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SYMMETRIC DUALITY IN MULTI-OBJECTIVE PROGRAMMING

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Abstract

A pair of multi-objective programming problems is shown to be symmetric dual by associating a vector-valued infinite game to the given pair. This symmetric dual pair seems to be more general than those studied in the literature.

1. Introduction

Dorn [6] introduced symmetric duality in nonlinear programming by defining a program and its dual to be symmetric if the dual of the dual is the original problem. In the past, the symmetric duality has been studied extensively in the literature, notably by Dantzig *et al.* [5], Mond [7] and Mond and Weir [9].

Recently, Weir and Mond [13] studied symmetric duality in the context of multi-objective programming by introducing a multi-objective analogue of the primal-dual pair presented in Mond [8]. Although the multi-objective primal-dual pair constructed in [13] subsumes the single objective symmetric duality [7] as a special case, the construction of [13] seems to be somewhat restricted because the same parameter $\lambda \in R^p$ (vector multiplier corresponding to various objectives) is present in both primal and dual. Further, the proof of the main duality result in [13] assumes this λ to be fixed in the dual problem.

The main aim of this paper is to present a pair of multi-objective programming problems (P) and (D) with λ as variable in both programs and to establish symmetric duality by associating a vector-valued infinite game to this pair.

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Although this construction seems to be more natural than that of [13] as it does not require λ to be fixed in the dual problem, it lacks the weak duality theorem as illustrated in Section 3. However, the case of single objective symmetric duality [7] is fully subsumed here as well, because (P) and (D) then reduce to the primal-dual pair of Dantzig *et al.* [5].

2. Preliminaries and statement of problems

Let R^n be an n-dimensional Euclidean space and R^n_+ be its non-negative orthant. For $z, w \in R^n$, by $z \leqq w$ we mean $z_i \leqq w_i$ for all i, and $z \leq w$ means $z_i \leqq w_i$ for all i and $z_s > w_s$ for at least one s, $1 \leqq s \leqq n$. By $z < w$, we mean $z_i < w_i$ for all i. Let $e = (1, 1, ..., 1)^{\mathrm{T}} \in R^p$ and $\Lambda = \{\lambda \in R^p_+ : \lambda^{\mathrm{T}} e = 1\}$.

We now consider the vector-valued two-person zero-sum game $G : \{X, Y, K\}$, where

(i) $X = x \in R^m_+ : p_k(x) \leqq 0, k = 1, 2, \dots, s$ is the space of strategies for player I

(ii) $Y = y \in R^m_+ : q_k(x) \geqq 0, r = 1, 2, \dots, t$ is the space of strategies for player II

(iii) $K : X \times Y \to R^p$ given by $K(X, Y) = (K_1(x, y), K_2(x, y), \dots, K_p(x, y))$, is the payoff to player I and $-K(x, y)$ is the payoff to player II.

In this representation it is assumed that player I solves the "min-max problem" and player II solves the "max-min problem" in the sense of Definition 3 given below. Also the symbol "V-max" stands for vector maximisation and V-min stands for vector minimisation.

The following definition will be needed in this sequel.

DEFINITION 1. *(Corley* [1] *): A point $(\bar{x}, \bar{y}) \in X \times Y$ is said to be an equilibrium point of the game G if*

$$K(x, \bar{y}) \not\geq K(\bar{x}, \bar{y}) \text{ for all } x \in X$$
$$\text{and } K(\bar{x}, y) \not\leq K(\bar{x}, \bar{y}) \text{ for all } y \in Y$$

DEFINITION 2. *(Tanino, Nakayama and Sawaragi* [12] *): Let $f : R^n \to R^p$. A point $\bar{x} \in X$ is said to be an* efficient solution *of the vector maximisation problem: V-max $f(x)$ over $x \in X$, if there does not exist any $x \in X$ such that $f(x) \geq f(\bar{x})$.*

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DEFINITION 3. *(Rödder* [10] *): A point* $(x^0, y^0) \in X \times Y$ *is called a solution of the max-min problem if*

(i) y^0 *is an efficient solution of* $V\text{-}\min_{y \in Y} K(x^0, y)$
(ii) $K(x^0, y^0) \not\leq K(x, y)$ *for all* $x \in X$ *and* $y \in Y$.

DEFINITION 4. *(Rödder* [10] *): A point* $(x^0, y^0) \in X \times Y$ *is called a solution of the min-max problem if*

(i) x^0 *is an efficient solution of* $V\text{-}\max_{x \in X} K(x, y^0)$
(ii) $K(x^0, y^0) \not\geq K(x, y)$ *for all* $x \in X$ *and* $y \in Y$.

