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On the Spectrum of an $n! \times n!$ Matrix Originating from Statistical Mechanics

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Abstract. Let $R_n(\alpha)$ be the $n! \times n!$ matrix whose matrix elements $[R_n(\alpha)]_{\sigma\rho}$, with σ and ρ in the symmetric group $\mathfrak{S}_n$, are $\alpha^{\ell(\sigma\rho^{-1})}$ with $0 < \alpha < 1$, where $\ell(\pi)$ denotes the number of cycles in $\pi \in \mathfrak{S}_n$. We give the spectrum of R_n and show that the ratio of the largest eigenvalue λ_0 to the second largest one (in absolute value) increases as a positive power of n as $n \to \infty$.

1 Introduction

Langlands [1] recently introduced matrices $R_n(\alpha)$ depending on a positive integer n and a continuous parameter $\alpha \in (0,1)$. They are $n! \times n!$, and their elements are naturally labeled and defined by permutations σ and ρ in the symmetric group $\mathfrak{S}_n$. These elements are $[R_n(\alpha)]_{\sigma\rho} = \alpha^{\ell(\sigma\rho^{-1})}$ where $\ell(\pi)$ counts the number of cycles[1] in the permutation π. His interest in these matrices was to know whether the ratio $|\lambda_0/\lambda_1|$ of the two largest eigenvalues of R_n (ordered using their absolute values) was increasing with n. An explicit computation of some of the eigenvalues for $\alpha = \frac{1}{2}$ led him to suggest that, for that α, the ratio $|\lambda_0/\lambda_1|$ might increase linearly with n. That the ratio $|\lambda_0/\lambda_1|$ increases as a power of n is true, and the purpose of this short note is to prove it, as stated in the following proposition.

Proposition 1 *Let $R_n(\alpha)$ be the $n! \times n!$ matrices with $[R_n(\alpha)]_{\sigma\rho} = \alpha^{\ell(\sigma\rho^{-1})}$ and $0 < \alpha < 1$. Let μ denote both a partition of n and the usual irreducible representation of $\mathfrak{S}_n$ associated with this partition, and let d_μ be the dimension of the representation μ.*

(a) *The eigenvalues of $R_n(\alpha)$ are in one-to-one correspondence with the partitions of n. The eigenvalue λ_μ associated with the partition $\mu = \{\mu_1, \mu_2, \dots, \mu_l\}$ has multiplicity at least d_μ^2 and is equal to*

$$\lambda_\mu = (-1)^{l(l-1)/2} \prod_{1 \le i \le l} (1-\alpha)^{(i-1)} \alpha^{(\mu_i - i + 1)}$$

where $\alpha^{(n)} = \Gamma(n+\alpha)/\Gamma(\alpha)$.

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[1]We stress that $\ell(\pi)$ is used here to denote the number of cycles in π and not its length (that is the minimum number of transpositions necessary to express π.) This usage is suggested by the tie between π and the partition $\mu_\pi = (\mu_1, \mu_2, \dots, \mu_l)$ of n that lists the number μ_i of elements in the i-th cycle of π. The length l of the partition μ_π is $\ell(\pi)$.

(b) *Let $\Lambda_\mu = |\lambda_\mu|$ and order these absolute values such that $\Lambda_0 \geq \Lambda_1 \geq \cdots$. Then the first inequality is strict, and for large n the ratio Λ_0/Λ_1 behaves as*

$$\frac{\Lambda_0}{\Lambda_1} \sim \begin{cases} n^{2\alpha} \cdot \Gamma(1-\alpha)/\Gamma(1+\alpha), & 0 < \alpha \leq \frac{1}{2}, \\ n/(1-\alpha), & \frac{1}{2} \leq \alpha < 1. \end{cases}$$

Langlands introduced the matrices R_n in a problem of statistical mechanics, but the proof of the proposition totally ignores this origin and relies (not surprisingly) on the properties of the representations of the symmetric group. It follows three steps:

(i) the reformulation of the diagonalisation of R_n using a determinantal formula,
(ii) the computation of the two determinants, and
(iii) the ordering of the eigenvalues leading to (b).

The text copies this structure.

