC(1). Let $Y_n = \mathrm{CR}(f_n)$, $g_n = (f_n)|_{Y_n} : Y_n \to Y_n$, and $\tilde{\pi}_n^{n+1} = (\pi_n^{n+1})|_{Y_{n+1}} : Y_{n+1} \to Y_n$ for every $n \geq 1$. Then (Y_n, g_n), $n \geq 1$, are chain recurrent SFTs, and by Lemma 4.2,

$$\tilde{\pi} = (\tilde{\pi}_n^{n+1} : (Y_{n+1}, g_{n+1}) \to (Y_n, g_n))_{n\geq 1}$$

satisfies MLC(1). Letting $(Y, g) = \lim_{\tilde{\pi}} (Y_n, g_n)$ and

$$R = \{x = (x_n)_{n\geq 1} \in X : x_n \in \mathrm{CR}(f_n),\ \text{for all } n \geq 1\},$$

we have $R = \mathrm{CR}(f)$ by Lemma 3.1, and as in the proof of Lemma 3.1, the inclusion $i : Y \to \mathrm{CR}(f)$ is a topological conjugacy $i : (Y, g) \to (\mathrm{CR}(f), f|_{\mathrm{CR}(f)})$. By this, again without loss of generality, we may assume that f and f_n, $n \geq 1$, are chain recurrent.

Since $h_{\mathrm{top}}(f) > 0$, there is $C^\dagger \in \mathcal{C}(f)$ such that $h_{\mathrm{top}}(f|_{C^\dagger}) > 0$. A proof of this fact is as follows. By the variational principle, there is an ergodic f-invariant Borel probability

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measure μ on X such that the measure-theoretical entropy $h_\mu(f)$ is positive. Since $f|_{\mathrm{supp}(\mu)}\colon \mathrm{supp}(\mu) \to \mathrm{supp}(\mu)$, the restriction of f to the support of μ, is transitive, there is $C^\dagger \in \mathcal{C}(f)$ such that $\mathrm{supp}(\mu) \subset C^\dagger$. By the variational principle again, we obtain

$$h_{\mathrm{top}}(f|_{C^\dagger}) \geq h_{\mathrm{top}}(f|_{\mathrm{supp}(\mu)}) \geq h_\mu(f|_{\mathrm{supp}(\mu)}) = h_\mu(f) > 0.$$

Then, by Lemma 3.2, there is $C_* = (C_n)_{n\geq 1} \in \mathcal{C}_\pi$ such that $C^\dagger = [C_*]$. Letting $\Gamma = \prod_{n\geq 1} \pi_n^{n+1}(C_{n+1})$, a closed F-invariant subset of $\prod_{n\geq 1} X_n$ where $F = \prod_{n\geq 1} f_n$ is the product map, since $C^\dagger \subset \Gamma$, we have

$$\begin{aligned}0 < h_{\mathrm{top}}(f|_{C^\dagger}) = h_{\mathrm{top}}(F|_{C^\dagger}) \leq h_{\mathrm{top}}(F|_\Gamma)\\ = h_{\mathrm{top}}\bigg(\prod_{n\geq 1}(f_n)|_{\pi_n^{n+1}(C_{n+1})}\bigg) = \sum_{n\geq 1} h_{\mathrm{top}}((f_n)|_{\pi_n^{n+1}(C_{n+1})})\end{aligned}$$

implying $h_{\mathrm{top}}((f_n)|_{\pi_n^{n+1}(C_{n+1})}) > 0$ for some $n \geq 1$.

Note that $\mathcal{C}(f_m)$, $m \geq 1$, are finite sets, and for any $m \geq 1$ and $D \in \mathcal{C}(f_m)$, $(f_m)|_D\colon D \to D$ is transitive (see §2.4). Let us prove the following claim.

CLAIM. *There is $C'_* = (C'_m)_{m\geq n} \in \prod_{m\geq n} \mathcal{C}(f_m)$ with the following properties:*

(1) $C'_n = C_n$;
(2) $\pi_n^{n+1}(C_{n+1}) \subset \pi_n^{n+1}(C'_{n+1})$;
(3) $\pi_m^{m+1}(C'_{m+1}) \subset C'_m$ *for every* $m \geq n$;
(4) $\pi_m^{m+1}(C'_{m+1}) = \pi_m^{m+2}(C'_{m+2})$ *for all* $m \geq n$.

Proof of the claim.

Step 1. Let $C'_n = C_n$. Take $D_{n+1} \in \mathcal{C}(f_{n+1})$ such that

$$\pi_n^{n+1}(C_{n+1}) \subset \pi_n^{n+1}(D_{n+1}) \subset C'_n \tag{P1}$$

and $\pi_n^{n+1}(D_{n+1})$ is maximal among

$$\{\pi_n^{n+1}(E_{n+1})\colon E_{n+1} \in \mathcal{C}(f_{n+1}), \pi_n^{n+1}(C_{n+1}) \subset \pi_n^{n+1}(E_{n+1}) \subset C'_n\}$$

with respect to the inclusion relation.

Step 2. Note that $(f_n)|_{\pi_n^{n+1}(D_{n+1})}$ is transitive, and take a transitive point $p_1 \in \pi_n^{n+1}(D_{n+1})$, that is, $\pi_n^{n+1}(D_{n+1}) = \omega(p_1, f_n)$, the ω-limit set. Since

$$p_1 \in \pi_n^{n+1}(D_{n+1}) \subset \pi_n^{n+1}(X_{n+1}) = \pi_n^{n+2}(X_{n+2}),$$

we have $p_1 = \pi_n^{n+2}(q_1)$ for some $q_1 \in X_{n+2}$. Take $C'_{n+1} \in \mathcal{C}(f_{n+1})$ with $\pi_{n+1}^{n+2}(q_1) \in C'_{n+1}$. Then choose $D_{n+2} \in \mathcal{C}(f_{n+2})$ such that

$$\pi_{n+1}^{n+2}(q_1) \in \pi_{n+1}^{n+2}(D_{n+2}) \subset C'_{n+1} \tag{P2}$$

and $\pi_{n+1}^{n+2}(D_{n+2})$ is maximal among

$$\{\pi_{n+1}^{n+2}(E_{n+2})\colon E_{n+2} \in \mathcal{C}(f_{n+2}), \pi_{n+1}^{n+2}(q_1) \in \pi_{n+1}^{n+2}(E_{n+2}) \subset C'_{n+1}\}$$

with respect to the inclusion relation. By (P2), we have

$$p_1 \in \pi_n^{n+2}(D_{n+2}) \subset \pi_n^{n+1}(C'_{n+1}),$$

implying

$$\pi_n^{n+1}(D_{n+1}) \subset \pi_n^{n+2}(D_{n+2}) \subset \pi_n^{n+1}(C'_{n+1})$$

since $\pi_n^{n+1}(D_{n+1}) = \omega(p_1, f_n)$, and $\pi_n^{n+2}(D_{n+2})$ is f_n-invariant. By (P1) and $p_1 \in \pi_n^{n+1}(D_{n+1})$, we see that $\pi_n^{n+1}(C_{n+1}) \subset \pi_n^{n+1}(D_{n+1})$ and $p_1 \in C'_n \cap \pi_n^{n+1}(C'_{n+1})$; therefore,

$$\pi_n^{n+1}(C_{n+1}) \subset \pi_n^{n+1}(D_{n+1}) \subset \pi_n^{n+2}(D_{n+2}) \subset \pi_n^{n+1}(C'_{n+1}) \subset C'_n.$$