DEFINITION 5. *(Rödder* [10] *): A point* $(x^0, y^0) \in X \times Y$ *is called a generalised saddle point if* (x^0, y^0) *solves both max-min and min-max problems.*

LEMMA 1. *(Rödder* [10] *): The following statements are equivalent.*

(i) (x^0, y^0) *is a generalised saddle point of* $K(x, y)$ *in* $X \times Y$.
(ii) y^0 *solves* $V\text{-}\min_{y \in Y}(x^0, y)$ *and* x^0 *solves* $V\text{-}\max_{x \in X} K(x, y^0)$.
(iii) $K(x, y^0) \not\geq K(x^0, y^0) \forall x \in X$ *and* $K(x^0, y) \not\leq K(x^0, y^0) \forall y \in Y$

We now state the following two multi-objective programming problems (P) and (D) and establish the main duality theorem in Section 3:

(P): $V\text{-}\min(K_1(x, y) - x^{\mathrm{T}} \nabla_1[\mu^{\mathrm{T}} K(x, y)], \ldots, K_p(x, y) - x^{\mathrm{T}} \nabla_1[\mu^{\mathrm{T}} K(x, y)])$,
subject to

$$\nabla_1[\mu^{\mathrm{T}} K(x, y)] \leqq 0, \tag{1}$$

$$x \geqq 0,\ y \geqq 0,\ \mu \in \Lambda. \tag{2}$$

(D): $V\text{-}\min(K_1(u, v) - x^{\mathrm{T}} \nabla_2[\mu^{\mathrm{T}} K(u, v)], \ldots, K_p(u, v) - x^{\mathrm{T}} \nabla_2[\mu^{\mathrm{T}} K(u, v)])$,
subject to

$$\nabla_2[\alpha^{\mathrm{T}} K(u, v)] \geqq 0, \tag{3}$$

$$u \geqq 0,\ v \geqq 0,\ \alpha \in \Lambda. \tag{4}$$

Here $x, u \in R^m$; $y, v \in R^n$; $\mu, \alpha \in R^p$; and $K : R^m \times R^n \to R^p$.

3. Vector-valued infinite game and multi-objective programming

Corresponding to the multi-objective programming problems (P) and (D) as defined above, we introduce the vector-valued infinite game $VG : \{S, T, K\}$, where

(i) $S = \{x \in R^m : x \geqq 0\}$ is the strategy space for player I,
(ii) $T = \{y \in R^n : y \geqq 0\}$ is the strategy space for player II

and

(iii) $K : S \times T \to R^p$ defined by $K(x, y)$, is the payoff to player I. The payoff to player II will be taken as $-K(x, y)$.

The theorems given below give necessary and sufficient conditions for a pair $(\bar{x}, \bar{y}) \in S \times T$ to be an equilibrium point of the game VG.

THEOREM 1. *(Necessary conditions): Let $(\bar{x}, \bar{y})$ be an equilibrium point of the game VG. Then there exists $\overline{\mu} \in R^p_+, \overline{\mu} \neq 0$ and $\overline{\alpha} \in R^p_+, \overline{\alpha} \neq 0$ such that $(\bar{x}, \bar{y}, \overline{\mu})$ and $(\bar{x}, \bar{y}, \overline{\alpha})$ are efficient to multi-objective programming problems* (P) *and* (D) *respectively.*

PROOF. Since $(\bar{x}, \bar{y})$ is an equilibrium point of the game VG, it follows that

$$K(\bar{x}, \bar{y}) \nleq K(x, \bar{y}) \quad \forall x \in S \tag{5}$$

$$\text{and} \quad K(\bar{x}, \bar{y}) \ngeq K(\bar{x}, y) \quad \forall y \in T. \tag{6}$$

Now (5) implies that $\bar{x}$ is an efficient solution of the following problem:

$$(P)_{\bar{y}} \quad V\text{-}\max K(x, \bar{y}), \text{ subject to } x \geqq 0.$$

Hence there exists (Singh [11]) $\mu_0 \in R^p_+$, $\mu_0 \neq 0$ such that

$$\begin{aligned} \nabla_1[\mu_0^{\mathsf{T}} K(\bar{x}, \bar{y})] &\leqq 0, \\ \bar{x}^{\mathsf{T}} \nabla_1[\mu_0^{\mathsf{T}} K(\bar{x}, \bar{y})] &= 0, \\ \overline{x} &\geqq 0. \end{aligned}$$

Let $\overline{\mu} = (\mu_0/\mu_0^{\mathsf{T}} e)$ so that $\overline{\mu} \in \Lambda$.

