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POWERS OF CHORDAL GRAPHS

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Abstract

An undirected simple graph G is called chordal if every circle of G of length greater than 3 has a chord. For a chordal graph G, we prove the following: (i) If m is an odd positive integer, G^m is chordal. (ii) If m is an even positive integer and if G^m is not chordal, then none of the edges of any chordless cycle of G^m is an edge of G^r, $r < m$.

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An undirected simple graph G is called chordal if every cycle of G of length greater than 3 has a chord, that is, an edge joining two non-adjacent vertices of the cycle. The mth power, G^m, of a graph G is the graph with the same vertex set as G and in which two vertices are adjacent iff the distance between them in G is $\leqslant m$. Laskar and Shier [3] conjectured that an odd power of any chordal graph is chordal and showed by an example that an even power of a chordal graph need not be chordal. In this paper, we give an affirmative answer to their conjecture. The problem of determining a necessary and sufficient condition for an even power of a chordal graph to be chordal is open. However, in the special case when the even power is 2, Laskar and Shier [4] have recently solved the problem while in [1], the present authors have obtained a sufficient condition for the square of *any* graph to be chordal.

The following two properties of chordal graphs can easily be verified:

(i) Every induced subgraph of a chordal graph is chordal; and

(ii) the interior face of any cycle of a chordal graph is divided into triangles. Our terminology is as in [2].

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THEOREM 1. *Let G be any chordal graph and m, an odd positive integer. Then G^m is chordal.*

PROOF. Suppose G^m is not chordal. Then it has a chordless cycle $C = (v_1 v_2 \cdots v_k v_1)$ of length $k > 3$. By the definition of G^m, each edge $v_i v_{i+1}$ of C arises out of a $v_i - v_{i+1}$ path p_i in G of length $\leqslant m$. (The suffixes i in v_i and P_i are taken modulo k.)

Case (i). Assume that for each i, there is no $v_i - v_{i+1}$ path of length less than m in G so that $d(v_i, v_{i+1}) = m$, where d is the distance function on the vertex set of G. Hence each of the paths P_i is of length m. Let $P_i = v_i u_{i1} u_{i2} \cdots u_{i(m-1)} v_{i+1}$, $1 \leqslant i \leqslant k$, and $m = 2n + 1$. If u_{ij_i} is the first vertex along P_i where P_{i+1} meets it, by the definition of distance, the $u_{ij_i} - v_{i+1}$ subpath of the path P_i and $v_{i+1} - u_{ij_i}$ subpath of P_{i+1} must be of equal length, and hence the $v_i - u_{ij_i}$ subpath of P_i and the $u_{ij_i} - v_{i+2}$ subpath of P_{i+1} must be of equal length. (See Figure 1. In all our figures, paths of equal length are shown by a single path.) Hence $j_i \geqslant n + 1$ since otherwise the union of the $v_i - u_{ij_i}$ subpath of P_i and $u_{ij_i} - v_{i+2}$ subpath of P_{i+1} would give rise to a $v_i - v_{i+2}$ path of length $\leqslant 2n$ in G, and therefore $v_i v_{i+2}$ would be a chord of C in G^m contradicting the assumption that C is not chordal in G^m. Thus for each i, a $u_{(i-1)j_{(i-1)}} - u_{ij_i}$ subpath Q_i of P_i is determined. Necessarily, Q_i must contain the vertices u_{in} and $u_{i(n+1)}$. The union of the k paths Q_i, $1 \leqslant i \leqslant k$, is clearly a cycle Z in G.

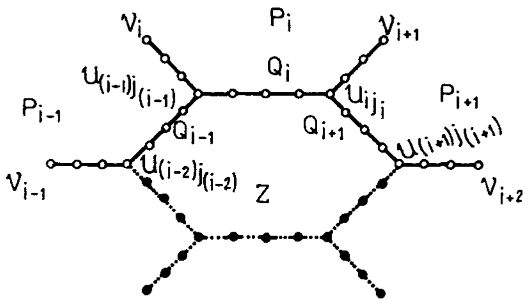

Figure 1

In what follows, we consider only the particular case when $Q_i = P_i$ for each i. However, the arguments would make it clear that our conclusion would remain valid even in the general case.

We prove the theorem by showing that in G, none of the (interior) vertices u_{it} of the path P_i is adjacent to any (interior) vertex u_{js} of P_j, $j \neq i$.

Subcase 1. $j \neq i + 1, i - 1, i + 2, i - 2$ (see Figure 2). If u_{it} is adjacent to u_{js}, then in this case, there would result a $v_p - v_q$ path of length $\leqslant 2n + 1$ in G where $p = i$ or $i + 1$ and $q = j$ or $j + 1$, thereby giving rise to a chord of C in G^m, a contradiction.