By the maximality of $\pi_n^{n+1}(D_{n+1})$ in Step 1, we obtain

$$\pi_n^{n+1}(D_{n+1}) = \pi_n^{n+2}(D_{n+2}) = \pi_n^{n+1}(C'_{n+1}). \tag{Q1}$$

Step 3. Note that $(f_{n+1})|_{\pi_{n+1}^{n+2}(D_{n+2})}$ is transitive, and take a transitive point $p_2 \in \pi_{n+1}^{n+2}(D_{n+2})$, that is, $\pi_{n+1}^{n+2}(D_{n+2}) = \omega(p_2, f_{n+1})$. Since

$$p_2 \in \pi_{n+1}^{n+2}(D_{n+2}) \subset \pi_{n+1}^{n+2}(X_{n+2}) = \pi_{n+1}^{n+3}(X_{n+3}),$$

we have $p_2 = \pi_{n+1}^{n+3}(q_2)$ for some $q_2 \in X_{n+3}$. Take $C'_{n+2} \in C(f_{n+2})$ with $\pi_{n+2}^{n+3}(q_2) \in C'_{n+2}$. Then choose $D_{n+3} \in C(f_{n+3})$ such that

$$\pi_{n+2}^{n+3}(q_2) \in \pi_{n+2}^{n+3}(D_{n+3}) \subset C'_{n+2} \tag{P3}$$

and $\pi_{n+2}^{n+3}(D_{n+3})$ is maximal among

$$\{\pi_{n+2}^{n+3}(E_{n+3})\colon E_{n+3} \in C(f_{n+3}), \pi_{n+2}^{n+3}(q_2) \in \pi_{n+2}^{n+3}(E_{n+3}) \subset C'_{n+2}\}$$

with respect to the inclusion relation. By (P3), we have

$$p_2 \in \pi_{n+1}^{n+3}(D_{n+3}) \subset \pi_{n+1}^{n+2}(C'_{n+2}),$$

implying

$$\pi_{n+1}^{n+2}(D_{n+2}) \subset \pi_{n+1}^{n+3}(D_{n+3}) \subset \pi_{n+1}^{n+2}(C'_{n+2})$$

since $\pi_{n+1}^{n+2}(D_{n+2}) = \omega(p_2, f_{n+1})$, and $\pi_{n+1}^{n+3}(D_{n+3})$ is f_{n+1}-invariant. By (P2) and $p_2 \in \pi_{n+1}^{n+2}(D_{n+2})$, we see that $\pi_{n+1}^{n+2}(q_1) \in \pi_{n+1}^{n+2}(D_{n+2})$ and $p_2 \in C'_{n+1} \cap \pi_{n+1}^{n+2}(C'_{n+2})$; therefore,

$$\pi_{n+1}^{n+2}(q_1) \in \pi_{n+1}^{n+2}(D_{n+2}) \subset \pi_{n+1}^{n+3}(D_{n+3}) \subset \pi_{n+1}^{n+2}(C'_{n+2}) \subset C'_{n+1}.$$

By the maximality of $\pi_{n+1}^{n+2}(D_{n+2})$ in Step 2, we obtain

$$\pi_{n+1}^{n+2}(D_{n+2}) = \pi_{n+1}^{n+3}(D_{n+3}) = \pi_{n+1}^{n+2}(C'_{n+2}). \tag{Q2}$$

Assertions (Q1) and (Q2) yield $\pi_n^{n+1}(C'_{n+1}) = \pi_n^{n+2}(C'_{n+2})$.

Step 4. Note that $(f_{n+2})|_{\pi_{n+2}^{n+3}(D_{n+3})}$ is transitive, and take a transitive point $p_3 \in \pi_{n+2}^{n+3}(D_{n+3})$, that is, $\pi_{n+2}^{n+3}(D_{n+3}) = \omega(p_3, f_{n+2})$. Since

$$p_3 \in \pi_{n+2}^{n+3}(D_{n+3}) \subset \pi_{n+2}^{n+3}(X_{n+3}) = \pi_{n+2}^{n+4}(X_{n+4}),$$

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we have $p_3 = \pi_{n+2}^{n+4}(q_3)$ for some $q_3 \in X_{n+4}$. Take $C'_{n+3} \in C(f_{n+3})$ with $\pi_{n+3}^{n+4}(q_3) \in C'_{n+3}$. Then choose $D_{n+4} \in C(f_{n+4})$ such that

$$\pi_{n+3}^{n+4}(q_3) \in \pi_{n+3}^{n+4}(D_{n+4}) \subset C'_{n+3} \tag{P4}$$

and $\pi_{n+3}^{n+4}(D_{n+4})$ is maximal among

$$\{\pi_{n+3}^{n+4}(E_{n+4})\colon E_{n+4} \in C(f_{n+4}), \pi_{n+3}^{n+4}(q_3) \in \pi_{n+3}^{n+4}(E_{n+4}) \subset C'_{n+3}\}$$

with respect to the inclusion relation. By (P4), we have

$$p_3 \in \pi_{n+2}^{n+4}(D_{n+4}) \subset \pi_{n+2}^{n+3}(C'_{n+3}),$$

implying

$$\pi_{n+2}^{n+3}(D_{n+3}) \subset \pi_{n+2}^{n+4}(D_{n+4}) \subset \pi_{n+2}^{n+3}(C'_{n+3})$$

since $\pi_{n+2}^{n+3}(D_{n+3}) = \omega(p_3, f_{n+2})$, and $\pi_{n+2}^{n+4}(D_{n+4})$ is f_{n+2}-invariant. By (P3) and $p_3 \in \pi_{n+2}^{n+3}(D_{n+3})$, we see that $\pi_{n+2}^{n+3}(q_2) \in \pi_{n+2}^{n+3}(D_{n+3})$ and $p_3 \in C'_{n+2} \cap \pi_{n+2}^{n+3}(C'_{n+3})$; therefore,

$$\pi_{n+2}^{n+3}(q_2) \in \pi_{n+2}^{n+3}(D_{n+3}) \subset \pi_{n+2}^{n+4}(D_{n+4}) \subset \pi_{n+2}^{n+3}(C'_{n+3}) \subset C'_{n+2}.$$

By the maximality of $\pi_{n+2}^{n+3}(D_{n+3})$ in Step 3, we obtain

$$\pi_{n+2}^{n+3}(D_{n+3}) = \pi_{n+2}^{n+4}(D_{n+4}) = \pi_{n+2}^{n+3}(C'_{n+3}). \tag{Q3}$$

Assertions (Q2) and (Q3) yield $\pi_{n+1}^{n+2}(C'_{n+2}) = \pi_{n+1}^{n+3}(C'_{n+3})$.