2 The Eigenvalues of $R_n(\alpha)$ as the Quotient of Two Determinants

We present Langlands' argument [1] casting the problem of finding the eigenvalues of R_n into that of computing two determinants. Let $\mathbb{C}\mathfrak{S}_n$ denote the group ring of the symmetric group over $\mathbb{C}$. It carries the regular representation of $\mathfrak{S}_n$ where every element $\sigma \in \mathfrak{S}_n$ is represented as an $n! \times n!$ permutation matrix. The matrix $R_n(\alpha)$ can be written as a sum of these permutation matrices by noticing that, given a fixed $\pi \in \mathfrak{S}_n$, there are precisely $n!$ pairs (σ, ρ) whose product $\sigma\rho^{-1}$ is π and all entries $[R_n]_{\sigma\rho}$, with (σ, ρ) one of these pairs, will be identical and equal to $\alpha^{\ell(\sigma\rho^{-1})}$. These entries are located where the 1's are in (the regular representation of) π. Therefore $R_n = \sum \alpha^{\ell(\pi)}\pi$. (Here π stands for both the element of $\mathfrak{S}_n$ and the permutation matrix representing the action of π on $\mathbb{C}\mathfrak{S}_n$.) With this form, it is easy to see that R_n belongs to the center of $\mathbb{C}\mathfrak{S}_n$.

A set $\mu = \{\mu_1, \mu_2, \dots, \mu_l\}$ is an ordered partition of n if $n = \sum \mu_i$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_l > 0$. All partitions will be ordered, and we drop the adjective from now on. (We use the notation and the results as stated in James' lecture notes [2].) It is known that the irreducible representations of $\mathfrak{S}_n$ (over $\mathbb{C}$) are in one-to-one correspondence with the partitions of n. Moreover the composition series of the regular representation has the form $\bigoplus_\mu d_\mu \mu$ where the sum is over partitions of n, μ stands both for the partition of n and the associated representation, and d_μ is the dimension of the representation μ. Since R_n is in the center of $\mathbb{C}\mathfrak{S}_n$, it will act as a constant times the identity I_{d_μ} on each of the subspaces μ, and the constant, an eigenvalue of R_n, will be denoted λ_μ. This eigenvalue λ_μ is

$$\lambda_\mu = \frac{\operatorname{tr}(\mu(R_n))}{\dim \mu} = \frac{\chi_\mu(R_n)}{\chi_\mu(e)}$$

where χ_μ is the character of the irreducible representation μ and e is the identity permutation in $\mathfrak{S}_n$. We see that R_n acts identically on each of the d_μ irreducible subspaces μ. The eigenvalue λ_μ will therefore be of multiplicity at least d_μ^2. (The degeneracy might be greater if $\lambda_\mu = \lambda_\nu$ for some $\mu \neq \nu$.)

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It is conventional to start computing the characters χ_{ι_μ} for the induced representation ι_μ. These are (usually) reducible, but the characters of the irreducible ones can be obtained easily from the induced ones'. For $\mu = \{\mu_1, \mu_2, \dots, \mu_l\}$ a partition of n, let $\mathfrak{S}_\mu \subset \mathfrak{S}_n$ be the subgroup $\mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots \times \mathfrak{S}_{\mu_l}$. The representation ι_μ is the one induced from the identity representation on $\mathfrak{S}_\mu$ to $\mathfrak{S}_n$: $\iota_\mu = 1_{\mathfrak{S}_\mu} \uparrow \mathfrak{S}_n$. The character χ_{ι_μ} is computed in the usual fashion. Choose a set $\{\sigma_1, \sigma_2, \dots, \sigma_r\} \subset \mathfrak{S}_n$ such that $\sigma_i \mathfrak{S}_\mu \cap \sigma_j \mathfrak{S}_\mu = \varnothing$ if $i \neq j$ and $\bigcup_{1 \le i \le r} \sigma_i \mathfrak{S}_\mu = \mathfrak{S}_n$. (Note that $r = |\mathfrak{S}_n|/|\mathfrak{S}_\mu|$.) Then $\chi_{\iota_\mu}(\rho) = \sum_{1 \le i \le r} 1_{\mathfrak{S}_\mu}(\sigma_i^{-1} \rho \sigma_i)$ with $1_{\mathfrak{S}_\mu}(\pi)$ being 1 if $\pi \in \mathfrak{S}_\mu$ and 0 otherwise. This gives the dimension of ι_μ as $\chi_{\iota_\mu}(e) = |\mathfrak{S}_n|/|\mathfrak{S}_\mu|$ and an efficient way to calculate $\chi_{\iota_\mu}(R_n)$. Indeed the inner sum of

$$\chi_{\iota_\mu}(R_n) = \sum_{1 \le i \le r} \sum_{\pi \in \mathfrak{S}_n} \alpha^{\ell(\pi)} 1_{\mathfrak{S}_\mu}(\sigma_i^{-1} \pi \sigma_i)$$

turns out to be independent of i, since if $\rho = \sigma_j \sigma_i^{-1}$, then the relabeling of the sum over π into a sum over $\pi' = \rho \pi \rho^{-1}$ transforms the inner sum for i into that for j. The two must be equal and

$$\chi_{\iota_\mu}(R_n) = \frac{|\mathfrak{S}_n|}{|\mathfrak{S}_\mu|} \sum_{\pi \in \mathfrak{S}_n} \alpha^{\ell(\pi)} 1_{\mathfrak{S}_\mu}(\pi) = \frac{n!}{\mu_1! \mu_2! \dots \mu_l!} \sum_{\pi \in \mathfrak{S}_\mu} \alpha^{\ell(\pi)}.$$

