J. Austral. Math. Soc. (*Series A*) **30** (1980), 50–54

ORTHODOX SEMIRINGS AND RINGS

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(Received 3 April 1979)

Communicated by T. E. Hall

Abstract

We show that in a regular ring $(R, +, \cdot)$, with idempotent set E, the following conditions are equivalent:

(i) $\forall e, f \in E, (e \cdot f = 0 \Rightarrow f \cdot e = 0)$.

(ii) $(R, \cdot)$ is orthodox.

(iii) $(R, \cdot)$ is a semilattice of groups.

These and other conditions are also considered for regular semigroups, and for semirings $(S, +, \cdot)$, in which $(S, +)$ is an inverse semigroup. Examples are given to show that they are not equivalent in these cases.

1980 Mathematics subject classification (*Amer. Math. Soc.*): primary 16 A 78; secondary 20 M 25.

1. Preliminaries

RESULT 1. (Chaptal (1966), Proposition 1.) *For a ring* $(R, +, \cdot)$, *the following conditions are equivalent.*

(i) $(R, \cdot)$ *is a union of groups.*

(ii) $(R, \cdot)$ *is an inverse semigroup.*

(iii) $(R, \cdot)$ *is a semilattice of groups.*

DEFINITION 2. A triple $(S, +, \cdot)$ is a *semiring* if S is a set, and $+, \cdot$ are binary operations satisfying

(i) $(S, +)$ is a semigroup,

(ii) $(S, \cdot)$ is a semigroup,

(iii) $a \cdot (b+c) = a \cdot b + a \cdot c$, $(a+b) \cdot c = a \cdot c + b \cdot c$, for all $a, b, c \in S$.

For any $a, b \in S$, we frequently denote $a \cdot b$ by ab.

DEFINITION 3. An element $a \in S$ is an *additive zero* if $x + a = a + x = x$ for all $x \in S$. An element $b \in S$ is a *multiplicative zero* if $x \cdot b = b \cdot x = b$ for all $x \in S$.

If $(S, +, \cdot)$ does not have an element which is both an additive and a multiplicative zero, form $S^0 = S \cup \{0\}$, where $x+0 = 0+x = x$, $x \cdot 0 = 0 \cdot x = 0$ for all $x \in S$. We shall henceforth assume $(S, +, \cdot)$ has an element 0 which is both an additive and a multiplicative zero.

Definition 4. In a semiring $(S, +, \cdot)$, we put $E^{[+]} = \{x \in S\colon x+x = x\}$ and $E^{[\cdot]} = \{e \in S\colon e \cdot e = e\}$ and for each $x \in S$ we define $V^{[+]}(x) = \{a \in S\colon x+a+x = x$ and $a+x+a = a\}$ and $V^{[\cdot]}(x) = \{b \in S\colon x \cdot b \cdot x = x$ and $b \cdot x \cdot x = b\}$. We denote by x'[respectively x^*] an element chosen from $V^{[+]}(x)$ [respectively $V^{[\cdot]}(x)$], when this set is nonempty.

A semiring $(S, +, \cdot)$ is said to be an *additively inverse semiring* if $(S, +)$ is an inverse semigroup.

Definition 5. A semigroup $(S, \cdot)$ is *orthodox* if it is regular and $E = \{e \in S\colon e \cdot e = e\}$ is a subsemigroup of S.

We shall require the following results.

Result 6. (Grillet (1970), Lemma 2(i).) *For any semiring $(S, +, \cdot)$, the set $E^{[+]}$ is an ideal of $(S, \cdot)$.*

Result 7. (Karvellas (1974), Theorem 3(ii) and Theorem 7.) *Take any additively inverse semiring $(S, +, \cdot)$.*

(i) *For all $x, y \in S$, $(x \cdot y)' = x' \cdot y = x \cdot y'$ and $x' \cdot y' = x \cdot y$.*

(ii) *If $a \in aS \cap Sa$ for all $a \in S$ then S is additively commutative (and hence additively a semilattice of commutative groups).*

We use the definitions and notation of Clifford and Preston (1961).

