On the Equation of the Parabolic Cylinder Functions.

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§1. *Introductory.*

Hermite, in 1864 (*Comptes Rendus*, vol. 58) introduced into analysis the polynomials defined by the relation

$$e^{-x^2} U_n = \frac{d^n}{dx^n}\left(e^{-x^2}\right),$$

where n is a positive integer. He showed that they satisfied the differential equation

$$\frac{d^2 U_n}{dx^2} - 2x\frac{dU_n}{dx} + 2nU_n = 0,$$

that they were orthogonal functions, and that an arbitrary function $f(x)$ could be expanded in the form

$$f(x) = a_0 U_0 + a_1 U_1 + \dots .$$

Five years later, Weber (*Math. Ann.*, vol. 1) in discussing the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 u = 0$$

reduced it to the type

$$\frac{d^2 y}{dx^2} + Zy = 0$$

where Z is a quadratic function of z, and showed that the functions of the parabolic cylinder were solutions of this equation. Various series which satisfy this equation have been found by Baer in 1883 (*Dissertation, Cüstrin*), and by Haentzschel in 1888 (*Zeitschrift für Mathematik*, vol. 33). In 1898, Markoff (*Bulletin de l'Academie de St Pétersbourg*, vol. 9) discusses the roots of the equation $e^{x^2}\frac{d^m}{dx^m}\left(e^{-x^2}\right) = 0.$

The next advance was made by E. T. Whittaker in 1902 (*Proc. London Math. Soc.*, vol. 35), who showed that Weber's functions were the same as Hermite's polynomials, and extended the latter to the case when n is not an integer. He solved the equation of the parabolic cylinder by means of a family of definite integrals, and showed that such previous results as Hermite's polynomials were but special cases of these. One of these definite integrals was defined as the standard solution of the equation, viz.,

$$D_n(z) = \frac{i\Gamma(n+1)}{2\pi} e^{-\frac{1}{4}z^2} z^n \int_\delta e^{-t-\frac{1}{2}(t^2/z^2)} (-t)^{-n-1} dt,$$

δ being a contour in the t-plane beginning and ending at infinity and encircling the origin. When the real part of n is negative, $D_n(z)$ can be expressed in the form

$$D_n(z) = \frac{1}{\Gamma(-n)} e^{-\frac{1}{4}z^2} z^n \int_0^\infty e^{-t-\frac{1}{2}(t^2/z^2)} t^{-n-1} dt.$$

The general solution of the differential equation of the parabolic cylinder was shown to be

$$aD_n(z) + bD_{-n-1}(iz)$$

where a and b are arbitrary constants. Recurrence formulae were also established, and the asymptotic expansion of $D_n(z)$ for large real positive values of z determined.

Adamoff, in 1906 (*Annales de l'Institut Polytechnique de St.-Pétersbourg*, vol. 5) discusses expansions of $D_n(z)$ when z is real and n is a large positive integer. In the following year, Myller-Lebedeff (*Math. Ann.*, vol. 64), in discussing integral equations, introduces the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial t} = 0,$$

and determines as solutions a series of polynomials $V_n(x, t)$, which are homogeneous in x and $\sqrt{t}$, and which are intimately connected with the polynomials of Hermite.

In 1910, G. N. Watson (*Proc. London Math. Soc.*, Series 2, vol. 8) obtains the asymptotic expansion of $D_n(z)$ when n is large and real and z is complex. He evaluates certain contour-integrals involving $D_n(z)$, and determines the expansion of an arbitrary function in a series of the functions $D_n(z)$.

Lastly, H. E. J. Curzon, in 1913 (*Proc. London Math. Soc.*, Series 2, vol. 12) establishes the connection between the Hermite functions of the first and second kind, viz. $D_n(z)$ and $D_{-n-1}(iz)$, and the Legendre functions of the first and second kind. This is done by means of relations involving integrals analogous to those connecting the Bessel and Legendre functions.

§ 2. *Contents of Paper.*

In the present paper two independent series-solutions of the differential equation of the parabolic cylinder are expressed as definite integrals, and $D_n(z)$ expressed in terms of these by means of a linear relation of which the values of the constants are determined. Recurrence formulae connecting these constants are then established. Next, the solution of the parabolic cylinder equation is represented as the solution of a homogeneous integral equation. The auto-functions of this integral equation are then shown to be given by the functions $D_n(z)$. Lastly, certain values of $D_n(z)$ are given, which values were computed by means of the above relation involving $D_n(z)$ and the series-solutions of the equation; and, in addition, graphs illustrating $D_n(z)$ for different values of n are appended.