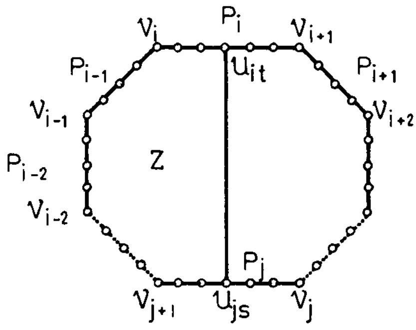

Figure 2

Subcase 2. $j = i - 2$ and $k \neq 4$. In this subcase, no vertex of the $u_{i(n+1)} - v_{i+1}$ subpath of P_i can be adjacent in G to any vertex $u_{(i-2)s}$ of the $v_{i-2} - u_{(i-2)n}$ subpath, say, P'_{i-2} of P_{i-2}, as such an adjacency would result in a $v_{i-2} - v_{i+1}$ path of length $\leqslant 2n + 1$ in G and force $v_{i+1}v_{i-2}$ as a chord of C in G^m, a contradiction. Similarly, no vertex u_{it} of the $v_i - u_{in}$ subpath of P_i can be adjacent in G to any vertex of P'_{i-2}. Consequently, no vertex u_{it} of P_i can be adjacent to any vertex $u_{(i-1)s}$ of P'_{i-2}, and for a similar reason, no vertex $u_{(i-2)r}$ of P_{i-2} can be adjacent to any vertex u_{iq}, $q \geqslant n + 1$.

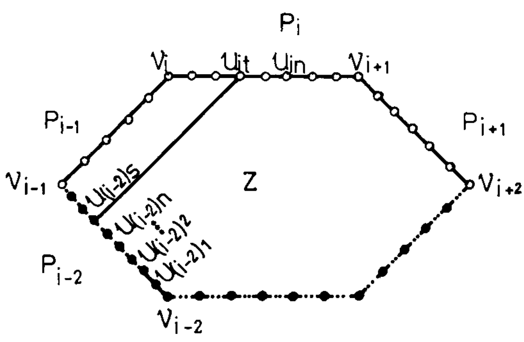

Figure 3

Thus we are left with the only possibility when u_{it}, $t \leqslant n$ is adjacent in G to $u_{(i-2)s}$, $s > n$. Suppose now $u_{it}u_{(i-2)s}$ is an edge of G. In case $d(v_i, u_{it}) \neq d(u_{(i-2)s}, v_{i-1})$, either $v_{i-1}v_{i+1}$ or $v_{i-2}v_i$ would be a chord of C in G^m. So suppose that $d(u_{(i-2)s}, v_{i-1}) = d(u_{it}, v_i) = t$ (see Figure 3). In this case, consider the cycle C_1 in G defined by the edge $u_{(i-2)s}u_{it}$ and the $u_{it} - u_{(i-2)s}$ path along Z containing the vertex v_{i+1}. If $k = 4$ so that Z is a union of four paths P_i, we apply the argument in subcase 3 below to obtain a contradiction. So assume that $k > 4$. As G is chordal, the edge $u_{(i-2)s}u_{it}$ in C_1 must be the side of a triangle whose third vertex is also in C_1. By subcase 1, this third vertex could only be $u_{i(t+1)}$ or

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$u_{(i-2)(s-1)}$. (The third vertex cannot be, for instance, u_{ip}, $p > t + 1$, as this would make u_{it} adjacent to u_{ip}, $p > t + 1$, contradicting the fact that P_i is a shortest $v_i - v_{i+1}$ path in G.) But then this reverts back to the possibility considered just above and again gives rise to a chord of C in G^m. For a similar reason no u_{ir} of P_i can be adjacent in G to any vertex $u_{(i+2)t}$ of P_{i+2}.

Subcase 3. $j = i - 2$, $k = 4$. As in subcase 2, if $d(v_i, u_{it}) \neq d(v_{i-1}, u_{(i-2)s})$, either $v_{i-2}v_i$ or $v_{i-1}v_{i+1}$ is a chord of C in G^m. So let $t = d(v_i, u_{it}) = d(v_{i-1}, u_{(i-2)s})$. The edge $u_{it}u_{(i-2)s}$ divides Z into two cycles Z_1 and Z_2 containing the $v_i - u_{it}$ and $u_{it} - v_{i+1}$ subpaths of P_i respectively (see Figure 4). As m is odd, either $t > m - t$ or $t < m - t$. If $t < m - t$ ($t > m - t$), consider the cycle Z_2 (Z_1). The side $u_{(i-2)s}u_{it}$ of Z_2 (Z_1) belongs to a triangle whose third vertex is also in Z_2 (Z_1). Once again, a chord of C in G^m is obtained.