The last sum factors into $\sum_{\pi \in \mathfrak{S}_\mu} \alpha^{\ell(\pi)} = \prod_{1 \le i \le l} \sum_{\pi \in \mathfrak{S}_{\mu_i}} \alpha^{\ell(\pi)}$ and if x_m stands for $\sum_{\pi \in \mathfrak{S}_m} \alpha^{\ell(\pi)}$ then

$$\chi_{\iota_\mu}(R_n) = n! \prod_{1 \le i \le l} (x_{\mu_i}/\mu_i!).$$

It remains to compute x_n.

The first step is to compute the number of permutations $\pi \in \mathfrak{S}_n$ that have a given set of cycle lengths. Suppose that i_j is the number of cycles of length j. The number of ways to break the set $\{1, 2, \dots, n\}$ into $\sum_j i_j$ cycles is $n!/\prod_j i_j!(j!)^{i_j}$. (The factor $i_j!$ assures that one does not distinguish between reorderings of cycles of identical lengths.) For each of these repartitions, one may choose arbitrarily the first element of each cycle. The reorderings of the remaining $(j-1)$ elements in each cycle provide all permutations π having i_j cycles of length j and their number is

$$n! \prod_j \frac{((j-1)!)^{i_j}}{i_j!(j!)^{i_j}} = n! \prod_j \frac{1}{i_j! j^{i_j}}.$$

This result is well known. (See [3].) Each of these permutations contribute $\alpha^{\ell(\pi)} = \prod_j \alpha^{i_j}$ to the sum $x_n = \sum_\pi \alpha^{\ell(\pi)}$. The second step is the final identification of this sum x_n through a generating function $f_\alpha(t) = \sum_{n=0}^\infty t^n x_n/n!$. Indeed f_α can be computed as a formal sum

$$f_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{n!} x_n = \sum_{n=0}^{\infty} \sum_{\left(\{i_j\} \text{ such that } n=\sum ji_j\right)} \prod_j \Big(\frac{\alpha t^j}{j}\Big)^{i_j} \frac{1}{i_j!}$$

$$= \prod_{j=1}^{\infty} e^{\alpha t^j/j} = \exp\big(-\alpha \ln(1-t)\big) = \frac{1}{(1-t)^\alpha}.$$

The x_n are extracted from f_α by differentiation

$$x_n = f_\alpha^{(n)}(t=0) = \prod_{i=0}^{n-1}(i+\alpha)$$

which closes the computation of χ_{ι_μ}.

The determinantal formula ([2], p. 74) expresses the irreducible representation associated to the partition $\mu = \{\mu_1, \mu_2, \dots, \mu_l\}$ as a sum (and subtraction) of induced representations through the determinant

$$\mu = |[\mu_i - i + j]|_{1\le i,j\le l}\,. \tag{1}$$

The expansion of the determinant leads to products $[\nu_1][\nu_2]\cdots[\nu_l]$ that are to be interpreted as the induced representation ι_ν with $\nu = \{\nu_1, \nu_2, \cdots, \nu_l\}$. If one of the $(\mu_i - i + j)$ is negative, the entry is set to zero and all the monomials $[\nu_1][\nu_2]\dots[\nu_l]$ involving it drop out of the sum. If one of the $(\mu_i - i + j)$ is 0, this entry is understood as the multiplicative identity. At first sight, this determinantal formula is weird as it contains sum *and* subtraction of representations. However, since the induced representations ι_μ are reducible, their expressions in terms of direct sums of irreducible representations, *i.e.*, their composition series, might be such that the number of times each irreducible one appears on the right hand side could turn out to be a non-negative integer (notwithstanding the minus signs in the determinant). The proof of (1) states that this is exactly so, namely, that all the integers in the composition series of $|[\mu_i - i + j]|_{1\le i,j\le l}$ are zero except one, that for μ, which turns out to be 1.