Since $\bar{y} \in T$, it follows that $(\bar{x}, \bar{y}, \overline{\mu})$ is feasible for (P) with $\bar{x}^{\mathsf{T}} \nabla_1 \bar{\mu}^{\mathsf{T}} K(\bar{x}, \bar{y}) = 0$. Now it remains to show that $(\bar{x}, \bar{y}, \overline{\mu})$ is efficient to (P). If possible let $(\bar{x}, \bar{y}, \overline{\mu})$

be not efficient to (P); then there exists (x_0, y_0, μ) which is feasible for (P) such that

$$K_i(x_0, y_0) - x_0^{\mathsf{T}}\nabla_1[\mu^{\mathsf{T}}K(x_0, y_0)] \leqq K_i(\bar{x}, \bar{y}) - \bar{x}^{\mathsf{T}}\nabla_1[\bar{\mu}^{\mathsf{T}}K(x, y)]$$

and

$$K_j(x_0, y_0) - x_0^{\mathsf{T}}\nabla_1[\mu^{\mathsf{T}}K(x_0, y_0)] < K_j(\bar{x}, \bar{y}) - \bar{x}^{\mathsf{T}}\nabla_1[\bar{\mu}^{\mathsf{T}}K(x, y)]$$

for at least j.

The above relations give $K(x_0, y_0) \leq K(\bar{x}, \bar{y})$ which contradicts the definition of a generalised saddle point. Hence $(\bar{x}, \bar{y}, \overline{\mu})$ is efficient to (P). Similarly from (6), we get that $(\bar{x}, \bar{y}, \overline{\alpha})$ is efficient to (D).

THEOREM 2. *(Sufficient conditions): Let $(\bar{x}, \bar{y}, \overline{\mu})$ and $(\bar{x}, \bar{y}, \overline{\alpha})$ be feasible for* (P) *and* (D) *respectively with*

$$\bar{x}^{\mathsf{T}}\nabla_1[\bar{\mu}^{\mathsf{T}}K(\bar{x}, \bar{y})] = 0 = \bar{y}^{\mathsf{T}}\nabla_1[\alpha^{-\mathsf{T}}K(\bar{x}, \bar{y})]$$

and $\overline{\mu} > 0$, $\overline{\alpha} > 0$. Also let, for each $i = 1, 2, \dots, p$, K_i be concave-convex. Then $(\bar{x}, \bar{y})$ is an equilibrium point of the game VG.

PROOF. We have to prove that

$$K(\bar{x}, \bar{y}) \not\leq K(x, \bar{y}) \qquad \forall x \in S,$$
$$K(\bar{x}, \bar{y}) \not\geq K(\bar{x}, y) \qquad \forall y \in T.$$

If possible, let $K(\bar{x}, \bar{y}) \leq K(\hat{x}, \bar{y})$ for some $\hat{x} \in S$. Therefore $\bar{\mu}^{\mathsf{T}}K(\bar{x}, \bar{y}) < \bar{\mu}^{\mathsf{T}}K(x, \bar{y})$. Now by the concavity of $\bar{\mu}^{\mathsf{T}}K$ at x, we have

$$(\hat{x} - \bar{x})^{\mathsf{T}}\nabla_1[\bar{\mu}^{\mathsf{T}}K(\bar{x}, \bar{y})] > 0$$

i.e.

$$\hat{x}^{\mathsf{T}}\nabla_1[\bar{\mu}^{\mathsf{T}}K(\bar{x}, \bar{y})] > \bar{x}^{\mathsf{T}}\nabla_1[\bar{\mu}^{\mathsf{T}}K(\bar{x}, \bar{y})]. \tag{7}$$

But (3) together with the hypothesis of the theorem yields

$$x^{\mathsf{T}}\nabla_1[\bar{\mu}^{\mathsf{T}}K(\bar{x}, \bar{y})] > 0,$$

which contradicts (1). Hence $K(\bar{x}, \bar{y}) \not\leq K(x, \bar{y})$, $\forall x \in S$. Similarly we can show that $K(\bar{x}, \bar{y}) \not\geq K(\bar{x}, y), \forall y \in T$.

COROLLARY 1. *If $\mu \geq 0$ and each K_i is strictly concave at $\bar{x}$, then Theorem 2 holds also.*

COROLLARY 2. *If $\alpha \geq 0$ and each K_i is strictly convex at $\bar{y}$, then Theorem 2 holds also.*

4. Symmetric duality

In this section, we shall prove a symmetric duality theorem for multi-objective programming problems (P) and (D). In this context, it may be remarked that the traditional weak duality theorem [13] does not hold good for multi-objective programming problems (P) and (D), as illustrated by the following example.