§ 3. *Series-solutions expressed as Definite Integrals.*

If we take Weber's differential equation for the parabolic cylinder, viz.,

$$\frac{d^2y}{dx^2} + Zy = 0$$

where Z is a quadratic function of z, and write it in the form adopted by Professor Whittaker, viz.,

$$\frac{d^2y}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)y = 0 \quad\quad (1)$$

we get, as he points out, the two independent solutions

$$1 - \frac{2n+1}{4}z^2 + \frac{4n^2+4n+3}{96}z^4 - \dots$$

and

$$z - \frac{2n+1}{12}z^3 + \frac{4n^2+4n+7}{480}z^5 - \dots .$$

These two series adopt a more symmetrical form if we make the substitution

$$y = e^{-\frac{1}{4}z^2} u.$$

This effected, (1) becomes

$$\frac{d^2u}{dz^2} - z\frac{du}{dz} + nu = 0,$$

and the two series-solutions become

$$e^{-\frac{1}{4}z^2}\left[1 - \frac{n}{2\,!}z^2 + \frac{n(n-2)}{4\,!}z^4 - \frac{n(n-2)(n-4)}{6\,!}z^6 + \ldots\ldots\right]$$

and

$$e^{-\frac{1}{4}z^2}\left[z - \frac{n-1}{3\,!}z^3 + \frac{(n-1)(n-3)}{5\,!}z^5 - \frac{(n-1)(n-3)(n-5)}{7\,!}z^7 + \ldots\right],$$

which, for convenience, we shall denote by $E_n(z)$ and $O_n(z)$ respectively.

Considering $E_n(z)$ first, we can represent this as a definite integral by expressing the various coefficients as Γ-functions.

Now

$$\int_0^\infty e^{-u}u^k \cos(cz\sqrt{u})\,du$$

$$= \int_0^\infty e^{-u}\,u^k\left[1 - \frac{c^2z^2u}{2\,!} + \frac{c^4z^4u^2}{4\,!} - \ldots\right]du$$

$$= \Gamma(k+1) - \Gamma(k+2)\frac{c^2z^2}{2\,!} + \Gamma(k+3)\frac{c^4z^4}{4\,!} - \ldots$$

$$= \Gamma(k+1)\left[1 - (k+1)c^2\frac{z^2}{2\,!} + (k+2)(k+1)c^4\frac{z^4}{4\,!} - \ldots\ldots\ldots\right].$$

If we compare this with the corresponding expansion in $E_n(z)$, we readily find that, for equality, $c = \sqrt{-2}$ and $k = -\frac{n}{2} - 1$.

Hence we get

$$\int_0^\infty e^{-u}u^{-\frac{n}{2}-1}\cosh(z\sqrt{2u})\,du$$

$$= \Gamma\left(-\frac{n}{2}\right)\left[1 - \frac{nz^2}{2\,!} + \frac{n(n-2)}{4\,!}z^4 - \ldots\ldots\right]$$

$$= \Gamma\left(-\frac{n}{2}\right)e^{\frac{1}{4}z^2}E_n(z)$$

$$\therefore \quad \mathrm{E}_n(z) = \frac{1}{\Gamma\left(-\frac{n}{2}\right)} e^{-\frac{1}{4}z^2} \int_0^\infty e^{-u} u^{-\frac{n}{2}-1} \cosh(z\sqrt{2u})du.$$

In an exactly similar fashion, by expanding $\sin(c'z\sqrt{u})$ in the integral

$$\int_0^\infty e^{-u} u^{k'} \sin(c'z\sqrt{u})du,$$

and comparing the result with $\mathrm{O}_n(z)$, we find that

$$\mathrm{O}_n(z) = \frac{1}{\sqrt{2}\,\Gamma\left(-\frac{n-1}{2}\right)} e^{-\frac{1}{4}z^2} \int_0^\infty e^{-u} u^{-\frac{n}{2}-1} \sinh(z\sqrt{2u})du.$$