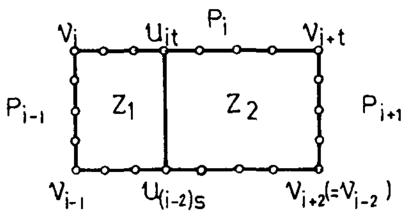

Figure 4

Subcase 4. $j = i - 1$. If $t \geqslant n + 1$ and $s \leqslant n$, it is obvious that no vertex u_{it} of P_i is adjacent to any vertex $u_{(i-1)s}$ of P_{i-1}. In the general case, we look at the cycle defined by the union of the edge $u_{(i-1)s}u_{it}$ and the $u_{it} - u_{(i-1)s}$ path of Z that contains the vertex v_{i+1} and repeat the argument of subcase 2 (paragraph 2). This would force both $u_{(i-1)s}$ and u_{it} to be adjacent to $u_{(i-1)(s-1)}$ or $u_{i(t+1)}$. Repetition of this argument with the successive edges thus obtained would end up with an edge $u_{ip}u_{(i-1)q}$, $p \geqslant n + 1$ and $q \leqslant n$ of G. But this is impossible as stated already.

Thus us have shown that no vertex u_{it} can be adjacent to any vertex u_{js}, $j \neq i$ in G. But this is impossible as G is chordal.

Case (ii). In this case, there exists at least one edge e of C which arises out of a path of length $< m$ in G. Then repetition of our argument given under case (i) would yield the same conclusion as in case (i) since the lengths of the various paths that will occur in the discussion of this case cannot exceed the corresponding lengths that appear in case (i).

LEMMA. *Let G be a chordal graph and let $C = (v_1v_2 \cdots v_kv_1)$ be a chordless cycle in G^{2m}, and let for each j (taken modulo k), the edge $e_j = v_jv_{j+1}$ of C arise out of a shortest path P_j in G. Suppose for some i, $1 \leqslant i \leqslant k$, P_i is of length $r < 2m$. Then no interior vertex of P_{i-1} is adjacent in G to any interior vertex of P_{i+1}.*

PROOF. Adopting the notation in the proof of Theorem 1, assume that $u_{(i-1)s}$ is adjacent to $u_{(i+1)t}$ in G. As C is chordless in G^{2m}, e_i cannot be an edge of G (as otherwise e_i would have to be a side of a triangle in the chordal graph G with third vertex in Z). In case $p = d(u_{(i-1)s}, v_i) > d(v_{i+1}, u_{(i+1)t})$, we have $d(v_{i-1}, v_{i+1}) \leqslant d(v_{i-1}, u_{(i-1)s}) + d(u_{(i-1)s}, u_{(i+1)t}) + d(u_{(i+1)t}, v_{i+1}) \leqslant (2m - p) + 1 + (p - 1) = 2m$. Hence $v_{i-1}v_{i+1}$ is a chord of C in G^{2m}, a contradiction. The case when $d(v_{i+1}, u_{(i+1)t}) > d(v_i, u_{(i-1)s})$ is similar. So let $d(v_i, u_{(i-1)s}) = d(v_{i+1}, u_{(i+1)t})$. Clearly, we need only consider the case when P_{i-1} and P_{i+1} are both of length $2m$.

Case 1. $u_{(i-1)s}$ is in the $u_{(i-2)j_{(i-2)}} - u_{(i-1)j_{(i-1)}}$ subpath of P_{i-1} and $u_{(i+1)t}$ is in the $u_{ij_i} - v_{i+1}$ subpath of P_i. For this possibility, consider the cycle C_1 in G defined by the union of the $u_{(i-1)s} - u_{(i-1)j_{(i-1)}}$ subpath of P_{i-1}, Q_i, $u_{ij_i} - u_{(i+1)t}$ subpath of P_{i+1} and the edge $x_i = u_{(i+1)t}u_{(i-1)s}$ (see Figure 5). x_i must belong to a triangle whose third vertex could only be $u_{(i-1)(s+1)}$ or $u_{(i+1)(t+1)}$ of C_1. But this reverts back to the case of unequal distances considered just above.

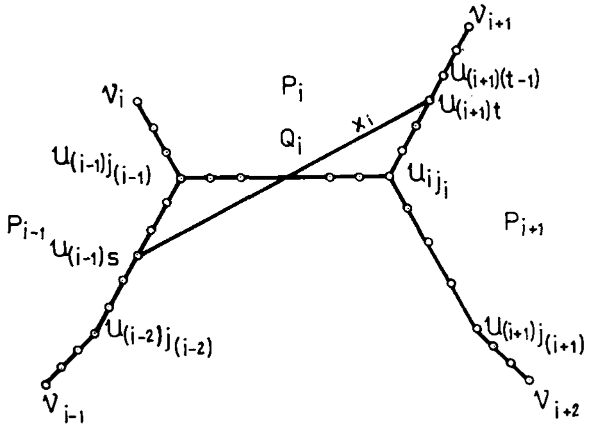

Figure 5

Case 2. $u_{(i-1)s}$ is in the $v_i - u_{(i-1)j_{(i-1)}}$ subpath of P_{i-1} and $u_{(i+1)t}$ is in the $u_{ij_i} - v_{i+2}$ subpath of P_{i+1}. This is similar to case 1.

