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EQUICONTINUITY OF FAMILIES OF CONVEX AND CONCAVE-CONVEX OPERATORS

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Introduction. J. M. Borwein has given in [**1**] a practical necessary and sufficient condition for a convex operator to be continuous at some point. Indeed J. M. Borwein has proved in his paper that a convex operator with values in an order topological vector space F (with normal positive cone F_+) is continuous at some point if and only if it is bounded from above by a mapping which is continuous at this point. This result extends a previous one by M. Valadier in [**16**] asserting that a convex operator is continuous at a point whenever it is bounded from above by an element in F on a neighbourhood of the concerned point. Note that Valadier's result is necessary if and only if the topological interior of F_+ is nonempty. Obviously both results above are generalizations of the classical one about real-valued convex functions formulated in this context exactly as Valadier's result (see for example [**5**]).

Our aim here is to study the equicontinuity of families of convex and concave-convex operators following the way opened by J. M. Borwein. Section one contains some preliminary material. Section two is devoted to some properties of equicontinuous families of convex and concave-convex operators. In particular we prove that a family of convex operators is equicontinuous at a point if and only if it is equilipschitzian at this point and we also give some generalizations of the Banach-Steinhaus theorem to convex operators.

In Section three we establish some conditions ensuring equicontinuity of families of convex and concave-convex operators and hence at the same time we provide a necessary and sufficient condition for a concave-convex operator to be continuous at a point. The results of this section generalize those given by R. T. Rockafellar in [**12**] for real-valued functions.

Section four closes with conditions of equicontinuity expressed in terms of continuous seminorms. A practical new necessary and sufficient condition for continuity of convex (or concave-convex) operators (see Corollary 4.5) is also proved.

Let us indicate that some related results on equilipschitzian operators can be found in [**2**] and [**3**].

1. Preliminaries. Throughout this paper E, F, G and H denote (real separated) topological vector spaces. We always assume that F_+ is a

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convex cone in F ($sF_+ + tF_+ \subset F_+$ for all real numbers $s, t \geqq 0$) and hence it induces an ordering in F by $y_1 \leqq y_2$ if $y_2 - y_1 \in F_+$. So F is an ordered topological vector space.

We adjoin an abstract greatest element infinity and a lowest one to F and we shall write $F^{\cdot} = F \cup \{+\infty\}$ and $\bar{F} = F \cup \{-\infty, +\infty\}$. This allows us to say that a mapping $f: \rightarrow F^{\cdot}$ is *convex* if

$$f(sx + ty) \leqq sf(x) + tf(y)$$

for all $x, y \in E$ and all positive numbers s, t satisfying $s + t = 1$. The set

$$\operatorname{dom} f = \{x \in E: f(x) \in F\}$$

is the domain of f. In the same way a mapping $f:C \rightarrow F$, where C is a convex subset in E, is convex if $\bar{f}:E \rightarrow F^{\cdot}$ defined by $\bar{f}(x) = f(x)$ if $x \in C$ and $\bar{f}(x) = +\infty$ if $x \notin C$ is convex.

1.1. *Definition*. One says that F_+ is a *normal cone* or F is *normal* if there exists a base of neighbourhoods $\{V\}$ of zero in F such that

$$V = (V - F_+) \cap (V + F_+).$$

Such neighbourhoods are said to be *full*. Many properties of normal cones may be found in [**11**] where it is proved that most of the usual ordered topological vector spaces are normal. In the sequel we shall always assume that F_+ is a normal convex cone.

Let us remark that, if F is locally convex and F_+ is normal, then there exists a base of convex, circled and full neighbourhoods of zero in F.

2. Properties of equicontinuous families of convex operators. In this section we shall consider some properties of equicontinuous families of convex operators.

2.1. *Definition*. We shall say that a family $(f_i)_{i \in I}$ of mappings from a topological space X into $F \cup \{-\infty, \infty\}$ is *upper equisemicontinuous* at a point $a \in X$ for which all f_i are finite on some neighbourhood of a in X if for every neighbourhood W of zero in F there exists a neighbourhood V of a in X such that

$$f_i(x) \in f_i(a) + W - F_+ \text{ for each } i \in I \text{ and each } x \in V.$$

In the same way we shall say that $(f_i)_{i \in I}$ is *lower equisemicontinuous* at a if the family $(-f_i)_{i \in I}$ is upper equisemicontinuous at a and that a mapping $f:X \rightarrow \bar{F}$ is upper (resp. lower) semicontinuous at a if $\{f\}$ is upper (resp. lower) equisemicontinuous at a.

For properties of semicontinuous vector-valued functions the reader is referred to [**10**].