Example: Let

$$K_1(x, y) = -x_1^2 - 30x_2^2 + 2y_1^2 + 50y_2^2,$$
$$K_2(x, y) = -3x_1^2 - 0.5x_2^2 + 5y_1^2 + 0.4y_2^2$$

where $x = (x_1, x_2)^{\mathsf T}$ and $y = (y_1, y_2)^{\mathsf T}$. Then $(x_1 = 0.2, x_2 = 0.3, y_1 = 0.0, y_2 = 0.0, \mu_1 = 0.25, \mu_2 = 0.75)$ and $(u_1 = 0.0, u_2 = 0.0, v_1 = 1.0, v_2 = 0.0, \alpha_1 = 0.5, \alpha_2 = 0.5)$ are feasible solutions for (P) and (D) respectively. Further for these feasible solutions, the primal and dual objective values for (P) and (D) are (-1.1225, 1.4525) and (-1.0, 2.0) respectively. But -1.1225 < -1.0 and 1.4525 < 2.0, and so the weak duality theorem between (P) and (D) does not hold good.

THEOREM 3. *(Symmetric Duality): Let $(\bar{x}, \bar{y}, \overline{\mu})$ be an efficient solution of* (P) *with $\overline{\mu} > 0$. Assume that the Hessian matrix $\nabla_{11}[\bar{\mu}^{\mathsf T} K]$ is negative definite. Let for each $i = 1, 2, \ldots, p$, $K_i(., \bar{y})$ be concave at $\bar{x}$ and $K_i(\bar{x}, .)$ be strictly convex at $\bar{y}$. Then there exists $\bar{\alpha} \in R_+^p$, $\bar{\alpha} \neq 0$ such that $(\bar{x}, \bar{y}, \overline{\alpha})$ is efficient to* (D).

PROOF. Since $(\bar{x}, \bar{y}, \overline{\mu})$ is an efficient solution of (P), it is a weak minimum. Hence there exists $\xi \in R^p$, $\delta \in R^m$, $\beta \in R^n$, $\gamma \in R^p$, $\eta \in R^p$ such that $(\bar{x}, \bar{y}, \overline{\mu})$ satisfies the following conditions ([3] and [4]): (For simplicity we write $\nabla_1[\bar{\mu}^{\mathsf T} K]$, $\nabla_{11}[\bar{\mu}^{\mathsf T} K]$ etc. instead of $\nabla_1[\bar{\mu}^{\mathsf T} K(\bar{x}, \bar{y})]$, $\nabla_{11}[\bar{\mu}^{\mathsf T} K(\bar{x}, \bar{y})]$ etc. respectively)

$$\nabla_1\Big[\xi^{\mathsf T} - \Big(\sum_i \xi_1\Big)\bar{x}^{\mathsf T} K\Big]\nabla_{11}[\bar{\mu}^{\mathsf T} K] + \delta^{\mathsf T}\nabla_{11}[\bar{\mu}^{\mathsf T} K] - \Big(\sum_i \xi_i \nabla_1[\bar{\mu}^{\mathsf T} K]\Big) \geqq 0, \tag{8}$$

$$\bar{x}^{\mathsf T}\nabla_1[\xi^{\mathsf T} K] - \Big(\sum_1 \xi_1\Big)\bar{x}^{\mathsf T}\left\{\nabla_{11}[\bar{\mu}^{\mathsf T} K]\right\}\bar{x} + \delta^{\mathsf T}\nabla_{11}[\bar{\mu}^{\mathsf T} K]\bar{x} - \Big(\sum_1 \xi_1\Big)\bar{x}^{\mathsf T}\nabla_1[\bar{\mu}^{\mathsf T} K] = 0, \tag{9}$$

$$\nabla_2[\xi^{\mathsf T} K] - \Big(\sum_i \xi_1\Big)\bar{x}^{\mathsf T}\nabla_{12}[\bar{\mu}^{\mathsf T} K] + \delta^{\mathsf T}\nabla_{12}[\bar{\mu}^{\mathsf T} K] - \beta = 0, \tag{10}$$

$$-\Big(\sum_i \xi_1\Big)\bar{x}^{\mathsf{T}}\nabla_1 K_i + \delta^{\mathsf{T}}\nabla_1 K_i - \gamma_i - \eta = 0, \qquad i = 1, 2, \dots, p, \tag{11}$$