Case 3. $u_{(i-1)s}$ is an interior vertex of Q_{i-1} and $u_{(i+1)t}$ is an interior vertex of Q_{i+1}. The edge $x_i = u_{(i-1)s}u_{(i+1)t}$ divides Z into two cycles Z_1 and Z_2 of G with $Q_i \in Z_1$ and $Q_i \notin Z_2$ (see Figure 6). Now consider the cycle Z_1 or Z_2 according as $t \geqslant m$ or $t < m$ and apply our by now familiar argument to the edge x_i. This would yield a chord for C in G^{2m}, a contradiction. (Note that the fact that $r < 2m$ will have to be used in our argument while considering the possibility of both $u_{(i-1)s}$ and $u_{(i+1)t}$ being adjacent to an interior vertex of Q_i.) This proves the lemma.

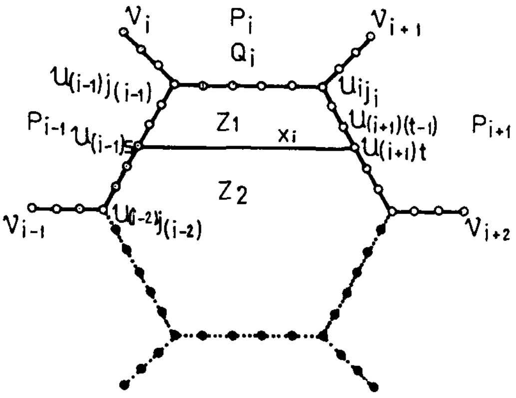

Figure 6

Theorem 2. *If G is chordal and G^{2m} is not chordal then none of the edges of any chordless cycle of G^{2m} is an edge of G^r, $r < 2m$.*

Proof. By contradiction. Suppose $C = (v_1 v_2 \cdots v_k v_1)$ is a chordless cycle in G^{2m} that has at least one edge $e_i = v_i v_{i+1}$ in G^r, $r < 2m$. We may suppose that $e_i \notin G^s$, $s < r$. The edge $v_i v_{i+1}$ of C arises out of a path $P_i = v_i u_{i1} u_{i2} \cdots u_{i(r-1)} v_{i+1}$ of length r in G.

We adopt again the notation in the proof of Theorem 1. As G is chordal and C is a chordless cycle in G^{2m}, length of $Q_i > 1$. Now it clearly suffices to show that no internal vertex u_{it} of Q_i can be adjacent to any vertex of Z not belonging to Q_i. For, this would prove that if x is an edge of Z whose one end vertex is u_{it}, then x cannot belong to a triangle whose third vertex is also in Z.

As length of $P_i < 2m$ and C is a chordless cycle in G^{2m}, u_{it} cannot be adjacent to any interior vertex of the $u_{(i+1)j_{(i+1)}} - u_{(i-2)j_{(i-2)}}$ path along Z not containing Q_i. So consider the case of u_{it} being adjacent to a vertex of Q_{i-1} or Q_{i+1}, say, $u_{(i-1)s}$ of Q_{i-1}. In this case, consider the cycle C_1 in G formed by the edge $x_i = u_{(i-1)s} u_{it}$ and the $u_{it} - u_{(i-1)s}$ path along Z containing u_{ij_i}. As G is chordal, x_i belongs to a triangle whose third vertex is also in C_1. However, by the lemma, this vertex cannot be in Q_{i+1} and hence it must be in Q_i or Q_{i-1} only. Repetition of the argument for the successive edges thus obtained would now yield, as $d(u_{rj_r}, v_{r+1}) \leqslant m - 1$ for each r, a $v_{i-1} - v_{i+1}$ path of length $\leqslant 2m$ in G. This contradiction proves the theorem.

Theorem 2 is a generalisation of the case when $m = 1$ considered by Laskar and Shier [3].

References

[1] R. Balakrishnan and P. Paulraja, 'Graphs whose squares are chordal,' *Indian J. Pure Appl. Math*. **12** (1981), 193–194 with erratum, same J. **12** (1981), 1062.
[2] J. A. Bondy and U. S. R. Murty, *Graph theory with applications* (Macmillan 1976).
[3] Renu Laskar and D. Shier, 'On chordal graphs,' *Congressus Numerantium*, **29** (1980), 579–588.
[4] Renu Laskar and D. Shier, *On powers and centres of chordal graphs*, Technical Report #357 (Department of Mathematical Sciences, Clemson University, South Carolina, February 1981).

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