Continuing inductively, we obtain a sequence $C'_* = (C'_m)_{m\geq n} \in \prod_{m\geq n} C(f_m)$. Then properties (1) and (2) are ensured in Steps 1 and 2. For any $k \geq 0$, $\pi_{n+k}^{n+k+1}(C'_{n+k+1}) \subset C'_{n+k}$ and $\pi_{n+k}^{n+k+1}(C'_{n+k+1}) = \pi_{n+k}^{n+k+2}(C'_{n+k+2})$ are established in Steps $k+2$ and $k+3$, respectively. Thus, C'_* satisfies the required properties, and so the claim has been proved. □

We continue the proof of Lemma 4.1. Define $C''_* = (C''_j)_{j\geq 1} \in \prod_{j\geq 1} C(f_j)$ by

$$C''_j = \begin{cases} C_j & \text{if } 1 \leq j < n, \\ C'_j & \text{if } n \leq j. \end{cases}$$

By properties (1) and (3) in the claim, we see that $C''_* \in C_\pi$. By Lemma 3.2, letting $C = [C''_*]$, we obtain $C \in C(f)$. Let

$$\pi'' = ((\pi_j^{j+1})|_{C''_{j+1}} : (C''_{j+1}, (f_{j+1})|_{C''_{j+1}}) \to (C''_j, (f_j)|_{C''_j}))_{j\geq 1}$$

and

$$\pi' = ((\pi_m^{m+1})|_{C'_{m+1}} : (C'_{m+1}, (f_{m+1})|_{C'_{m+1}}) \to (C'_m, (f_m)|_{C'_m}))_{m\geq n}.$$

Then $(C, f|_C)$ (respectively, $\lim_{\pi''}(C''_j, (f_j)|_{C''_j})$) is topologically conjugate to

$$\lim_{\pi''}(C''_j, (f_j)|_{C''_j})$$

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(respectively, $\lim_{\pi'}(C'_m, (f_m)|_{C'_m})$), so $(C, f|_C)$ is topologically conjugate to

$$\lim_{\pi'}(C'_m, (f_m)|_{C'_m}).$$

Let $(Y, g) = \lim_{\pi'}(C'_m, (f_m)|_{C'_m})$. By property (4) in the claim, π' satisfies MLC(1). Since $(f_m)|_{C'_m}$ has the shadowing property for each $m \geq 1$, due to Lemma 1.1, g has the shadowing property. Let us prove $h_{\mathrm{top}}(g) > 0$. Again by MLC(1) of π', we have $\hat{C}'_n = \pi_n^{n+1}(C'_{n+1})$ (see Lemma 2.1). Then a map $\phi\colon Y \to \pi_n^{n+1}(C'_{n+1})$, defined by $\phi(y) = y_n$ for all $y = (y_m)_{m\geq n} \in Y$, gives a factor map

$$\phi\colon (Y, g) \to (\pi_n^{n+1}(C'_{n+1}), (f_n)|_{\pi_n^{n+1}(C'_{n+1})}).$$

Property (2) in the claim ensures that $\pi_n^{n+1}(C_{n+1})$ is a closed f_n-invariant subset of $\pi_n^{n+1}(C'_{n+1})$, therefore,

$$h_{\mathrm{top}}(g) \geq h_{\mathrm{top}}((f_n)|_{\pi_n^{n+1}(C'_{n+1})}) \geq h_{\mathrm{top}}((f_n)|_{\pi_n^{n+1}(C_{n+1})}) > 0.$$

Thus, $f|_C$ has the shadowing property and satisfies $h_{\mathrm{top}}(f|_C) > 0$, completing the proof of the lemma. □

5. *Proof of Lemma 1.3*

In this section we prove Lemma 1.3. Let

$$\pi = (\pi_n^{n+1}\colon (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n\geq 1}$$

be an inverse sequence of equivariant maps with MLC(1) and suppose that (X_n, f_n) is a transitive SFT for each $n \geq 1$. For every $n \geq 1$, note that $\mathcal{D}(f_n)$ is a finite set, and let $m_n = |\mathcal{D}(f_n)|$. Then $m_n | m_{n+1}$ for all $n \geq 1$, and for any $n \geq 1$ and $E \in \mathcal{D}(f_n)$, $(f_n^{m_n})|_E\colon E \to E$ is mixing. The proof of the first lemma is similar to that of Lemma 4.1.

LEMMA 5.1. *There is $D_* = (D_n)_{n\geq 1} \in \mathcal{D}_\pi$ such that*

$$\tilde{\pi} = ((\pi_n^{n+1})|_{D_{n+1}}\colon D_{n+1} \to D_n)_{n\geq 1}$$

satisfies MLC(1).

Proof. We argue as in the proof of Lemma 4.1.

Step 1. Fix $D_1 \in \mathcal{D}(f_1)$ with $\pi_1^2(F_2) \subset D_1$ for some $F_2 \in \mathcal{D}(f_2)$. Take $E_2 \in \mathcal{D}(f_2)$ such that

$$\pi_1^2(E_2) \subset D_1 \tag{P1}$$

and $\pi_1^2(E_2)$ is maximal among

$$\{\pi_1^2(F_2)\colon F_2 \in \mathcal{D}(f_2), \pi_1^2(F_2) \subset D_1\}$$

with respect to the inclusion relation.

Step 2. Note that $(f_1^{m_2})|_{\pi_1^2(E_2)}$ is mixing, so $(f_1^{m_3})|_{\pi_1^2(E_2)}$ is transitive, and take a transitive point $p_1 \in \pi_1^2(E_2)$, that is, $\pi_1^2(E_2) = \omega(p_1, f_1^{m_3})$, the ω-limit set. Since

$$p_1 \in \pi_1^2(E_2) \subset \pi_1^2(X_2) = \pi_1^3(X_3),$$

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we have $p_1 = \pi_1^3(q_1)$ for some $q_1 \in X_3$. Take $D_2 \in \mathcal{D}(f_2)$ with $\pi_2^3(q_1) \in D_2$. Then choose $E_3 \in \mathcal{D}(f_3)$ such that

$$\pi_2^3(q_1) \in \pi_2^3(E_3) \subset D_2 \tag{P2}$$

and $\pi_2^3(E_3)$ is maximal among

$$\{\pi_2^3(F_3) \colon F_3 \in \mathcal{D}(f_3), \pi_2^3(q_1) \in \pi_2^3(F_3) \subset D_2\}$$

with respect to the inclusion relation. By (P2), we have

$$p_1 \in \pi_1^3(E_3) \subset \pi_1^2(D_2),$$

implying

$$\pi_1^2(E_2) \subset \pi_1^3(E_3) \subset \pi_1^2(D_2)$$

since $\pi_1^2(E_2) = \omega(p_1, f_1^{m_3})$, and $\pi_1^3(E_3)$ is $f_1^{m_3}$-invariant. By (P1) and $p_1 \in \pi_1^2(E_2)$, we see that $p_1 \in D_1 \cap \pi_1^2(D_2)$; therefore,

$$\pi_1^2(E_2) \subset \pi_1^3(E_3) \subset \pi_1^2(D_2) \subset D_1.$$

By the maximality of $\pi_1^2(E_2)$ in Step 1, we obtain

$$\pi_1^2(E_2) = \pi_1^3(E_3) = \pi_1^2(D_2). \tag{Q1}$$

Step 3. Note that $(f_2^{m_3})|_{\pi_2^3(E_3)}$ is mixing, so $(f_2^{m_4})|_{\pi_2^3(E_3)}$ is transitive, and take a transitive point $p_2 \in \pi_2^3(E_3)$, that is, $\pi_2^3(E_3) = \omega(p_2, f_2^{m_4})$. Since

$$p_2 \in \pi_2^3(E_3) \subset \pi_2^3(X_3) = \pi_2^4(X_4),$$

we have $p_2 = \pi_2^4(q_2)$ for some $q_2 \in X_4$. Take $D_3 \in \mathcal{D}(f_3)$ with $\pi_3^4(q_2) \in D_3$. Then choose $E_4 \in \mathcal{D}(f_4)$ such that

$$\pi_3^4(q_2) \in \pi_3^4(E_4) \subset D_3 \tag{P3}$$

and $\pi_3^4(E_4)$ is maximal among

$$\{\pi_3^4(F_4) \colon F_4 \in \mathcal{D}(f_4), \pi_3^4(q_2) \in \pi_3^4(F_4) \subset D_3\}$$

with respect to the inclusion relation. By (P3), we have

$$p_2 \in \pi_2^4(E_4) \subset \pi_2^3(D_3),$$

implying

$$\pi_2^3(E_3) \subset \pi_2^4(E_4) \subset \pi_2^3(D_3)$$

since $\pi_2^3(E_3) = \omega(p_2, f_2^{m_4})$, and $\pi_2^4(E_4)$ is $f_2^{m_4}$-invariant. By (P2) and $p_2 \in \pi_2^3(E_3)$, we see that $\pi_2^3(q_1) \in \pi_2^3(E_3)$ and $p_2 \in D_2 \cap \pi_2^3(D_3)$; therefore,

$$\pi_2^3(q_1) \in \pi_2^3(E_3) \subset \pi_2^4(E_4) \subset \pi_2^3(D_3) \subset D_2.$$

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By the maximality of $\pi_2^3(E_3)$ in Step 2, we obtain

$$\pi_2^3(E_3) = \pi_2^4(E_4) = \pi_2^3(D_3). \tag{Q2}$$

Assertions (Q1) and (Q2) yield $\pi_1^2(D_2) = \pi_1^3(D_3)$.