$$\delta^{\mathsf{T}}\nabla[\overline{\mu}^{\mathsf{T}}K] = 0, \tag{12}$$

$$\beta^{\mathsf{T}}\bar{y} = 0, \tag{13}$$

$$\gamma^{\mathsf{T}}\overline{\mu} = 0, \tag{14}$$

$$(\xi, \delta, \beta, \gamma) \neq 0, \tag{15}$$

$$(\xi, \delta, \beta, \gamma) \geqq 0. \tag{16}$$

Since $\overline{\mu} > 0$, it follows from (14) that $\gamma = 0$. Hence (11) becomes

$$(\delta - \sigma\bar{x})^{\mathsf{T}}\nabla_1 K_i - \eta = 0, \qquad i = 1, 2, \dots, p, \tag{17}$$

where $\sigma = \sum_{i=1}^{p}\xi_i$. (8) and (9) can be rewritten as

$$\nabla_1[(\xi - \sigma\overline{\mu})^{\mathsf{T}}K] + (\delta - \sigma\bar{x})^{\mathsf{T}}\nabla_{11}[\bar{\mu}^{\mathsf{T}}K] \geqq 0, \tag{18}$$

$$\bar{x}^{\mathsf{T}}\nabla_1[(\xi - \sigma\overline{\mu})^{\mathsf{T}}K] + (\delta - \sigma\bar{x})^{\mathsf{T}}\nabla_{11}[\bar{\mu}^{\mathsf{T}}K]\bar{x} = 0. \tag{19}$$

Now from (16), (18) and (19), it follows that

$$\delta - \sigma\bar{x})^{\mathsf{T}}\nabla_1[(\xi - \sigma\overline{\mu})^{\mathsf{T}}K] + (\delta - \sigma\bar{x})^{\mathsf{T}}\nabla_{11}[\bar{\mu}^{\mathsf{T}}K](\delta - \sigma\bar{x}) \geqq 0.$$

By using (17), the above inequality gives

$$\Big(\sum_i \xi_i - \sigma\sum_i \mu_i\Big)\eta + (\delta - \sigma\bar{x})^{\mathsf{T}}\nabla_{11}[\bar{\mu}^{\mathsf{T}}K](\delta - \sigma\bar{x}) \geqq 0,$$

which implies that

$$(\delta - \sigma\bar{x})^{\mathsf{T}}\nabla_{11}[\bar{\mu}^{\mathsf{T}}K](\delta - \sigma\bar{x}) > 0.$$

Since the Hessian matrix $\nabla_{11}[\bar{\mu}^{\mathsf{T}}K]$ is negative definite, it follows that

$$\delta - \sigma\bar{x} = 0$$

$$\Rightarrow \qquad \delta = \sigma\bar{x} \tag{20}$$

Let $\sigma = 0$. Then $\xi = 0$ and $\delta = 0$. Thus from (10) and (17), we have $\beta = 0$ and $\eta = 0$. Hence $\xi = 0$, $\delta = 0$, $\gamma = 0$, $\beta = 0$, $\eta = 0$ contradicts (15). Therefore, $\sigma > 0$, i.e. $\xi \geq 0$. Now (10) and (20) imply

$$\nabla_2[\bar{\xi}^{\mathsf{T}}K] \geqq 0 \text{ where } \overline{\xi} = \xi/\sigma. \tag{21}$$

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Also (10), (13) and (20) give

$$\bar{y}^{\mathrm{T}}\nabla_2[\bar{\xi}^{\mathrm{T}}K] = 0. \tag{22}$$

Thus from (21) and (22), it follows that $(\bar{x}, \bar{y}, \bar{\xi})$ is feasible for (D) with $\bar{y}^{\mathrm{T}}\nabla_2[\bar{\xi}^{\mathrm{T}}K] = 0$. Also from (20) and (12), we have

$$\bar{y}^{\mathrm{T}}\nabla_2[\bar{\xi}^{\mathrm{T}}K] = 0.$$

Now by applying Theorem 2, we have $(\bar{x}, \bar{y})$ is an equilibrium point of the game VG. Hence by Theorem 1, there exists $\overline{\alpha} \in R_+^p$, $\overline{\alpha} \neq 0$ such that $(\bar{x}, \bar{y}, \overline{\alpha})$ is efficient to (D). This proves Theorem 3.

REMARK 1. *In the case $p = 1$, the strict convexity of $K(\bar{x}, .)$ at $\bar{y}$ can be replaced by convexity.*

REMARK 2. *The symmetric dual formulations of* [13] *as well as the one presented here are restricted because the former has the same multipliers λ and μ in* (P) *and* (D) *and the latter does not admit the weak duality theorem in general.*

Thus the status of symmetric duality in multi-objective programming is not very satisfactory, and it needs to be studied further.

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