2. Orthodox semirings

Lemma 8. *Take any regular semigroup $(S, \cdot)$ with zero 0 and set of idempotents E. Then the following conditions are equivalent.*

(i) $\forall e \in E, \forall x \in S, (e \cdot x = 0 \Rightarrow x \cdot e = 0)$.

(ii) $\forall n \in \mathbf{N}, \forall x \in S, (x^n = 0 \Rightarrow x = 0)$.

(iii) $\forall x \in S, (x^2 = 0 \Rightarrow x = 0)$.

(iv) $\forall x, y \in S, (x \cdot y = 0 \Rightarrow y \cdot x = 0)$.

Proof: (i) $\Rightarrow$ (ii). Take any $x \in S$ with $x^n = 0$ for some $n > 1$. Take any inverse x^* of x in S. Then $x^* x^n = 0$ and so $(x^* x) x^{n-1} = 0$. But $x^* x \in E$ and thus $x^{n-1}(x^* x) = 0$.

https://doi.org/10.1017/S144678870002190X Published online by Cambridge University Press

Hence $x^{n-2}(x\cdot x^*\cdot x)=0$, i.e. $x^{n-1}=0$; and continuing this process, we have $x=0$.

(iii) $\Rightarrow$ (iv). Take any $x,y\in S$ with $xy=0$. Then $(yx)^2=y(xy)x=y\cdot 0\cdot x=0$ and thus $yx=0$. So (i), (ii), (iii), (iv) are equivalent.

THEOREM 9. *Let $(S,+,\cdot)$ be any additively inverse semiring in which $(S,\cdot)$ is regular. The the following conditions are equivalent.*

(i) $\forall e,f\in E^{[\cdot]}$, $(e\cdot f=0\Rightarrow f\cdot e=0)$.

(ii) $\forall e\in E^{[\cdot]}$, $\forall x\in S$, $(e\cdot x=0\Rightarrow x\cdot e=0)$.

(iii) $\forall n\in \mathbf{N}, \forall x\in S$, $(x^n=0\Rightarrow x=0)$.

(iv) $\forall x\in S$, $(x^2=0\Rightarrow x=0)$.

(v) $\forall x,y\in S$, $(x\cdot y=0\Rightarrow y\cdot x=0)$.

Further, each is implied by

(vi) *$(S,\cdot)$ is orthodox.*

PROOF: (i) $\Rightarrow$ (ii). Take $e\in E^{[\cdot]}$, $x\in S$, with $e\cdot x=0$. Then

$$\begin{aligned}(e+(xe)')^2 &= e(e+(xe)')+(xe)'(e+(xe)')\\ &= e\cdot e+(exe)'+(xee)'+xexe\\ &= e+0'+(xe)'+0\\ &= e+(xe)'.\end{aligned}$$

Thus $e+(xe)'\in E^{[\cdot]}$. Now $(e+(xe)')(xx^*)=exx^*+(xexx^*)'=0+0'=0$. But $e+(xe)'$, $xx^*\in E^{[\cdot]}$, so $(xx^*)(e+(xe)')=0$ and thus $xx^*e+(xx^*xe)'=0$. Hence $xx^*e+(xe)'=0$; and so $xx^*e+(xe)'+xx^*e=xx^*e$, and $(xe)'+xx^*e+(xe)'=(xe)'$. Since $(S,+)$ is inverse, $xx^*e=xe$. Now $exx^*=0$ and thus $xe=xx^*e=0$. Hence by Lemma 8, we have that (i), (ii), (iii), (iv), (v) are equivalent.

(vi) $\Rightarrow$ (i). Take $e,f\in E^{[\cdot]}$ with $e\cdot f=0$. Since $f\cdot e\in E^{[\cdot]}$,

$$(fe)=(fe)^2=f(ef)e=f\cdot 0\cdot e=0.$$

EXAMPLE 10. In an arbitrary regular semigroup $(S,\cdot)$, condition (i) of Theorem 9 does not imply condition (ii), and $(S,\cdot)$ being orthodox does not imply condition (ii). To see this, we may take any Brandt semigroup $S=\mathscr{M}^0(G,I,I,\Delta)$ in which $|I|\geqslant 2$.