2.2. PROPOSITION. *A family $(f_i)_{i \in I}$ of convex operators from E into $F^{\cdot}$ is equicontinuous at a point a (with $f_i(a) \in F$ for each $i \in I$) if and only if it is upper equisemicontinuous at a.*

Proof. It is enough to show that the condition is sufficient. Let W be any neighbourhood of zero in F. Choose a full circled neighbourhood W_0 of zero in F with $W_0 \subset W$ and a circled neighbourhood V of zero in E satisfying

(2.1) $f_i(a + x) - f_i(a) \in W_0 - F_+$ for all $x \in V$ and $i \in I$.

As f_i is convex, by relation (2.1) we have for each $x \in V$ and each $i \in I$

$$f_i(a + x) - f_i(a) \in f_i(a) - f_i(a - x) + F_+$$
$$\subset W_0 + F_+ + F_+ = W_0 + F_+.$$

Making use of relation (2.1) once again we obtain

$$f_i(a + x) - f_i(a) \in (W_0 - F_+) \cap (W_0 + F_+) = W_0 \subset W$$

for each $x \in V$ and each $i \in I$.

Remark. This generalizes a result in **[10]** proving that upper semicontinuous convex operators are continuous.

A similar result also holds for families of biconvex operators.

2.3. *Definition.* Let C and D be two convex subsets of G and H respectively. A mapping $f:C \times D \to F$ is said to be *biconvex* (resp. *concave-convex*) if for each $(x, y) \in C \times D$ the mappings $f(x, \cdot)$ and $f(\cdot, y)$ (resp. $-f(\cdot, y)$) are convex.

2.4. PROPOSITION. *Let C and D be two open convex subsets of G and H respectively. A family $(f_i)_{i \in I}$ of biconvex operators from $C \times D$ into F is equicontinuous at a point $(c, d) \in C \times D$ if and only if it is upper equisemicontinuous at (c, d).*

Proof. The condition is obviously necessary. Consider now any full circled neighbourhood W_0 of zero in F. Choose a circled neighbourhood W of zero with $W + W + W \subset W_0$ and a circled neighbourhood $U \times U'$ of (0, 0) in $G \times H$ satisfying

$$(c, d) + U \times U' \subset C \times D \quad \text{and}$$
$$f_i((c, d) + U \times U') \subset f_i(c, d) + W - F_+.$$

By assumption and by Proposition 2.2 the family of convex operators

$(f_i(\cdot\,, d))_{i\in I}$ is equicontinuous at c and hence there exists a neighbourhood V of zero in G such that $c + V \subset C$ and

$$f_i(c + V, d) \subset f_i(c, d) + W \text{ for every } i \in I.$$

Therefore, as $f_i(c + x, \cdot)$ is convex, we have for each $(x, y) \in (U \cap V) \times U'$ and each $i \in I$

$$\begin{aligned} f_i(c + x, d + y) &\in 2[f_i(c + x, d) - f_i(c, d)] \\ &+ f_i(c, d) + f_i(c, d) - f_i(c + x, d - y) + F_+ \\ &\subset f_i(c, d) + W + W + W + F_+ \\ &\subset f_i(c, d) + W_0 + F_+ \end{aligned}$$

and hence

$$f_i(c + x, d + y) - f_i(c, d) \in (W_0 + F_+) \cap (W_0 - F_+) = W_0$$

for all $i \in I$ and $(x, y) \in (U \cap V) \times U'$.

In order to show that for convex operators equicontinuity is equivalent to equi-Lipschitz continuity let us consider the following notions.

For each closed circled neighbourhood W of a topological vector space Y we shall put

$$\rho_W(y) = \inf\{t > 0 : y \in tW\}$$

and hence $y \in W$ if and only if $\rho_W(y) \leqq 1$.

Following the generalization of Lipschitz mappings introduced in [**14**] we shall say that a family $(f_i)_{i\in I}$ of mappings from an open subset X_0 of a topological vector space X into a topological vector space Y is *equilipschitzian around* a point $a \in X_0$ if for each closed circled neighbourhood W of zero in Y there exists a closed circled neighbourhood V of zero in X and a neighbourhood U of zero in X such that

$$\rho_W(f_i(x) - f_i(y)) \leqq \rho_V(x - y)$$

for all $x, y \in a + U \subset X_0$ and $i \in I$.

Obviously this definition extends the classical one related to normed or locally convex vector spaces.