Step 4. Note that $(f_3^{m_4})|_{\pi_3^4(E_4)}$ is mixing, so $(f_3^{m_5})|_{\pi_3^4(E_4)}$ is transitive, and take a transitive point $p_3 \in \pi_3^4(E_4)$, that is, $\pi_3^4(E_4) = \omega(p_3, f_3^{m_5})$. Since

$$p_3 \in \pi_3^4(E_4) \subset \pi_3^4(X_4) = \pi_3^5(X_5),$$

we have $p_3 = \pi_3^5(q_3)$ for some $q_3 \in X_5$. Take $D_4 \in \mathcal{D}(f_4)$ with $\pi_4^5(q_3) \in D_4$. Then choose $E_5 \in \mathcal{D}(f_5)$ such that

$$\pi_4^5(q_3) \in \pi_4^5(E_5) \subset D_4 \tag{P4}$$

and $\pi_4^5(E_5)$ is maximal among

$$\{\pi_4^5(F_5)\colon F_5 \in \mathcal{D}(f_5), \pi_4^5(q_3) \in \pi_4^5(F_5) \subset D_4\}$$

with respect to the inclusion relation. By (P4), we have

$$p_3 \in \pi_3^5(E_5) \subset \pi_3^4(D_4),$$

implying

$$\pi_3^4(E_4) \subset \pi_3^5(E_5) \subset \pi_3^4(D_4)$$

since $\pi_3^4(E_4) = \omega(p_3, f_3^{m_5})$, and $\pi_3^5(E_5)$ is $f_3^{m_5}$-invariant. By (P3) and $p_3 \in \pi_3^4(E^4)$, we see that $\pi_3^4(q_2) \in \pi_3^4(E_4)$ and $p_3 \in D_3 \cap \pi_3^4(D_4)$; therefore,

$$\pi_3^4(q_2) \in \pi_3^4(E_4) \subset \pi_3^5(E_5) \subset \pi_3^4(D_4) \subset D_3.$$

By the maximality of $\pi_3^4(E_4)$ in Step 3, we obtain

$$\pi_3^4(E_4) = \pi_3^5(E_5) = \pi_3^4(D_4). \tag{Q3}$$

Assertions (Q2) and (Q3) yield $\pi_2^3(D_3) = \pi_2^4(D_4)$.

Continuing inductively, we obtain a sequence $D_* = (D_n)_{n\geq 1} \in \prod_{n\geq 1} \mathcal{D}(f_n)$. For any $n \geq 1$, $\pi_n^{n+1}(D_{n+1}) \subset D_n$ and $\pi_n^{n+1}(D_{n+1}) = \pi_n^{n+2}(D_{n+2})$ are established in Steps $n+1$ and $n+2$, respectively. Thus, $D_* \in \mathcal{D}_\pi$, and

$$\tilde{\pi} = ((\pi_n^{n+1})|_{D_{n+1}}\colon D_{n+1} \to D_n)_{n\geq 1}$$

satisfies MLC(1), completing the proof. □

The next lemma relates the previous lemma to a method developed in [15]. Let $\pi = (\pi_n^{n+1}\colon (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n\geq 1}$ be an inverse sequence of equivariant maps with MLC(1) and let $(X, f) = \lim_\pi (X_n, f_n)$. Suppose that (X_n, f_n), $n \geq 1$, are transitive SFTs, and for $D_* = (D_n)_{n\geq 1} \in \mathcal{D}_\pi$,

$$\tilde{\pi} = ((\pi_n^{n+1})|_{D_{n+1}}\colon D_{n+1} \to D_n)_{n\geq 1}$$

satisfies MLC(1). By Lemma 3.3, letting $D = [D_*]$, we have $D \in \mathcal{D}(f)$. Let $d, d_n, n \geq 1$, be the metrics on X, X_n.

LEMMA 5.2. *For any $\epsilon > 0$, there is $\delta > 0$ such that every δ-pseudo-orbit $(x^{(i)})_{i\geq 0}$ of f with $x^{(0)} \in D$ is ϵ-shadowed by some $x \in D$.*

Proof. Fix any $N > 0$ and $\epsilon' > 0$. Note that $f_{N+1} \colon X_{N+1} \to X_{N+1}$ has the shadowing property, and $\mathcal{D}(f_{N+1})$ is a clopen partition of X_{N+1}; therefore, there is $\delta' > 0$ such that, for any $E_{N+1} \in \mathcal{D}_{N+1}$, every δ'-pseudo-orbit $(y_{N+1}^{(i)})_{i\geq 0}$ of f_{N+1} with $y_{N+1}^{(0)} \in E_{N+1}$ is ϵ'-shadowed by some $y_{N+1} \in E_{N+1}$.

If $\delta > 0$ is small enough, then, for every δ-pseudo-orbit $\xi = (x^{(i)})_{i\geq 0}$ of f with $x^{(0)} \in D$, $\xi_{N+1} = (x_{N+1}^{(i)})_{i\geq 0}$ is a δ'-pseudo-orbit of f_{N+1} with $x_{N+1}^{(0)} \in D_{N+1}$, which is ϵ'-shadowed by some $x_{N+1} \in D_{N+1}$. Since

$$\pi_N^{N+1}(x_{N+1}) \in \pi_N^{N+1}(D_{N+1}) = \hat{D}_N$$

by MLC(1) of $\tilde{\pi}$ (see Lemma 2.1), there is $x = (x_n)_{n\geq 1} \in D = [D_*]$ such that $x_n = \pi_n^{N+1}(x_{N+1})$ for each $1 \leq n \leq N$. Then, for any $i \geq 0$ and $1 \leq n \leq N$, we have

$$\begin{aligned} d_n(f^i(x)_n, x_n^{(i)}) &= d_n(f_n^i(x_n), x_n^{(i)}) \\ &= d_n(f_n^i(\pi_n^{N+1}(x_{N+1})), \pi_n^{N+1}(x_{N+1}^{(i)})) \\ &= d_n(\pi_n^{N+1}(f_{N+1}^i(x_{N+1})), \pi_n^{N+1}(x_{N+1}^{(i)})) \end{aligned}$$

with $d_{N+1}(f_{N+1}^i(x_{N+1}), x_{N+1}^{(i)}) \leq \epsilon'$. Therefore, for every $\epsilon > 0$, if N is large enough, and then ϵ' is sufficiently small, we have $d(f^i(x), x^{(i)}) \leq \epsilon$ for all $i \geq 0$, that is, ξ is ϵ-shadowed by $x \in D$. Since ξ is arbitrary, the lemma has been proved. □