§ 4. *Expression of $D_n(z)$ in terms of $E_n(z)$ and $O_n(z)$.*

From the theory of linear differential equations there must be a linear relation connecting $\mathrm{D}_n(z)$—the standard solution of the differential equation in Professor Whittaker's paper—and $\mathrm{E}_n(z)$ and $\mathrm{O}_n(z)$ above, say,

$$\mathrm{D}_n(z) = a_n \mathrm{E}_n(z) + b_n \mathrm{O}_n(z),$$

where a_n and b_n are functions of n, but, of course, do not involve z. In order to get this relation, we note that

$$\left.\begin{aligned} \mathrm{E}_n(z) &= \frac{1}{\Gamma\left(-\frac{n}{2}\right)} e^{-\frac{1}{4}z^2} \mathrm{I_E} \\ \mathrm{O}_n(z) &= \frac{1}{2^{\frac{1}{2}}\Gamma\left(-\frac{n-1}{2}\right)} e^{-\frac{1}{4}z^2} \mathrm{I_O} \\ \mathrm{D}_n(z) &= \frac{1}{\Gamma(-n)} e^{-\frac{1}{4}z^2} z^n \mathrm{I_D} \end{aligned}\right\} \quad \cdots\cdots\cdots\cdots (2)$$

where

$$\mathrm{I_E} = \int_0^\infty e^{-u} u^{-\frac{n}{2}-1} \cosh(z\sqrt{2u})du$$

$$\mathrm{I_O} = \int_0^\infty e^{-u} u^{-\frac{n}{2}-1} \sinh(z\sqrt{2u})du$$

$$\mathrm{I_D} = \int_0^\infty e^{-t-\frac{1}{2}(t^2/z^2)} t^{-n-1}\, dt.$$

If in I_E and I_O we substitute $t^2 = 2z^2u$, we get

$$I_E = 2^{\frac{n}{2}+1}\int_0^\infty e^{-\frac{1}{2}(t^2/z^2)} t^{-n-1} z^n \left[\tfrac{1}{2}(e^t + e^{-t})\right] dt$$

and
$$I_O = 2^{\frac{n}{2}+1}\int_0^\infty e^{-\frac{1}{2}(t^2/z^2)} t^{-n-1} z^n \left[\tfrac{1}{2}(e^t - e^{-t})\right] dt$$

whence
$$I_E - I_O = 2^{\frac{n}{2}+1} z^n \int_0^\infty e^{-t-\frac{1}{2}(t^2/z^2)} t^{-n-1}\, dt$$

$$= 2^{\frac{n}{2}+1} z^n I_D.$$

Using (2) above, we get the relation

$$D_n(z) = \frac{\Gamma\left(-\frac{n}{2}\right)}{2^{\frac{n}{2}+1}\Gamma(-n)} E_n(z) - \frac{\Gamma\left(-\frac{n-1}{2}\right)}{2^{\frac{n+1}{2}}\Gamma(-n)} O_n(z).$$

Putting $z = -\frac{n}{2}$ in the identity

$$\Gamma(z)\Gamma(z+\tfrac{1}{2}) = \Gamma(2z)(2\pi)^{\frac{1}{2}} 2^{\frac{1}{2}-2z}$$

we get the simpler form

$$D_n(z) = \frac{2^{\frac{n}{2}}\sqrt{\pi}}{\Gamma\left(-\frac{n-1}{2}\right)} E_n(z) - \frac{2^{\frac{n+1}{2}}\sqrt{\pi}}{\Gamma\left(-\frac{n}{2}\right)} O_n(z),$$

which gives us the desired relation.

§ 5. *Recurrence formulae connecting a_n and b_n.*

a_n and b_n are not independent, for if we substitute

$$a_n E_n(z) + b_n O_n(z)$$

for $D_n(z)$ in the recurrence formula

$$D_n(z) - zD_{n-1}(z) + (n-1)D_{n-2}(z) = 0$$

we get

$$a_n E_n(z) - zb_{n-1}O_{n-1}(z) + (n-1)a_{n-2}E_{n-2}(z)$$

and

$$b_n O_n(z) - za_{n-1}E_{n-1}(z) + (n-1)b_{n-2}O_{n-2}(z)$$

each equal to 0, the former being an even function of z and the latter an odd.