2.5. PROPOSITION. *Let $(f_i)_{i\in I}$ be a family of convex operators from an open convex subset C of E into F. This family is equicontinuous at a point $a \in C$ if and only if it is equilipschitzian around a.*

Proof. Suppose that the family $(f_i)_{i\in I}$ is equicontinuous at a. For every $x \in C_0 := C - a$ put

$$g_i(x) = f_i(x + a) - f_i(a).$$

The family $(g_i)_{i\in I}$ of convex operators from C_0 into F is equicontinuous at

0 with $g_i(0) = 0$. Let W be any closed circled neighbourhood of zero in F. Choose a full circled neighbourhood W_0 of zero in F with $W_0 \subset W$ and by equicontinuity of $(g_i)_{i \in I}$ at zero a closed circled neighbourhood V of zero in E with $V + V \subset C_0$ and such that

$$g_i(x) - g_i(x') \in W_0 \text{ for all } x, x' \in V + V \text{ and } i \in I.$$

Let $x, y \in V$. For each real number $r > \rho_V(x - y)$ put

$$z_r = y + r^{-1}(y - x).$$

Then $z_r \in V + V$ and if we put $t: = (1 + r)^{-1}r < 1$ we may write

$$y = (1 - t)x + tz_r$$

and hence as g_i is convex we have for each $i \in I$

$$g_i(y) \in (1 - t)g_i(x) + tg_i(z_r) - F_+$$
$$= g_i(x) + t(g_i(z_r) - g_i(x)) - F_+$$

which implies

$$g_i(y) - g_i(x) \in t(g_i(z_r) - g_i(x)) - F_+ \subset tW_0 - F_+.$$

Therefore making use of the symmetry of the latter relation we have for each $i \in I$

$$g_i(y) - g_i(x) \in t(W_0 - F_+) \cap (W_0 + F_+) = tW_0 \subset rW$$

for every $r > \rho_V(x - y)$. So we have

$$\rho_W(g_i(y) - g_i(x)) \leqq \rho_V(y - x) \text{ for all } x, y \in V \text{ and } i \in I$$

and the proof is complete.

The following two propositions are generalizations of the Banach-Steinhaus theorem to convex operators.

2.6. PROPOSITION. *Assume that E is a Baire topological vector space. Let $(f_i)_{i \in I}$ be a family of continuous convex operators from an open convex set $C \subset E$ into F. If there exists a neighbourhood X of a point $a \in C$ with $X \subset C$ such that for each $x \in X$ the family $(f_i(x) - f_i(a))_{i \in I}$ is topologically bounded in F, then the family $(f_i)_{i \in I}$ is equilipschitzian around a.*

Proof. Let W_0 be any circled neighbourhood of zero in F and let W be a closed circled neighbourhood of zero with $W + W \subset W_0$. Choose an open circled neighbourhood X_0 of zero in E included in $X - a$. For each $i \in I$ define a convex operator $g_i: X - a \to F$ by putting

$$g_i(x) = f_i(x + a) - f_i(a) \text{ for every } x \in X - a.$$

For each integer $n \geqq 1$ consider the closed set V_n in X_0 with respect to the induced topology on X_0 and containing zero defined by

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$$V_n = \bigcap_{i \in I} \{x \in X_0 : g_i(x) \in 2nW,\ g_i(-x) \in 2nW\}.$$

As for each $x \in X_0$ the family $(g_i(x))_{i \in I}$ is topologically bounded in F, it follows that

$$X_0 = \bigcup_{n \geqq 1} V_n$$

and hence there exists an integer k such that

$$\mathrm{int}_E\, V_k \neq \emptyset.$$

So the set

$$U = \frac{1}{2}V_k - \frac{1}{2}V_k = \frac{1}{2}V_k + \frac{1}{2}V_k$$

is a symmetric neighbourhood of zero in E and making use of the convexity of g_i we have for each $i \in I$

$$g_i\left(\frac{1}{k}\ U\right) \subset \frac{1}{k}\ g_i(U) - F_+$$

$$\subset \frac{1}{2k}\ g_i(V_k) + \frac{1}{2k}\ g_i(V_k) - F_+ \subset W_0 - F_+.$$

Thus by Proposition 2.2 the family $(f_i)_{i \in I}$ is equicontinuous at a and hence by Proposition 2.5 it is equilipschitzian around a.

If F is locally convex we may weaken the assumption on E as follows.

2.7. PROPOSITION. *Assume that E is a barreled vector space and that F is locally convex. Let $(f_i)_{i \in I}$ be a family of continuous convex operators from an open convex set $C \subset E$ into F. If there exists a neighbourhood X of a in E with $X \subset C$ such that for each $x \in X$ the family $(f_i(x) - f_i(a))_{i \in I}$ is topologically bounded in F, then the family $(f_i)_{i \in I}$ is equilipschitzian around a.*

Proof. Let W be any convex circled neighbourhood of zero in F. Making a translation as in the proof of Proposition 2.6 we may suppose $a = 0$ and $f_i(a) = 0$. Consider a closed convex neighbourhood X_0 of zero in E included in X and denote by g_i the restriction of f_i to X_0. If we put

(2.2) $$V = \bigcap_{i \in I} \mathrm{cl}_E\, g_i^{-1}(W - F_+),$$

then V is obviously a closed subset containing zero and moreover V is convex since each g_i is convex. Let us show that V is absorbing (radial). Let x be any point in E and let t be a real positive number with $tx \in X_0$.