For the proof of Lemma 1.3, we need a sequence of lemmas. Let $f \colon X \to X$ be a chain transitive continuous map. For $\epsilon, \delta > 0$, we denote by $\mathcal{D}^{\epsilon,\delta}(f)$ the set of $D \in \mathcal{D}(f)$ for which every δ-pseudo-orbit $(x_i)_{i\geq 0}$ of f with $x_0 \in D$ is ϵ-shadowed by some $x \in D$. We set

$$\mathcal{D}_{\mathrm{sh}}(f) = \bigcap_{\epsilon>0} \bigcup_{\delta>0} \mathcal{D}^{\epsilon,\delta}(f).$$

LEMMA 5.3. *Let $f \colon X \to X$ be a chain transitive continuous map and let $\epsilon, \delta > 0$. For any $D \in \mathcal{D}(f)$, if $D \in \mathcal{D}^{\epsilon,\delta}(f)$, then $f(D) \in \mathcal{D}^{\epsilon,\delta}(f)$.*

Proof. Let $D \in \mathcal{D}^{\epsilon,\delta}(f)$. For any δ-pseudo-orbit $\xi = (x_i)_{i\geq 0}$ of f with $x_0 \in f(D)$, take $y \in D$ with $f(y) = x_0$ and consider

$$(y, x_0, x_1, x_2, \ldots),$$

a δ-pseudo-orbit of f, which is ϵ-shadowed by $x \in D$. Then ξ is ϵ-shadowed by $f(x) \in f(D)$. Since ξ is arbitrary, we obtain $f(D) \in \mathcal{D}^{\epsilon,\delta}(f)$, proving the lemma. □

LEMMA 5.4. *Let $f \colon X \to X$ be a chain transitive continuous map and let $\epsilon, \gamma > 0$. If $\mathcal{D}_{\mathrm{sh}}(f) \cap \mathcal{D}^{\epsilon,\gamma}(f) \neq \emptyset$, then $\mathcal{D}(f) = \mathcal{D}^{\epsilon,\delta}(f)$ holds for every $0 < \delta < \gamma$.*

Proof. Fix $D_0 \in \mathcal{D}_{\mathrm{sh}}(f) \cap \mathcal{D}^{\epsilon,\gamma}(f)$, $x \in D_0$, and a sequence $0 < \epsilon_1 > \epsilon_2 > \cdots \to 0$. Since $D_0 \in \mathcal{D}_{\mathrm{sh}}(f)$, there is a sequence $0 < \delta_1 > \delta_2 > \cdots \to 0$ such that $D_0 \in \mathcal{D}^{\epsilon_n,\delta_n}(f)$ for every $n \geq 1$. Given any $D \in \mathcal{D}(f)$ and any δ-pseudo-orbit $\xi = (x_i)_{i\geq 0}$ of f with $x_0 \in D$, since f is chain transitive, for every $n \geq 1$, there is a δ_n-chain $(x_i^{(n)})_{i=0}^{k_n}$ of f with $x_0^{(n)} = x$ and $x_{k_n}^{(n)} = x_0$. Then, for each $n \geq 1$, because $D_0 \in \mathcal{D}^{\epsilon_n,\delta_n}(f)$ and $x_0^{(n)} = x \in D_0$, we have $x_n \in D_0$ with $d(f^i(x_n), x_i^{(n)}) \leq \epsilon_n$ for all $0 \leq i \leq k_n$. Since

$$d(f^{k_n}(x_n), x_0) = d(f^{k_n}(x_n), x_{k_n}^{(n)}) \leq \epsilon_n,$$

$n \geq 1$, $\lim_{n\to\infty} \epsilon_n = 0$, and $\delta < \gamma$, there exists $N > 0$ such that for all $n \geq N$,

$$\xi_n = (f^{k_n}(x_n), x_1, x_2, \ldots)$$

is a γ-pseudo-orbit of f with $f^{k_n}(x_n) \in f^{k_n}(D_0)$. For any $n \geq N$, because $D_0 \in \mathcal{D}^{\epsilon,\gamma}(f)$, by Lemma 5.3, we have $f^{k_n}(D_0) \in \mathcal{D}^{\epsilon,\gamma}(f)$, so ξ_n is ϵ-shadowed by some $y_n \in f^{k_n}(D_0)$. Taking a sequence $N \leq n_1 < n_2 < \cdots$ such that $\lim_{j\to\infty} y_{n_j} = y$ for some $y \in X$, we easily see that ξ is ϵ-shadowed by y. Note that, for each $n \geq N$, $\{f^{k_n}(x_n), y_n\} \subset f^{k_n}(D_0)$ and so $f^{k_n}(x_n) \sim_f y_n$. Because $\sim_f$ is closed in X^2, by

$$\lim_{j\to\infty} (f^{k_{n_j}}(x_{n_j}), y_{n_j}) = (x_0, y),$$

we obtain $x_0 \sim_f y$ and thus $y \in D$. In other words, ξ is ϵ-shadowed by $y \in D$. Since ξ and then $D \in \mathcal{D}(f)$ are arbitrary, we conclude that $\mathcal{D}(f) = \mathcal{D}^{\epsilon,\delta}(f)$. □

As a consequence of Lemma 5.4, we obtain the following corollary.

COROLLARY 5.1. *For any chain transitive continuous map $f: X \to X$, if $\mathcal{D}_{\mathrm{sh}}(f) \neq \emptyset$, then $\mathcal{D}(f) = \mathcal{D}_{\mathrm{sh}}(f)$.*

The next lemma is needed for the proof of Lemma 5.6.

LEMMA 5.5. *Let $f: X \to X$ be a chain transitive continuous map and let $D \in \mathcal{D}(f)$. If, for any $x \in D$ and $\epsilon > 0$, there is $\delta(x, \epsilon) > 0$ such that every $\delta(x, \epsilon)$-pseudo-orbit $(x_i)_{i\geq 0}$ of f with $x_0 = x$ is ϵ-shadowed by some $y \in D$, then $D \in \mathcal{D}_{\mathrm{sh}}(f)$.*

Proof. Fix $\epsilon > 0$ and, for any $x \in D$, take $\delta(x, \epsilon/2) > 0$ as in the assumption. Then, for each $x \in D$, there is $0 < \delta(x) < \epsilon/2$ such that, for every $\delta(x)$-pseudo-orbit $\xi = (x_i)_{i\geq 0}$ of f with $d(x, x_0) < \delta(x)$,

$$\xi' = (x, x_1, x_2, x_3, \ldots)$$

is a $\delta(x, \epsilon/2)$-pseudo-orbit of f, $\epsilon/2$-shadowed by some $y \in D$. This clearly implies that ξ is ϵ-shadowed by $y \in D$. Take a finite subset $F \subset D$ such that

$$D \subset \bigcup_{x\in F} B_{\delta(x)}(x)$$

where $B_{\delta(x)}(x)$ denotes the $\delta(x)$-ball. Let $\delta = \min\{\delta(x) : x \in F\}$. It follows that every δ-pseudo-orbit $(x_i)_{i\geq 0}$ of f with $x_0 \in D$ is a $\delta(x)$-pseudo-orbit of f with $x_0 \in B_{\delta(x)}(x)$

for some $x \in F$, and so ϵ-shadowed by some $y \in D$. Since $\epsilon > 0$ is arbitrary, we obtain $D \in \mathcal{D}_{\mathrm{sh}}(f)$, proving the lemma. □

LEMMA 5.6. *Let $f: X \to X$ be a chain transitive continuous map with the shadowing property. Let Y be a compact metric space and let $g: Y \to Y$ be a chain transitive continuous map. If there is a factor map $\pi: (Y, g) \to (X, f)$, and if $\mathcal{D}_{\mathrm{sh}}(g) \neq \emptyset$, then $\mathcal{D}_{\mathrm{sh}}(f) \neq \emptyset$.*

Proof. Fix $D \in \mathcal{D}(f)$. By Lemma 5.5, it suffices to show that, for any $x \in D$ and $\epsilon > 0$, there exists $\delta > 0$ such that every δ-pseudo-orbit $(x_i)_{i\geq 0}$ of f with $x_0 = x$ is ϵ-shadowed by some $q \in D$, because this implies $D \in \mathcal{D}_{\mathrm{sh}}(f)$.