Substituting in these equations the values of $E_n(z)$ and $O_n(z)$, and equating the several coefficients to zero, we have these relations connecting a_n and b_n,

$$a_n + (n-1)a_{n-2} = 0$$
$$b_n + nb_{n-2} = 0$$
$$a_n + b_{n-1} = 0.$$

§ 6. *Expression of the solution of the parabolic-cylinder equation as the solution of a homogeneous integral equation.*

Reverting to the original form of the equation

$$\frac{d^2y}{dz^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}z^2)y = 0 \qquad (1)$$

if we assume

$$y = \int e^{izt}\phi dt$$

where ϕ is a function of t, we have

$$\frac{d^2y}{dz^2} = -\int e^{izt}t^2\phi dt.$$

Now

$$\begin{aligned} z^2y &= \int e^{izt}z^2\phi dt \\ &= \left[\frac{e^{izt}z\phi}{i}\right] + \int e^{izt}iz\frac{d\phi}{dt}dt \\ &= \left[\frac{e^{izt}z\phi}{i}\right] + \left[e^{izt}\frac{d\phi}{dt}\right] - \int e^{izt}\frac{d^2\phi}{dt^2}dt. \end{aligned}$$

Since the functions inside the square brackets can be made to vanish by suitable choice of limits, we have finally

$$z^2y = -\int e^{izt}\frac{d^2\phi}{dt^2}dt.$$

Substituting in (1), we get for that equation

$$\int e^{izt}\left[t^2\phi - (n + \tfrac{1}{2})\phi - \tfrac{1}{4}\frac{d^2\phi}{dt^2}\right]dt = 0$$

whence ϕ must satisfy the differential equation

$$\tfrac{1}{4}\frac{d^2\phi}{dt^2} + (n + \tfrac{1}{2} - t^2)\phi = 0,$$

or, s being substituted for $2t$, ϕ must satisfy

$$\frac{d^2\phi}{ds^2} + (n + \tfrac{1}{2} - \tfrac{1}{4}s^2)\phi = 0,$$

which is obviously our initial equation (1).

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If we call the solution of this last equation $\phi = y_n(s)$, then we get as the solution of (1)

$$y_n(z) = \lambda_n \int_a^b e^{\frac{izs}{2}} y_n(s) ds.$$

Limits of integration that satisfy the conditions are obviously $-\infty$ and $+\infty$, so that we have

$$y_n(z) = \lambda_n \int_{-\infty}^{\infty} e^{\frac{izs}{2}} y_n(s) ds.$$

§ 7. *Determination of λ_n in the integral equation by means of Fourier's Double-Integral Theorem.*

From the equation

$$y_n(z) = \lambda_n \int_{-\infty}^{+\infty} e^{\frac{izs}{2}} y_n(s) ds$$

we get

$$y_n(s) = \lambda_n \int_{-\infty}^{+\infty} e^{\frac{isu}{2}} y_n(u) du$$

$$\therefore \quad \frac{1}{\lambda_n^2} y_n(z) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} du\, e^{\frac{is}{2}(z+u)} y_n(u)$$

$$= \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} du \cos\frac{s(z+u)}{2} y_n(u)$$

$$+ i \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} du \sin\frac{s(z+u)}{2} y_n(u).$$

Now the former of these integrals is equal to

$$2\int_0^{\infty} ds \int_{-\infty}^{\infty} du \cos\frac{s(z+u)}{2} y_n(u), \text{ since } \cos(-\theta) = \cos\theta$$

$$= 4\int_0^{\infty} ds \int_{-\infty}^{\infty} du \cos s(z+u) y_n(u), \text{ on replacing } \frac{s}{2} \text{ by } s.$$

Similarly the latter of the integrals vanishes, since

$$\sin(-\theta) = -\sin\theta.$$

Hence

$$\frac{1}{\lambda_n^2} y_n(z) = 4\int_0^{\infty} ds \int_{-\infty}^{\infty} du \cos s(z+u) y_n(u).$$

If we put $-z$ for z, this becomes

$$\frac{1}{\lambda_n^2} y_n(-z) = 4\int_0^{\infty} ds \int_{-\infty}^{\infty} du \cos s(s-z) y_n(u)$$

$$= 4\int_0^{\infty} ds \int_{-\infty}^{\infty} du \cos s(z-u) y_n(u).$$

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Now from Fourier's Double-Integral Theorem,

$$\pi f(x)=\int_0^\infty d\mu\int_{-\infty}^{+\infty}f(\beta)\cos\mu(\beta-x)d\beta,$$

we get
$$\int_0^\infty ds\int_{-\infty}^\infty du\cos s(z-u)y_n(u)=\pi y_n(z).$$

Hence
$$\frac{1}{\lambda_n{}^2}y_n(-z)=4\pi y_n(z).$$

But, as shown in the next paragraph, $y_n(z)$ is always either an odd or an even function of z and is of the form $e^{-\frac{1}{4}z^2}f_n(z)$ where $f_n(z)$ is of degree n in z. Therefore