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As the family $(g_i(tx))_{i \in I}$ is topologically bounded in F we can select a real number $s > 1$ such that $g_i(tx) \in sW$ for every $i \in I$. Then for each $i \in I$ we have

$$g_i(s^{-1}tx) \in s^{-1}g_i(tx) - F_+ \subset W - F_+$$

and hence $s^{-1}tx \in V$ which implies that V is absorbing. Therefore V is a neighbourhood of zero in E and by relation (2.2) we have for each $i \in I$

(2.3) $\quad V \subset \mathrm{cl}_E\, g_i^{-1}(W - F_+).$

As g_i is continuous, $g_i^{-1}(W - F_+)$ is a convex set with nonempty interior in E. So if V_0 denotes an open circled neighbourhood of zero included in V, by relation (2.3) the inclusion $V_0 \subset g_i^{-1}(W - F_+)$ is satisfied for each $i \in I$ and hence we obtain for each $i \in I$

$$g_i(V_0) \subset W - F_+.$$

Therefore by Proposition 2.2 the family $(f_i)_{i \in I}$ is equicontinuous at a and hence by Proposition 2.5 it is equilipschitzian around a.

The two preceding propositions admit many important consequences. The first one whose proof is obvious gives a condition under which conditions in Proposition 2.6 or 2.7 are also necessary.

2.8. COROLLARY. *Assume that E is a Baire space or barreled space and that F is a normed space. Let $(f_i)_{i \in I}$ be a family of continuous convex operators from an open convex subset C of E into F. Then the family $(f_i)_{i \in I}$ is equilipschitzian around a point $a \in C$ if and only if there exists a neighbourhood $X \subset C$ of a such that the family $(\|f_i(x) - f_i(a)\|)_{i \in I}$ is bounded in $\mathbf{R}$ for each $x \in X$.*

2.9. COROLLARY. *Let G and H be two metrizable topological vector spaces, C an open convex subset in G and D an open subset in H. Assume that G is a Baire space or that G is barreled and F is locally convex. Let $f:C \times D \to F$ a separately continuous mapping which is convex with respect to the first variable, that is $f(\cdot, y)$ is convex for each $y \in D$. Then f is jointly continuous on $C \times D$.*

Proof. Let (c, d) be any point in $C \times D$. Define a separately continuous mapping g from $C_0 \times D_0 := (C - c) \times (D - d)$ into F by putting

$$g(x, y) = f(c + x, d + y) - f(c, d + y)$$

for every $(x, y) \in C_0 \times D_0$.

For each $y \in D_0$ the mapping $g(\cdot, y)$ is convex on C_0 and $g(0, y) = 0$. Let $(x_n, y_n)_{n \in \mathbf{N}}$ be any sequence in $C_0 \times D_0$ converging to $(0, 0)$ and let W be any neighbourhood of zero in F. For each $x \in C_0$ the set $\{g(x, y_n):n \in \mathbf{N}\}$ is topologically bounded in F since $\{0\} \cup \{y_n:n \in \mathbf{N}\}$ is compact in H

and hence by Proposition 2.6 or 2.7 the family $(g(\cdot, y_n))_{n \in \mathbf{N}}$ is equicontinuous at zero. Therefore there exists a neighbourhood V of zero in G such that $g(V, y_n) \subset W$ for each $n \in \mathbf{N}$ and hence there is an integer k such that $g(x_n, y_n) \in W$ for every $n \geqq k$ and the proof is complete.

3. Equicontinuity of families of convex and concave-convex operators. In this section we shall give some conditions ensuring equicontinuity of families of convex and concave-convex operators in the line of Borwein's continuity condition recalled in the introduction of the paper.

The proof of Proposition 3.1 is largely similar to the one of Proposition 2.3 in [**1**].

3.1. PROPOSITION. *Let $(f_i)_{i \in I}$ be a family of convex operators from E into $F^{\cdot}$ and let a be a point in E with $f_i(a) \in F$ for each $i \in I$. This family is equicontinuous at a if and only if there exist an element $k \in F$ and a family $(M_i)_{i \in I}$ of mappings from E into $F^{\cdot}$ which is upper equisemicontinuous at a and such that*

$$f_i(x) - f_i(a) \leqq M_i(x) \text{ for all } x \in E \text{ and } i \in I$$

and

$$M_i(a) \leqq k \text{ for all } i \in I$$

Proof. We have only to prove that the condition is sufficient. For each $i \in I$ put