Let d_Y denote the metric on Y. For any $\epsilon' > 0$, Lemma 5.4 with $\mathcal{D}_{\mathrm{sh}}(g) \neq \emptyset$ implies $\mathcal{D}(g) = \mathcal{D}^{\epsilon',\delta'}(g)$ for some $\delta' > 0$. For any $\gamma > 0$, take $0 < \epsilon'' < \epsilon/2$ such that $d(x, y) \leq \epsilon''$ implies

$$\pi^{-1}(y) \subset B_\gamma(\pi^{-1}(x)) = \{z \in Y : d_Y(z, \pi^{-1}(x)) < \gamma\}$$

for all $y \in X$. Since f has the shadowing property, there is $\delta > 0$ such that every δ-pseudo-orbit $\xi = (x_i)_{i\geq 0}$ of f with $x_0 = x$ is ϵ''-shadowed by some $y \in X$. Take $z \in \pi^{-1}(y)$ and note that $d(x, y) = d(x_0, y) \leq \epsilon''$. By the choice of ϵ'', we obtain $w \in \pi^{-1}(x)$ such that $d_Y(z, w) < \gamma$. If γ is sufficiently small, then

$$\xi' = (y_i)_{i\geq 0} = (w, g(z), g^2(z), g^3(z), \ldots)$$

is a δ'-pseudo-orbit of g. By $\mathcal{D}(g) = \mathcal{D}^{\epsilon',\delta'}(g)$, ξ' is ϵ'-shadowed by some $p \in Y$ with $w \sim_g p$. Here, $w \sim_g p$ implies $\pi(w) \sim_f \pi(p)$, so, putting $q = \pi(p)$, we have $x \sim_f q$, that is, $q \in D$. Note that

$$d(q, x_0) = d(q, x) = d(\pi(p), \pi(w)) = d(\pi(p), \pi(y_0)),$$

and

$$\begin{aligned} d(f^i(q), x_i) &\leq d(f^i(q), f^i(y)) + d(f^i(y), x_i) \\ &= d(f^i(\pi(p)), f^i(\pi(z))) + d(f^i(y), x_i) \\ &= d(\pi(g^i(p)), \pi(g^i(z))) + d(f^i(y), x_i) \\ &= d(\pi(g^i(p)), \pi(y_i)) + d(f^i(y), x_i) \\ &\leq d(\pi(g^i(p)), \pi(y_i)) + \epsilon'' \\ &< d(\pi(g^i(p)), \pi(y_i)) + \epsilon/2 \end{aligned}$$

for each $i \geq 1$. Since ξ' is ϵ'-shadowed by p, we have $d_Y(g^i(p), y_i) \leq \epsilon'$, $i \geq 0$, so if ϵ' is sufficiently small, then $d(f^i(q), x_i) \leq \epsilon$ for all $i \geq 0$, that is, ξ is ϵ-shadowed by $q \in D$. Since ξ is arbitrary, this shows the existence of δ, and thus the lemma has been proved. □

For any chain transitive continuous map $f: X \to X$, we denote by $T(f)$ the set of transitive points for f:

$$T(f) = \{x \in X : X = \omega(x, f)\}$$

where $\omega(\cdot, f)$ denotes the ω-limit set.

LEMMA 5.7. *Let $f: X \to X$ be a chain transitive continuous map. For any $D \in \mathcal{D}(f)$, if $D \in \mathcal{D}_{\mathrm{sh}}(f)$, then $D \cap T(f)$ is a dense G_δ-subset of D.*

Proof. Let $\{U_n : n \geq 1\}$ be a countable basis for the topology of X. Then

$$D \cap T(f) = \bigcap_{n \geq 1} \bigcap_{j \geq 0} \bigg[D \cap \bigcup_{k \geq j} f^{-k}(U_n) \bigg],$$

a G_δ-subset of D. For any $n \geq 1$, take $p_n \in U_n$ and $\epsilon_n > 0$ such that

$$B_{\epsilon_n}(p_n) = \{y \in X : d(p_n, y) < \epsilon_n\} \subset U_n.$$

Let $n \geq 1$ and $j \geq 0$. For any $x \in D$ and $0 < \epsilon < \epsilon_n$, since $D \in \mathcal{D}_{\mathrm{sh}}(f)$, we have $D \in \mathcal{D}^{\epsilon,\delta}(f)$ for some $\delta > 0$. Then the chain transitivity of f gives a δ-chain $(x_i)_{i=0}^k$ of f with $x_0 = x$, $x_k = p_n$, and also $k \geq j$. By $x \in D$ and $D \in \mathcal{D}^{\epsilon,\delta}(f)$, we obtain $d(f^i(y), x_i) \leq \epsilon$ for all $0 \leq i \leq k$ for some $y \in D$. Note that $d(y, x) = d(y, x_0) \leq \epsilon$, $d(f^k(y), p_n) = d(f^k(y), x_k) \leq \epsilon < \epsilon_n$, and so $f^k(y) \in U_n$, implying

$$y \in D \cap \bigcup_{k \geq j} f^{-k}(U_n).$$

Since $x \in D$ and $0 < \epsilon < \epsilon_n$ are arbitrary, this shows that

$$D \cap \bigcup_{k \geq j} f^{-k}(U_n)$$

is dense in D. Since $n \geq 1$ and $j \geq 0$ are arbitrary, we conclude that $D \cap T(f)$ is a dense G_δ-subset of D, completing the proof. □

To prove Lemma 1.3, we use the method in [**15**]. The next lemma is a modification of [**15**, Lemma 2.6].

LEMMA 5.8. *Let $f: X \to X$ be a chain transitive continuous map and let $D \in \mathcal{D}(f)$. If $D \in \mathcal{D}_{\mathrm{sh}}(f)$, then, for any $y, z \in D$ and $\epsilon > 0$, there is $w \in D$ such that $d(z, w) \leq \epsilon$ and* $\limsup_{k\to\infty} d(f^k(y), f^k(w)) \leq \epsilon$.

Proof. Given any $\epsilon > 0$, take $\delta > 0$ so small that $D \in \mathcal{D}^{\epsilon,\delta}(f)$. For this δ, choose $N > 0$ as in property (3) of $\sim_{f,\delta}$ (see §2.2.3). Note that $y, z \in D$ implies $y \sim_f z$ and so $y \sim_{f,\delta} z$. Since $y \sim_{f,\delta} f^{mN}(y)$, we have $z \sim_{f,\delta} f^{mN}(y)$. Then the choice of N gives a δ-chain $\alpha = (y_i)_{i=0}^{mN}$ of f with $y_0 = z$ and $y_{mN} = f^{mN}(y)$. Let

$$\beta = (f^{mN}(y), f^{mN+1}(y), \ldots)$$

and $\xi = \alpha\beta = (x_i)_{i \geq 0}$. Then ξ is a δ-pseudo-orbit of f with $x_0 = z \in D$, so is ϵ-shadowed by some $w \in D$. Note that $d(z, w) = d(x_0, w) \leq \epsilon$. Also, we have

$$d(f^i(y), f^i(w)) = d(x_i, f^i(w)) \leq \epsilon$$

for every $i \geq mN$, so $\limsup_{k\to\infty} d(f^k(y), f^k(w)) \leq \epsilon$. This completes the proof. □

Let $f: X \to X$ be a continuous map. For $n \geq 2$ and $r > 0$, we say that an n-tuple $(x_1, x_2, \ldots, x_n) \in X^n$ is *r-distal* if

$$\inf_{k\geq 0} \min_{1\leq i<j\leq n} d(f^k(x_i), f^k(x_j)) \geq r.$$

Then the following lemma is a consequence of [15, Lemmas 2.4 and 2.5].