$$y_n(-z)=(-1)^n y_n(z).$$

Hence
$$\frac{1}{\lambda_n{}^2}=4\pi(-1)^n,\ \text{ or }\ \lambda_n=\frac{1}{2\sqrt{\pi}(-1)^{\frac{n}{2}}},$$

so that the homogeneous integral equation may be written in the form

$$y_n(z)=\frac{1}{2\sqrt{\pi}(-1)^{\frac{n}{2}}}\int_{-\infty}^\infty e^{\frac{isz}{2}}y_n(s)ds.$$

§ 8. *Identity of $D_n(z)$ and $y_n(z)$.*

We next show that for integral values of n, $D_n(z)$ represents the auto-functions of the homogeneous integral equation

$$\frac{1}{\lambda_n}y_n(z)=\int_{-\infty}^\infty e^{\frac{isz}{2}}y_n(s)ds.$$

Consider the case where n is even and positive. Then since $D_n(z)=a_nE_n(z)+b_nO_n(z)$ (see § 4), and in this case b_n vanishes, since $\Gamma(-p)=\infty$, it will be necessary and sufficient to show that

$$\frac{1}{\lambda_n}E_n(z)=\int_{-\infty}^\infty e^{\frac{izs}{2}}E_n(s)ds.$$

Now

$$\int_{-\infty}^\infty e^{\frac{izs}{2}}E_n(s)ds=2e^{-\frac{1}{4}z^2}\int_{-\infty}^\infty e^{-x^2}\left[1-\frac{n}{2\,!}(2x+iz)^2+\frac{n(n-2)}{4\,!}(2x+iz)^4-\dots\right]dx$$

where $2x=s-iz$.

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If we take as the general term z^r, where r of course is even, this will appear in the expansions of $(2x+iz)^r$, $(2x+iz)^{r+2}$, ... $(2x+iz)^n$. The terms involving z^r therefore are

$$4e^{-\frac{1}{4}z^2}\int_0^\infty e^{-x^2}\Big[(-1)^{\frac{r}{2}}\frac{n(n-2)\ldots(n-r+2)}{r\,!}(iz)^r$$

$$+(-1)^{\frac{r+2}{2}}{}_{r+2}C_2\frac{n(n-2)\ldots(n-r)}{(r+2)\,!}(iz)^r(2x)^2$$

$$+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+(-1)^{\frac{n-2}{2}}{}_{n-2}C_{n-r-2}\frac{n(n-2)\ldots4}{(n-2)\,!}(iz)^r(2x)^{n-r-2}$$

$$+(-1)^{\frac{n}{2}}{}_nC_{n-r}\frac{n(n-2)\ldots2}{n\,!}(iz)^r(2x)^{n-r}\Big]dx.$$

Using the identity

$$\int_0^\infty e^{-x^2}x^{2n}dx=\frac{1\,.\,3\ldots(2n-1)}{2^{n+1}}\sqrt{\pi},$$

we find that the above is equal to

$$2\sqrt{\pi}(-1)^{\frac{r}{2}}(iz)^re^{-\frac{1}{4}z^2}\Big[\frac{n(n-2)\ldots(n-r+2)}{r\,!}-\frac{n(n-2)\ldots(n-r)}{r\,!}$$

$$+\frac{n(n-2)\ldots(n-r-2)}{r\,!\,2\,!}-\ldots\ldots\ldots\ldots$$

$$+(-1)^{\frac{n-r-2}{2}}\frac{n(n-2)\ldots4}{r\,!\left(\frac{n-r-2}{2}\right)!}+(-1)^{\frac{n-r}{2}}\frac{n(n-r)\ldots2}{r\,!\left(\frac{n-r}{2}\right)!}\Big]$$

$$=\frac{2\sqrt{\pi}(-1)^{\frac{r}{2}}(iz)^re^{-\frac{1}{4}z^2}}{r\,!}n(n-2)\ldots(n-r+2)$$

$$\left[1-(n-r)+\frac{(n-r)(n-r-2)}{2\,!}-\ldots+(-1)^{\frac{n-r}{2}}\frac{(n-r)(n-r-2)\ldots2}{\left(\frac{n-r}{2}\right)!}\right].$$

If we put $2p=n$ and $2q=r$, the expansion inside the brackets becomes

$$1-(p-q)\,.\,2+\frac{(p-q)(p-q-1)}{2\,!}\,.\,2^2-\ldots,$$

which is obviously the expansion of $(1-2)^{p-q}$, *i.e.* $(-1)^{\frac{n-r}{2}}$.