$$g_i(x) = f_i(x) - f_i(a) \text{ for every } x \in E.$$

Let W_0 be any neighbourhood of zero in F. Choose a circled neighbourhood W of zero with $W + W \subset W_0$ and a real number $t > 1$ such that

$$(3.1) \quad k \in (t - 1)W.$$

As the family $(M_i)_{i \in I}$ is upper equisemicontinuous at a there exists a circled neighbourhood V of zero in E with

$$(3.2) \quad M_i(a + V) - M_i(a) \subset W - F_+ \text{ for all } i \in I.$$

Therefore by relations (3.1) and (3.2) we have for each $i \in I$ and each $v \in V$

$$\begin{aligned} g_i(a + v) - g_i(a) &= g_i(a + v) \in M_i(a + v) - F_+ \\ &= (M_i(a + v) - M_i(a)) + M_i(a) - F_+ \\ &\subset W + k - F_+ \subset W + (t - 1)W - F_+ \\ &= t\left(\frac{1}{t}W + \left(1 - \frac{1}{t}\right)W - F_+\right) \\ &\subset t(W_0 - F_+). \end{aligned}$$

As g_i is convex we obtain

$$g_i\left(a + \frac{1}{t} V\right) - g_i(a) \subset W_0 - F_+ \text{ for each } i \in I$$

and hence by Proposition 2.2 the family $(g_i)_{i \in I}$ is equicontinuous at a and the proof is complete.

The following two corollaries are direct consequences of Proposition 3.1.

3.2. COROLLARY. *Let $(f_i)_{i \in I}$ be a family of convex operators from E into $F^{\cdot}$ and let a be a point in E with $f_i(a) \in F$ for each $i \in I$. Assume that there exists a mapping $M:E \to F^{\cdot}$ finite on some neighbourhood of a, upper semicontinuous at this point and such that*

$$f_i(x) - f_i(a) \leqq M(x) \text{ for all } x \in E \text{ and } i \in I.$$

Then the family $(f_i)_{i \in I}$ is equicontinuous at a.

3.3. COROLLARY. *A convex operator $f:E \to F^{\cdot}$ is continuous at a point $a \in \operatorname{dom} f$ if and only if there exists a mapping $M:E \to F^{\cdot}$ finite on some neighbourhood of a, upper semicontinuous at a and such that*

$$f(x) \leqq M(x) \text{ for every } x \in E.$$

Consider now the equicontinuity of families of biconvex or concave-convex operators.

3.4. PROPOSITION. *Let C and D be two open convex subsets in G and H respectively, let (c, d) be a point in $C \times D$ and let $(f_i)_{i \in I}$ be a family of biconvex or concave-convex operators from $C \times D$ into F. Assume that there exist two elements k and $l \in F$, a family $(M_i)_{i \in I}$ of mappings from $C \times D$ into F which is upper equisemicontinuous at (c, d) and a family $(m_i)_{i \in I}$ of mappings from $C \times D$ into F which is lower equisemicontinuous at (c, d) such that*

$$m_i(x, y) \leqq f_i(x, y) - f_i(c, d) \leqq M_i(x, y)$$
$$\text{for all } i \in I \text{ and } (x, y) \in C \times D$$

and

$$l \leqq m_i(c, d) \text{ and } M_i(c, d) \leqq k \text{ for all } i \in I.$$

Then for each neighbourhood W_0 of zero in F there exist a real number $t > 1$, a neighbourhood X of zero in G and a neighbourhood Y of zero in H such that

$$f_i(c + sx, d + sy) - f_i(c, d) \in stW_0$$

for all $i \in I$, $(x, y) \in X \times Y$ and $s \in [0, 1]$.

Proof. By putting

$$g_i(x, y) = f_i(c + x, d + y) - f_i(c, d)$$

for every $(x, y) \in (C - c) \times (D - d)$ we may assume that $(c, d) = (0, 0)$ and $f_i(c, d) = 0$. Let W_0 be a full neighbourhood of zero in F. Choose a circled neighbourhood W of zero in F with

$$W + W + W + W + W + W \subset W_0.$$

As the family $(M_i)_{i \in I}$ is upper equisemicontinuous at (0, 0) and the family $(m_i)_{i \in I}$ is lower equisemicontinuous at (0, 0), there exists a circled neighbourhood X_0 of zero in G and a circled neighbourhood Y_0 of zero in H such that

$$m_i(x, y) - m_i(0, 0) \in W + F_+ \text{ and}$$

$$M_i(x, y) - M_i(0, 0) \in W - F_+$$

for all $(x, y) \in X_0 \times Y_0$, $i \in I$. Select a real number $t_0 > 1$ such that