Taking the character on both sides for $\mu = \{\mu_1, \mu_2, \dots, \mu_l\}$ leads to an expression for $\chi_\mu(R_n)$ as an alternating sum of various $\chi_{\iota_\mu}(R_n)$. Because $\chi_{\iota_\mu}(R_n)$ is itself a product

$$\frac{1}{n!}\chi_{\iota_\mu}(R_n) = \prod_{1\le i\le l} \frac{x_{\mu_i}}{\mu_i!},$$

the character χ_μ for the irreducible representation μ evaluated at R_n is expressible as the determinant

$$n!\left|\frac{\prod_{k=0}^{\mu_i-i+j-1}(k+\alpha)}{(\mu_i-i+j)!}\right|_{1\le i,j\le l}$$

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and the eigenvalue λ_μ as

$$\lambda_\mu = \frac{\left| \frac{\prod_{k=0}^{\mu_i - i + j - 1}(k+\alpha)}{(\mu_i - i + j)!} \right|_{1 \le i,j \le l}}{\left| \frac{1}{(\mu_i - i + j)!} \right|_{1 \le i,j \le l}}. \tag{2}$$

Following the rule stated for the representations in (1) with one of the ν_i negative, the determinant elements with a factorial of a negative $(\mu_i - i + j)$ are set to 0, those with a factorial of a zero $(\mu_i - i + j)$ to 1.

3 Computing the Quotient of the Two Determinants

We use the notation $\alpha^{(n)}$ for the rising factorial (or Pochhammer symbol) defined for non-negative integers n as

$$\alpha^{(n)} = \prod_{i=0}^{n-1}(i + \alpha) \qquad \text{with} \qquad \alpha^{(0)} = 1.$$

This definition can be extended to all integers through

$$\alpha^{(n)} = \frac{\Gamma(n + \alpha)}{\Gamma(\alpha)}.$$

We shall use this definition for $\alpha \in (0, 1)$ in which case the latter expression avoids the singularities of the function Γ. One can easily verify that the rising factorial satisfies

$$\frac{1}{(1 - \alpha)^{(n)}} = (-1)^n \alpha^{(-n)}. \tag{3}$$

The numerator appearing in λ_μ is then

$$N_\mu = \left| \frac{\alpha^{(\mu_i - i + j)}}{(\mu_i - i + j)!} \right|_{1 \le i,j \le l}.$$

Our strategy for computing the ratio in (2) is to express N_μ as $\lambda_\mu \cdot D_\mu$ where D_μ is the denominator in the expression for λ_μ. This procedure consists of $(l - 1)$ steps, each of these involving several column operations on the determinant N_μ. These column operations are chosen to bring the rising factorials $\alpha^{(n_{ij})}$ in column C_j in N_μ closer to those $\alpha^{(n_{i1})}$ in column C_1. Step m of this procedure consists of the following $(l - m)$ column operations performed in that order

$$\begin{gathered} C_l \to C_l - C_{l-1}, \\ C_{l-1} \to C_{l-1} - C_{l-2}, \\ \vdots \\ C_{m+1} \to C_{m+1} - C_m. \end{gathered}$$

C_j always refers to the j-th column of N_μ. Similarly we shall also use R_i for the i-th row of N_μ and $(N_\mu)_{ij} = (C_j)_i = (R_i)_j$. These manipulations are made somewhat complicated by the fact that elements of N_μ and D_μ for which $\mu_i - i + j$ is non-positive have a special value, namely 1 if $\mu_i - i + j = 0$ and 0 otherwise. Let us first compute N_μ/D_μ in the cases when $\mu_i - i + j$ is positive for all i and j, that is when $\mu_i \geq i$ for all i. One can easily compute the effect of step 1 on the columns of the determinant N_μ. The column C_1 is unchanged and C_j, $j > 1$, becomes

$$\begin{aligned}(C_j)_i \to (C_j)_i - (C_{j-1})_i &= \frac{\alpha^{(\mu_i - i + j)}}{(\mu_i - i + j)!} - \frac{\alpha^{(\mu_i - i + j - 1)}}{(\mu_i - i + j - 1)!}\\ &= \frac{(\alpha - 1)\alpha^{(\mu_i - i + j - 1)}}{(\mu_i - i + j)!}\\ &= -\frac{(1-\alpha)^{(1)}\alpha^{(\mu_i - i + j - 1)}}{(\mu_i - i + j)!}.\end{aligned}$$

As claimed this first step has brought the rising factorials $\alpha^{(\mu_i - i + j)}$ in $(C_j)_i$ "closer" to $\alpha^{(\mu_i - i + 1)}$, which is the one in $(C_1)_i$. This has been done at the expense of introducing a factor $-(1-\alpha)^{(1)}$ which, luckily, is common to all elements in C_j.