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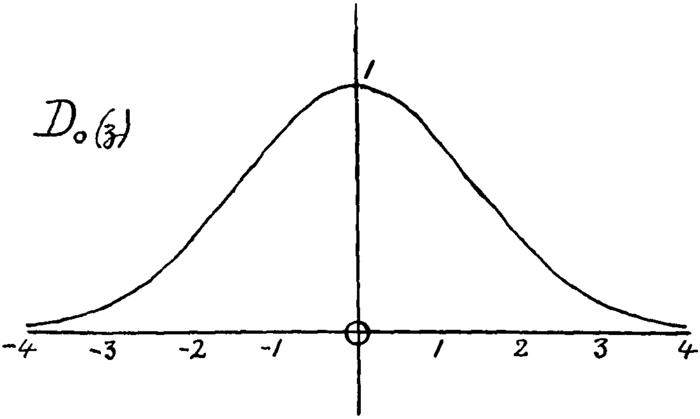
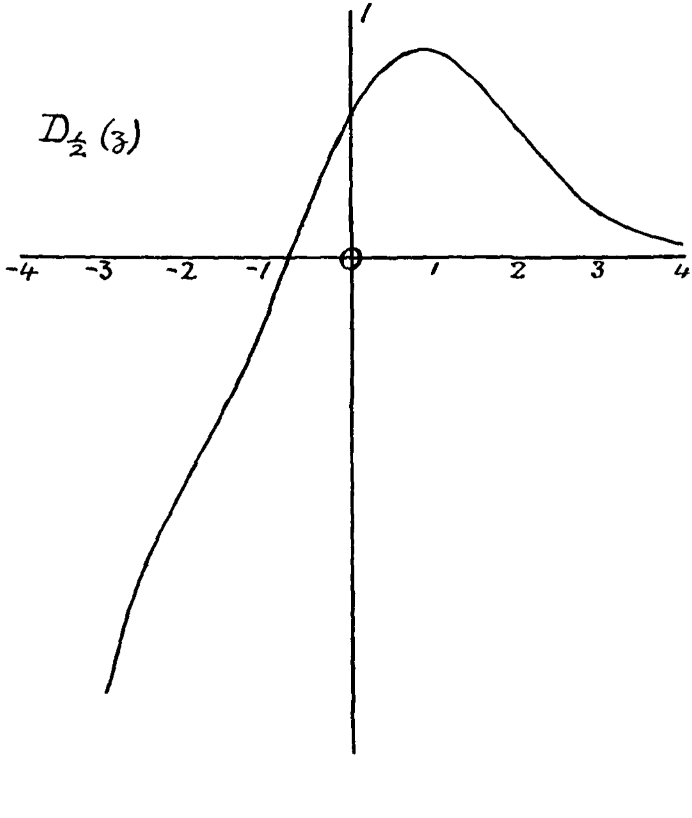
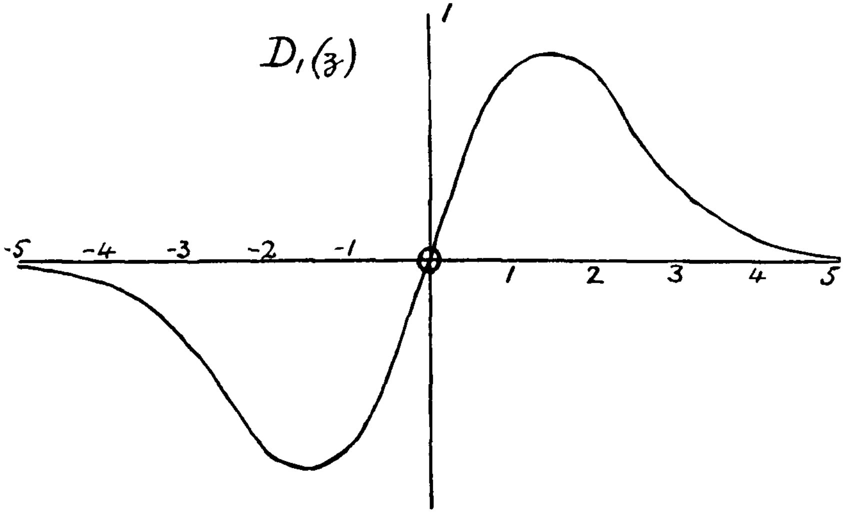
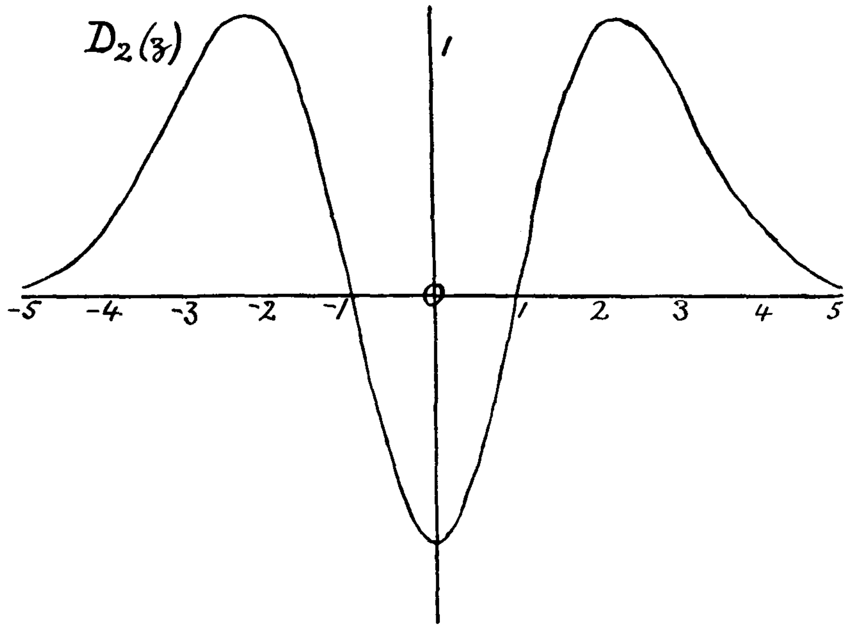