$$k - l \in (t_0 - 1)W.$$

By Propositions 2.5 and 3.1 the families of convex (or concave) operators

$$\{f_i(\cdot\,, 0): i \in I\} \text{ and } \{f_i(0, \cdot): i \in I\}$$

are equilipschitzian around zero and hence there exist two real positive numbers t_1 and t_2 and two circled neighbourhoods X_1 of zero in G and Y_1 of zero in H such that

$$(3.3) \quad f_i(sx, 0) \in st_1 W \text{ and } f_i(0, sy) \in st_2 W$$

for all $i \in I$, $(x, y) \in X_1 \times Y_1$ and $s \in [0, 1]$. Put $X = X_0 \cap X_1$, $Y = Y_0 \cap Y_1$ and $t = \max(t_0, t_1, t_2)$ and fix $i \in I$, $(x, y) \in X \times Y$ and $s \in [0, 1]$. On the one hand we have

$$(3.4) \quad f_i(sx, sy) = (f_i(sx, sy) - f_i(sx, 0)) + (f_i(sx, 0) - f_i(0, 0))$$
$$\in s(f_i(sx, y) - f_i(sx, 0) - F_+) + st_1 W.$$

Moreover if we write

$$f_i(sx, y) - f_i(sx, 0)$$
$$= (f_i(sx, y) - M_i(0, 0)) + (M_i(0, 0) - m_i(0, 0))$$
$$+ (m_i(0, 0) - f_i(sx, 0))$$
$$\subset (M_i(sx, y) - M_i(0, 0)) + (k - l)$$
$$+ (m_i(0, 0) - m_i(sx, 0)) - F_+$$

we obtain

$$f_i(sx, y) - f_i(sx, 0) \in W + (t_0 - 1)W + W - F_+$$

and hence by relation (3.4)

$$(3.5)\quad f_i(sx, sy) \in st\left(\frac{1}{t}W + \frac{t_0 - 1}{t}W + \frac{1}{t}W + \frac{t_1}{t}W\right) - F_+$$

$$(3.6)\quad \subset stW_0 - F_+.$$

On the other hand by convexity of $f_i(sx, \cdot)$ we have

$$f_i(sx, sy) = [f_i(sx, sy) - f_i(sx, 0)] + f_i(sx, 0)$$
$$\in -[f_i(sx, -sy) - f_i(sx, 0)] + f_i(sx, 0) + F_+$$

and hence making use of relations (3.3) and (3.5) we obtain

$$f_i(sx, sy) \in st(W + W + W + W + W + W)$$
$$+ F_+ \subset stW_0 + F_+.$$

So by relation (3.6) we have

$$f_i(sx, sy) \in st(W_0 - F_+) \cap (W_0 + F_+) = stW_0$$

for all $i \in I$, $(x, y) \in X \times Y$ and $s \in [0, 1]$ and the proposition is proved.

3.5. COROLLARY. *A family $(f_i)_{i \in I}$ of biconvex or concave-convex operators from an open convex set $C \times D \subset G \times H$ into F is equilipschitzian at a point $(c, d) \in C \times D$ in the sense that for each closed circled neighbourhood W of zero in F there exist a neighbourhood U of zero in $G \times H$ and a closed circled neighbourhood V of zero in $G \times H$ such that*

$$\rho_W(f_i(c + x, d + y) - f_i(c, d)) \leqq \rho_V(x, y)$$

for all $(x, y) \in U$, $i \in I$, if and only if there are two elements k and $l \in F$ and two families $(m_i)_{i \in I}$ and $(M_i)_{i \in I}$ of mappings satisfying the conditions of Proposition 3.4.

Proof. The condition is obviously necessary. Let us show that it is sufficient. By Proposition 3.4 there exist a real number $t > 1$ and a closed circled neighbourhood U_0 of (0, 0) in $G \times H$ such that

$$(3.7)\quad f_i((c, d) + s(x, y)) - f_i(c, d) \in stW$$

for all $i \in I$, $(x, y) \in U_0$ and $s \in [0, 1]$. Therefore for each $(x, y) \in \frac{1}{2}U_0$, each $\epsilon \in]0, 1/2]$ and each $i \in I$ we have by relation (3.7)

$$f_i(c + x, d + y) - f_i(c, d)$$
$$= f_i((c, d) + (\epsilon + \rho_{U_0}(x, y)) \cdot$$

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$$((\epsilon + \rho_{U_0}(x, y))^{-1}(x, y))) - f_i(c, d)$$
$$\in t(\epsilon + \rho_{U_0}(x, y))W$$

and hence for $V: = t^{-1}U_0$ we have

$$\rho_W(f_i(c + x, d + y) - f_i(c, d)) \leqq t\rho_{U_0}(x, y) = \rho_V(x, y)$$

for all $i \in I$ and $(x, y) \in U: = \frac{1}{2} U_0$.

The next corollary gives a sufficient and necessary condition for continuity of biconvex or concave-convex operators.