EXAMPLE 11. Let $(S,+,\cdot)$ be a regular ring in which $(S,\cdot)$ is not orthodox. Put $T=S\cup\{a\}$, where $a\notin S$ and define $s+a=a+s=s$, $a+a=a$, $s\cdot a=a\cdot s=a=a\cdot a$, for all $s\in S$. Then $(T,+,\cdot)$ is a semiring in which $(T,+)$ is inverse, $(T,\cdot)$ is regular, and a is the additive and multiplicative zero of T. Hence $(T,+,\cdot)$ satisfies condition (i) of Theorem 9, but is not orthodox.

https://doi.org/10.1017/S144678870002190X Published online by Cambridge University Press

3. Orthodox rings

Definition 12. A semigroup $(S,\cdot)$ is a *ring-semigroup* if there exists a binary operation $+$ on S such that $(S,+,\cdot)$ is a ring.

Theorem 13. *In a regular ring-semigroup* $(S,\cdot)$, *the following are equivalent.*

(i) $(S,\cdot)$ is *orthodox*.

(ii) $\forall e,f\in E$, $(e\cdot f=0\Rightarrow f\cdot e=0)$.

(iii) $\forall e\in E$, $\forall x\in S$, $(e\cdot x=0\Rightarrow x\cdot e=0)$.

(iv) $\forall n\in \mathbf{N}$, $\forall x\in S$, $(x^n=0\Rightarrow x=0)$.

(v) $\forall x\in S$, $(x^2=0\Rightarrow x=0)$.

(vi) $\forall x,y\in S$, $(x\cdot y=0\Rightarrow y\cdot x=0)$.

(vii) $(S,\cdot)$ is *inverse*.

Proof. (vi) $\Rightarrow$ (vii). Take any $e,f\in E$. Then $e(f-ef)=0$ and $(f-fe)e=0$. So $(f-ef)e=0$ and $e(f-fe)=0$. Thus $fe=efe=ef$. Since any inverse semigroup is orthodox, the theorem now follows from Theorem 9.

The above theorem does not hold if $(S,+,\cdot)$ is a semiring in which $(S,+)$ is inverse [orthodox] and $(S,\cdot)$ is orthodox [inverse], as is shown by the following examples.

Example 14. (i) Let $(S,+)$ be a semilattice with $|S|\geqslant 2$ and define $x\cdot y=x$ for all $x,y\in S$. Then $x\cdot(y+z)=x$ and $x\cdot y+x\cdot z=x+x=x$. Also $(x+y)\cdot z=x+y$ and $x\cdot z+y\cdot z=x+y$. So $(S,+,\cdot)$ is a semiring in which $(S,+)$ is inverse and $(S,\cdot)$ is othodox (in fact a left zero band) but not inverse.

(ii) Let $(S\backslash\{0\},\cdot)$ be a group and define $x+y=x$ for all $x,y\in S$. Then $x\cdot(y+z)=x\cdot y$ and $x\cdot y+x\cdot z=x\cdot y$. Also $(x+y)\cdot z=x\cdot z$ and $x\cdot z+y\cdot z=x\cdot z$. Hence $(S,+,\cdot)$ is a semiring in which this time $(S,\cdot)$ is inverse and $(S,+)$ is orthodox (in fact a left zero band) but not inverse.

Remark 15. From Result 1 and Theorem 13, the following are equivalent for a regular ring-semigroup.

(i) $\forall e,f\in E$, $(e\cdot f=0\Rightarrow f\cdot e=0)$.

(ii) $(S,\cdot)$ is orthodox.

(iii) $(S,\cdot)$ is inverse.

(iv) $(S,\cdot)$ is a semilattice of groups.

Since any ring $(S,+,\cdot)$ satisfying (iv) is regular and has no nonzero nilpotent elements, by Kovacs (1956), Theorem 2, the conditions (i)–(iv) are also equivalent to the condition

(v) R is a subdirect sum of division rings.

From Result 1, for any ring $(R,+,\cdot)$, if $(R,\cdot)$ is a union of groups then $(R,\cdot)$ is a semilattice of groups. In Example 14(i), the semiring $(S,+,\cdot)$ has $(S,+)$ a semilattice, and $(S, \cdot)$ both orthodox and a union of groups. However, $(S, \cdot)$ is not a semilattice of groups.

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