LEMMA 5.9. *Suppose that a continuous map $f: X \to X$ is chain transitive and has the shadowing property. If $h_{\mathrm{top}}(f) > 0$, then, for any $n \geq 2$, there is $r_n > 0$ such that, for every $D \in \mathcal{D}(f)$, there is an r_n-distal n-tuple $(x_1, x_2, \ldots, x_n) \in X^n$ with $\{x_1, x_2, \ldots, x_n\} \subset D$.*

We recall a simplified version of Mycielski's theorem [27, Theorem 1]. A topological space is said to be *perfect* if it has no isolated point.

LEMMA 5.10. *Let X be a perfect complete metric space. If R_n is a residual subset of X^n for each $n \geq 2$, then there is a Mycielski set S which is dense in X and satisfies $(x_1, x_2, \ldots, x_n) \in R_n$ for any $n \geq 2$ and distinct $x_1, x_2, \ldots, x_n \in S$.*

Finally, we complete the proof of Lemma 1.3.

Proof of Lemma 1.3. Due to Lemmas 1.2 and 2.4, (Y, g) is topologically conjugate to $(Z, h) = \lim_\pi (X_j, f_j)$ where (X_j, f_j), $j \geq 1$, are SFTs, and

$$\pi = (\pi_j^{j+1}: (X_{j+1}, f_{j+1}) \to (X_j, f_j))_{j\geq 1}$$

is an inverse sequence of equivariant maps with MLC(1). Since g is transitive and so is h, we have $Z \in \mathcal{C}(h)$, so by Lemma 3.2, $Z = [C_*]$ for some $C_* = (C_j)_{j\geq 1} \in \mathcal{C}_\pi$. Then any $z = (x_j)_{j\geq 1} \in Z$ satisfies $x_j \in C_j$ for all $j \geq 1$, and (Z, h) is topologically conjugate to $\lim_{\pi'}(C_j, (f_j)|_{C_j})$, where

$$\pi' = ((\pi_j^{j+1})|_{C_{j+1}}: (C_{j+1}, (f_{j+1})|_{C_{j+1}}) \to (C_j, (f_j)|_{C_j}))_{j\geq 1}.$$

Note that $(C_j, (f_j)|_{C_j})$, $j \geq 1$, are transitive SFTs, and by Lemma 3.4, π' satisfies MLC(1); therefore, without loss of generality, we may assume that f_j is transitive for every $j \geq 1$. Then, by Lemma 5.1, there is $D_* = (D_j)_{j\geq 1} \in \mathcal{D}_\pi$ such that

$$\tilde{\pi} = ((\pi_j^{j+1})|_{D_{j+1}}: D_{j+1} \to D_j)_{j\geq 1}$$

satisfies MLC(1). Letting $E = [D_*] \in \mathcal{D}(h)$, by Lemma 5.2, we see that $E \in \mathcal{D}_{\mathrm{sh}}(h)$, which implies $\mathcal{D}_{\mathrm{sh}}(h) \neq \emptyset$ and so $\mathcal{D}_{\mathrm{sh}}(g) \neq \emptyset$. From Lemma 5.6 and Corollary 5.1, it follows that $\mathcal{D}(f) = \mathcal{D}_{\mathrm{sh}}(f)$. For any $D \in \mathcal{D}(f)$, since $D \in \mathcal{D}_{\mathrm{sh}}(f)$, D satisfies the conclusions of Lemmas 5.7 and 5.8. Note that the conclusion of Lemma 5.9 is also satisfied. Similarly to the proof of [15, Theorem 1.1], it can be shown that there exists a sequence of positive numbers $(\delta_n)_{n\geq 2}$ for which

$$D^n \cap [T(f)^n \cap \mathrm{DC1}_n^{\delta_n}(X, f)]$$

https://doi.org/10.1017/etds.2021.127 Published online by Cambridge University Press

is a residual subset of D^n for all $D \in \mathcal{D}(f)$ and $n \geq 2$. By Lemma 5.10, we conclude that every $D \in \mathcal{D}(f)$ contains a dense Mycielski subset S which is included in $T(f)$ and distributionally n-δ_n-scrambled for all $n \geq 2$, completing the proof. □

6. *A remark on the chain components under shadowing*

Given any continuous map $f: X \to X$, $\mathcal{C}(f)$ can be seen as a quotient space of $\mathrm{CR}(f)$ with respect to the closed $(f \times f)$-invariant equivalence relation $\leftrightarrow_f$ in $\mathrm{CR}(f)^2$. Then $\mathcal{C}(f) = \mathrm{CR}(f)/\leftrightarrow_f$ is a compact metric space.

In the case of $\dim X = 0$, if f has the shadowing property, then by Lemmas 1.2 and 2.4, (X, f) is topologically conjugate to $\lim_\pi(X_n, f_n)$, where (X_n, f_n), $n \geq 1$, are SFTs, and

$$\pi = (\pi_n^{n+1}: (X_{n+1}, f_{n+1}) \to (X_n, f_n))_{n\geq 1}$$

is an inverse sequence of equivariant maps with MLC(1). Without loss of generality, we consider the case where $(X, f) = \lim_\pi(X_n, f_n)$. For any $C \in \mathcal{C}(f)$, by Lemma 3.2, we have $C = [C_*]$ for some $C_* = (C_n)_{n\geq 1} \in \mathcal{C}_\pi$. As in the proof of Lemma 4.1, it can be shown that for each $N > 0$, there is $C'_* = (C'_m)_{m\geq N} \in \prod_{m\geq N} \mathcal{C}(f_m)$ with the following properties:

(1) $C'_N = C_N$;
(2) $\pi_m^{m+1}(C'_{m+1}) \subset C'_m$ for every $m \geq N$;
(3) $\pi_m^{m+1}(C'_{m+1}) = \pi_m^{m+2}(C'_{m+2})$ for all $m \geq N$.

Define $C''_* = (C''_n)_{n\geq 1} \in \prod_{n\geq 1} \mathcal{C}(f_n)$ by

$$C''_n = \begin{cases} C_n & \text{if } 1 \leq n < N, \\ C'_n & \text{if } N \leq n. \end{cases}$$

Properties (1) and (2) ensure $C''_* \in \mathcal{C}_\pi$. By Lemma 3.2, letting $C'' = [C''_*]$, we obtain $C'' \in \mathcal{C}(f)$, and by property (3), similarly to the proof of Lemma 4.1, it can be seen that $f|_{C''}: C'' \to C''$ has the shadowing property. Note that for any neighborhood U of C in $\mathrm{CR}(f)$, by property (1) above, if N is sufficiently large, then $C'' \subset U$. Thus, letting

$$\mathcal{C}_{\mathrm{sh}}(f) = \{C \in \mathcal{C}(f): f|_C \text{ has the shadowing property}\},$$

we conclude that $\mathcal{C}_{\mathrm{sh}}(f)$ is dense in $\mathcal{C}(f)$. In other words, we obtain the following theorem.