3.6. COROLLARY. *Let f be a biconvex or concave-convex operator from an open convex subset $C \times D \subset G \times H$ into F. Then f is lipschitzian at a point $(c, d) \in C \times D$ if and only if there are a mapping $m:C \times D \to F$ lower semicontinuous at (c, d) and a mapping $M:C \times D \to F$ upper semicontinuous at (c, d) satisfying*

$$m(x, y) \leqq f(x, y) \leqq M(x, y) \text{ for each } (x, y) \in C \times D.$$

Proof. The condition is seen to be necessary by taking $m = f$ and $M = f$. To prove that it is also sufficient it is enough to apply Corollary 3.5 with

$$m'(\cdot, \cdot) \equiv m - f(c, d) \text{ and } M'(\cdot, \cdot) \equiv M - f(c, d).$$

To close this section let us give a condition (a little stringent when int $F_+ = \emptyset$) ensuring equi-Lipschitz continuity around a point of families of biconvex or concave-convex operators.

3.7. PROPOSITION. *Let $(f_i)_{i \in I}$ be a family of biconvex or concave-convex operators from an open convex subset $C \times D$ of $G \times H$ into F. Assume that there exist two elements m and M in F satisfying*

$$m \leqq f_i(x, y) \leqq M \text{ for all } i \in I \text{ and } (x, y) \in C \times D.$$

Then the family $(f_i)_{i \in I}$ is equilipschitzian around any point in $C \times D$.

Proof. Let (c, d) be any point in $C \times D$. Let W be a closed circled neighbourhood of zero in F. Choose a closed circled neighbourhood W' of zero with $W' + W' \subset W$. By Corollary 3.2 and Proposition 2.5 the families

$$\{f_i(x, \cdot):i \in I, x \in C\} \text{ and } \{f_i(\cdot, y):i \in I, y \in D\}$$

are equilipschitzian around d and c respectively and hence there exist a closed circled neighbourhood $U \times V$ of zero in $G \times H$ and a neighbourhood $X \times Y$ of (c, d) in $C \times D$ such that

$$f_i(x, y) - f_i(x, y') \in \rho_V(y - y')W'$$

for all $i \in I$, $x \in C$ and $(y, y') \in Y \times Y$ and

$$f_i(x, y) - f_i(x', y) \in \rho_U(x - x')W'$$

for all $i \in I$, $y \in D$ and $(x, x') \in X \times X$. So if we write

$$f_i(x, y) - f(x', y') = (f_i(x, y) - f_i(x', y)) + (f_i(x', y) - f_i(x', y'))$$

we see that

$$f_i(x, y) - f_i(x', y') \in \rho_U(x - x')W' + \rho_V(y - y')W' \subset \rho_{\frac{1}{2}(U \times V)}(x - x', y - y')W$$

and hence

$$\rho_W(f_i(x, y) - f_i(x', y')) \leqq \rho_{\frac{1}{2}(U \times V)}(x - x', y - y')$$

for all $i \in I$, (x, y) and $(x', y') \in X \times Y$.

Remark. The above condition is also necessary whenever $\mathrm{int}(F_+) \neq \emptyset$.

4. Continuous seminorms and equicontinuity of convex and concave-convex operators. This section is devoted to some necessary and sufficient conditions for equicontinuity of families of biconvex or concave-convex operators when F is locally convex (again F_+ is assumed to be normal). In this case (see [**11**]) there exists a base of neighbourhoods of zero which are circled, convex and full. Moreover one easily verifies that for each circled full neighbourhood W of zero in F the following property is satisfied:

(4.1) $\quad x \leqq y \leqq z \Rightarrow \rho_W(y) \leqq \max(\rho_W(x), \rho_W(z))$,

and ρ_W is a seminorm whenever W is convex.

Any seminorm satisfying property 4.1 will be called an *order-monotone seminorm* and hence the directed family of all continuous order-monotone seminorms on F generates the topology of F.

Before we proceed to those conditions we need the following definition.

4.1. *Definition*. A family $(h_i)_{i \in I}$ of functions from a topological space X into $\mathbf{R}$ will be called *locally equimajorized around* a point $a \in X$ if there exists a real number r and a neighbourhood A of a in X such that

(4.2) $\quad h_i(x) \leqq r$ for all $i \in I$ and $x \in A$.