$D_0(z)$
1
-4
-3
-2
-1
1
2
3
4


$D_{\frac{1}{2}}(z)$
1
-4
-3
-2
-1
1
2
3
4


$D_1(z)$
1
-5
-4
-3
-2
-1
1
2
3
4
5


$D_2(z)$
1
-5
-4
-3
-2
-1
1
2
3
4
5

Hence we finally get

$$\int_{-\infty}^{\infty} e^{\frac{izs}{2}} \mathrm{E}_n(s) ds = 2\sqrt{\pi}\, e^{-\frac{1}{4}z^2} \sum_{r=0,2,..n} (-1)^{\frac{n+r}{2}} \frac{n(n-2)\dots(n-r+2)}{r\,!} z^r.$$

But this is equal to $\frac{1}{\lambda_n}\mathrm{E}_n(z)$, since $\frac{1}{\lambda_n} = 2\sqrt{\pi}(-1)^{\frac{n}{2}}$, and

$$\mathrm{E}_n(z) = e^{-\frac{1}{4}z^2}\left[1 - \frac{n}{2\,!}z^2 + \dots + (-1)^{\frac{r}{2}} \frac{n(n-2)\dots(n-r+2)}{r\,!} z^r + \dots\right].$$

Parity of reasoning establishes the case when n is odd, and also when n is negative. Hence for integral values of n, the autofunctions of the integral equation are given by $\mathrm{D}_n(z)$.

§ 9. *Values of $D_n(z)$ when $n = \frac{1}{2}$.*

z	$\mathrm{D}_n(z)$	z	$\mathrm{D}_n(z)$
−3.0	−1.767855	0.1	0.639178
−2.5	−1.241135	0.2	0.690620
−2.0	−0.904956	0.3	0.735234
−1.5	−0.577806	0.4	0.772672
−1.0	−0.195001	0.5	0.802702
−0.9	−0.112590	0.6	0.825217
−0.8	−0.029289	0.7	0.840232
−0.7	0.054253	0.8	0.847884
−0.6	0.137314	0.9	0.848423
−0.5	0.219123	1.0	0.842203
−0.4	0.298877	1.5	0.727895
−0.3	0.375762	2.0	0.534014
−0.2	0.448977	2.5	0.337318
−0.1	0.517752	3.0	0.184882
0	0.581368		