THEOREM 6.1. *Let $f: X \to X$ be a continuous map with the shadowing property. If* $\dim X = 0$*, then* $\mathcal{C}(f) = \overline{\mathcal{C}_{\mathrm{sh}}(f)}$.

This theorem gives a positive answer to a question by Moothathu [**26**] in the zero-dimensional case. Note that, for any $C \in \mathcal{C}_{\mathrm{sh}}(f)$, by the shadowing property of $f|_C$, we have $C = \overline{M(f|_C)}$, where $M(f|_C)$ denotes the set of minimal points for $f|_C$ (see [**26**] for details).

As a complement to Theorem 6.1, we give an example of a continuous map $f: X \to X$ with the following properties:

(1) X is a Cantor space;
(2) f has the shadowing property;

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(3) $C(f)$ is a Cantor space;

(4) $C_{\mathrm{sh}}(f)$ is a countable set and so is a meager subset of $C(f)$.

Example 6.1. For any closed interval $I = [a, b]$ and $c \in (0, 1/2)$, let $\hat{I} = \{a, b\}$, $I_c^{(0)} = [a, a + c(b - a)]$, and $I_c^{(1)} = [b - c(b - a), b]$. Let $(c_j)_{j\geq 1}$ be a sequence of positive numbers with $1/2 > c_1 > c_2 > \cdots$. For any $s = (s_j)_{j\geq 1} \in \{0, 1\}^{\mathbb{N}}$, let

$$i(s) = \bigcap_{j\geq 0} I(s, j),$$

where $I(s, j)$ is defined by $I(s, 0) = [0, 1]$, and $I(s, j + 1) = I(s, j)_{c_{j+1}}^{(s_{j+1})}$ for every $j \geq 0$. Let

$$C = \{i(s) : s \in \{0, 1\}^{\mathbb{N}}\} \subset [0, 1]$$

and note that $i : \{0, 1\}^{\mathbb{N}} \to C$ is a homeomorphism, so C is a Cantor space. For any $j \geq 1$, let

$$\hat{I}_j = \bigcup_{s\in\{0,1\}^{\mathbb{N}}} [I(s, j)]\hat{}$$

and note that $\hat{I}_1 \subset \hat{I}_2 \subset \cdots$. Also, let $A_1 = \hat{I}_1$, $A_{j+1} = \hat{I}_{j+1} \setminus \hat{I}_j$, $j \geq 1$, and

$$A = \bigsqcup_{j\geq 1} A_j \subset C.$$

Let $\sigma : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ be the shift map. For each $k \geq 1$, define

$$\Sigma_k = \{x = (x_i)_{i\geq 1} \in \{0, 1\}^{\mathbb{N}} : \text{ for all } i \geq 1, \text{ for all } j \in \{i + 1, \ldots, i + k\}, \\ x_i = 1 \Rightarrow x_j = 0\},$$

which is a mixing SFT, so $\sigma|_{\Sigma_k} : \Sigma_k \to \Sigma_k$ has the shadowing property. Note that $\Sigma_1 \supset \Sigma_2 \supset \cdots$ and consider

$$\Sigma_\infty = \bigcap_{k\geq 1} \Sigma_k = \{0^\infty, 10^\infty\} \cup \{0^m 10^\infty : m \geq 1\}.$$

Then it is easily seen that $\sigma|_{\Sigma_\infty} : \Sigma_\infty \to \Sigma_\infty$ does not have the shadowing property. Let $\hat{\Sigma}_k = i(\Sigma_k)$, $k \geq 1$, and $\hat{\Sigma}_\infty = i(\Sigma_\infty)$; here, $i : \{0, 1\}^{\mathbb{N}} \to C$ is the homeomorphism defined above. Let

$$X = [(C \setminus A) \times \hat{\Sigma}_\infty] \sqcup \bigsqcup_{k\geq 1} [A_k \times \hat{\Sigma}_k],$$

which is a perfect compact subset of $C \times C$ and so is a Cantor space.

Let $\hat{\sigma} = i \circ \sigma \circ i^{-1} : C \to C$ and

$$f = (\mathrm{id}_C \times \hat{\sigma})|_X : X \to X.$$

To ensure the shadowing property of f, we define a sequence of positive numbers $(c_j)_{j\geq 1}$ with $1/2 > c_1 > c_2 > \cdots$ as follows. Fix a sequence of positive numbers $(\epsilon_k)_{k\geq 1}$ with

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$\lim_{k\to\infty} \epsilon_k = 0$. Denote by $\pi: X \to C$ the projection onto the first coordinate. Let

$$B_k = \bigsqcup_{j=1}^{k} A_j,$$

$k \geq 1$. For each $k \geq 1$, since $\pi^{-1}(B_k)$ is a finite disjoint union of SFTs,

$$f|_{\pi^{-1}(B_k)}: \pi^{-1}(B_k) \to \pi^{-1}(B_k)$$

has the shadowing property, implying the existence of $\delta'_k > 0$ such that every δ'_k-pseudo-orbit $(x_i)_{i\geq 0}$ of $f|_{\pi^{-1}(B_k)}$ is $\epsilon_k/2$-shadowed by some $x \in \pi^{-1}(B_k)$. Fix any $c_1 \in (0, 1/2)$ and assume that c_k, $k \geq 1$, is given. For any $\delta_k \in (0, \epsilon_k/2)$, if $c_{k+1} \in (0, c_k)$ is small enough, then X is contained in the δ_k-neighborhood of $\pi^{-1}(B_k)$. Then, for every δ_k-pseudo-orbit $(y_i)_{i\geq 0}$ of f, we have

$$d(x_i, y_i) = \inf\{d(y_i, z_i): z_i \in \pi^{-1}(B_k)\} \leq \delta_k$$

for all $i \geq 0$ for some $x_i \in \pi^{-1}(B_k)$. Since

$$d(f(x_i), x_{i+1}) \leq d(f(x_i), f(y_i)) + d(f(y_i), y_{i+1}) + d(y_{i+1}, x_{i+1})$$

for every $i \geq 0$, if δ_k is small enough, then $(x_i)_{i\geq 0}$ is a δ'_k-pseudo-orbit of $f|_{\pi^{-1}(B_k)}$, $\epsilon_k/2$-shadowed by some $x \in \pi^{-1}(B_k)$. This implies

$$d(f^i(x), y_i) \leq d(f^i(x), x_i) + d(x_i, y_i) \leq \epsilon_k/2 + \epsilon_k/2 = \epsilon_k$$

for all $i \geq 0$, that is, $(y_i)_{i\geq 0}$ is ϵ_k-shadowed by x. By defining $(c_j)_{j\geq 1}$ in this way, we conclude that f has the shadowing property.

Note that $X = \mathrm{CR}(f)$ and

$$C_{\mathrm{sh}}(f) = \{\pi^{-1}(u): u \in A\},$$

which is a countable set. It remains to show that $\mathcal{C}(f)$ is a Cantor space. Let $\pi_{\leftrightarrow_f}: X \to \mathcal{C}(f)$ be the quotient map. For any $x, y \in X$, we easily see that $x \leftrightarrow_f y$ if and only if $\pi(x) = \pi(y)$. This implies that there is a continuous map $h: \mathcal{C}(f) \to C$ with $\pi = h \circ \pi_{\leftrightarrow_f}$, which is bijective and so is a homeomorphism. Thus, $\mathcal{C}(f)$ is a Cantor space.

Acknowledgements. The author would like to thank the anonymous reviewer for helpful comments and suggestions. The author is a JSPS Research Fellow. This work was supported by JSPS KAKENHI grant number JP20J01143.

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