4.2. Proposition. *Assume that F is locally convex. Let $(f_i)_{i \in I}$ be a family of convex operators from an open convex set $C \subset E$ into F. Then this family is equicontinuous at a point $a \in C$ if and only if for each continuous seminorm p on F the family of functions $(p(f_i(\cdot) - f_i(a))_{i \in I}$ from C into $\mathbf{R}$ is locally equimajorized at the point a.*

Proof. It is easily verified that the condition is necessary. By putting

$$g_i(x) = f_i(a + x) - f_i(a) \text{ for every } x \in C - a$$

we may assume that $a = 0$ and $f_i(a) = 0$. Let p be any continuous order-monotone seminorm on F and ϵ be any positive real number. Choose a real number $r \geqq \epsilon$ and a circled neighbourhood V of zero such that

(4.3) $V \subset C$ and $p(f_i(x)) \leqq r$ for all $i \in I$ and $x \in V$.

By convexity of the operators f_i we have for each $i \in I$ and each $x \in V$

$$-\frac{\epsilon}{r} f_i(-x) \leqq f_i(\frac{\epsilon}{r} x) \leqq \frac{\epsilon}{r} f_i(x)$$

and hence by relation (4.1)

$$p\left(f_i\left(\frac{\epsilon}{r} x\right)\right) \leqq \frac{\epsilon}{r} \max(p(f_i(x)), p(f_i(-x))).$$

Therefore making use of relation (4.3) we obtain

$$p(f_i(x)) \leqq \epsilon \text{ for all } i \in I \text{ and } x \in \frac{\epsilon}{r} V$$

and the proof is complete.

Remark. A similar result also holds in terms of ρ_W without assuming that F is locally convex.

4.3. PROPOSITION. *Assume that F is locally convex. Let $(f_i)_{i \in I}$ be a family of biconvex or concave-convex operators from an open convex subset $C \times D$ into F. Then this family is equicontinuous at a point $(c, d) \in C \times D$ if and only if for each continuous seminorm p on F the family of functions $(p(f_i(\cdot) - f_i(c, d)))_{i \in I}$ is locally equimajorized around (c, d).*

Proof. It is clearly enough to prove that the condition is sufficient. We may suppose that $c = 0$, $d = 0$ and $f_i(c, d) = 0$. Let p be any continuous order-monotone seminorm on F and let ϵ be any positive number. Choose a real number $r > 0$ and a circled neighbourhood $X \times Y$ of zero in $G \times H$ such that

(4.4) $X \times Y \subset C \times D$ and $p(f_i(x, y)) \leqq r$

for all $i \in I$ and $(x, y) \in X \times Y$.

By convexity of the operators $f_i(x, \cdot)$ we have for each $i \in I$ and each $(x, y) \in X \times Y$

$$-\frac{\epsilon}{4r}(f_i(x, -y) - f_i(x, 0))$$

$$\leqq f_i(x, \frac{\epsilon}{4r} y) - f_i(x, 0)$$

$$\leqq \frac{\epsilon}{4r} (f_i(x, y) - f_i(x, 0))$$

and hence making use of relations (4.1) and (4.4) we obtain

$$(4.5) \quad p(f_i(x, \frac{\epsilon}{4r} y) - f_i(x, 0)) \leqq \frac{\epsilon}{4r} \cdot 2r = \frac{\epsilon}{2}.$$

Moreover by assumption and by Proposition 4.2 the family of convex (or concave) operators $(f_i(\cdot, 0))_{i \in I}$ is equicontinuous at zero and hence there exists a neighbourhood X' of zero in G such that

$$(4.6) \quad p(f_i(x, 0)) \leqq \frac{\epsilon}{2} \text{ for all } i \in I \text{ and } x \in X'.$$

So if we write

$$f_i(x, \frac{\epsilon}{4r} y) = (f_i(x, \frac{\epsilon}{4r} y) - f_i(x, 0)) + f_i(x, 0)$$

we see by relations (4.5) and (4.6) that

$$p(f_i(x, y)) \leqq \epsilon \text{ for all } i \in I, x \in X \cap X' \text{ and } y \in \frac{\epsilon}{4r} Y$$

and hence the family $(f_i)_{i \in I}$ is equicontinuous at (0, 0).

The following two corollaries are direct consequences of Proposition 4.3.

4.4. COROLLARY. *Assume that F is a normed space (always with F_+ normal). Then a family $(f_i)_{i \in I}$ of biconvex or concave-convex operators from an open convex set $C \times D \subset G \times H$ into F is equicontinuous at a point $(c, d) \in C \times D$ if and only if the family of functions $(\|f_i(\cdot) - f_i(c, d)\|)_{i \in I}$ is locally equimajorized at (c, d).*

4.5. COROLLARY. *Assume that F is a normed space. Then a biconvex or concave-convex operator f from an open convex subset $C \times D \subset G \times H$ into F is continuous at a point $(c, d) \in C \times D$ if and only if there exists a neighbourhood of (c, d) in $C \times D$ on which the real function $\|f(\cdot, \cdot)\|$ is majorized.*

Obviously similar results also hold for convex or concave operators and by taking $F_+ = \{0\}$ one recovers well known results for linear or bilinear operators.

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