One proves by induction that, after step m, columns C_j have become

$$(C_j)_i = \frac{(-1)^m (1-\alpha)^{(m)} \alpha^{(\mu_i - i + j - m)}}{(\mu_i - i + j)!}, \qquad \text{if } j > m.$$

The condition $j > m$ characterizes the columns C_j that are changed at each step through (and including) step m. The last time a column C_j is changed is at step $(j-1)$. Once steps 1 through $(l-1)$ have been performed, the columns of N_μ have therefore been brought to

$$(C_j)_i = (-1)^{j-1}(1-\alpha)^{(j-1)} \cdot \alpha^{(\mu_i - i + 1)} \cdot \frac{1}{(\mu_i - i + j)!}. \tag{4}$$

Each column C_j has a common factor $(-1)^{j-1}(1-\alpha)^{(j-1)}$. Each row R_i has a common factor $\alpha^{(\mu_i - i + 1)}$. Taking these factors out of the determinant, the element ij of N_μ becomes $1/(\mu_i - i + j)!$ and N_μ has been written as $\lambda_\mu \cdot D_\mu$, where the eigenvalue λ_μ is the product of all these factors

$$\lambda_\mu = \prod_{1 \leq j \leq l} \underbrace{(-1)^{j-1}(1-\alpha)^{(j-1)}}_{\text{from the columns}} \cdot \prod_{1 \leq i \leq l} \underbrace{\alpha^{(\mu_i - i + 1)}}_{\text{from the rows}} \tag{5}$$

if $\mu_i \geq i$ for all $1 \leq i \leq l$.

We now face the cases when some μ_i does not satisfy $\mu_i \geq i$. The row R_i will then contain one element equal to 1 and maybe some 0's to its left (there will be $-\min(0, \mu_i - i + 1)$ zero elements in R_i). If $J = i - \mu_i$, this row has the form

$$R_i = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 & \frac{\alpha^{(1)}}{1!} & \cdots & \frac{\alpha^{(j-J)}}{(j-J)!} & \cdots & \frac{\alpha^{(l-J)}}{(l-J)!} \end{bmatrix}$$

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with the "1" appearing in column J. Steps 1 to $(l-1)$ transform this row into

$$R_i = \begin{bmatrix} 0 \; 0 \; \cdots \; 0 \; 1 \; \frac{(-1)(1-\alpha)^{(J)}}{1!(1-\alpha)^{(J-1)}} \; \cdots \; \frac{(-1)^{j-J}(1-\alpha)^{(j-1)}}{(j-J)!(1-\alpha)^{(J-1)}} \; \cdots \; \frac{(-1)^{l-J}(1-\alpha)^{(l-1)}}{(l-J)!(1-\alpha)^{(J-1)}} \end{bmatrix}.$$

Using (3), the non-zero elements of R_i can be rewritten as

$$(R_i)_j = \frac{(-1)^{j-J}(1-\alpha)^{(j-1)}}{(j-J)!(1-\alpha)^{(J-1)}} = \frac{(-1)^{j-1}(1-\alpha)^{(j-1)}\alpha^{(-J+1)}}{(j-J)!}, \quad \text{for } j \geq J,$$

and comparing with the row i of D_μ and using $\mu_i - i + 1 = -J + 1$,

$$(R_i)_j = (-1)^{j-1}(1-\alpha)^{(j-1)} \cdot \alpha^{(\mu_i - i+1)} \cdot (D_\mu)_{ij}.$$

This expression is identical to the form (4) obtained when $\mu_i \geq i$ for all i that allows for the factorization $N_\mu = \lambda_\mu \cdot D_\mu$. (The only difference here is that $(\mu_i - i + 1)$ is non-positive. As noted before the rising factorial $\alpha^{(\mu_i - i+1)}$ is nonetheless well defined and the factorization takes place again.) The expression (5) for λ_μ holds therefore for any partition μ of n and part (a) of the proposition is proved.

4 Ordering the Eigenvalues

To finish the proof of the proposition we must be able to identify the two largest $|\lambda_\mu|$. For that purpose we prove the following lemma.

Lemma 2 *Let μ be a partition of n into m elements with $m \geq 2$, and let the μ_i equal to 1 be written explicitly: $\mu = \{\mu_1, \mu_2, \dots, \mu_l, 1, 1, \dots, 1\}$ with 1 repeated $(m-l)$ times and $m \geq l \geq 2, \mu_l \geq 2$. Let $\nu = \{\nu_1 = \mu_1, \nu_2 = \mu_2, \dots, \nu_{l-1} = \mu_{l-1}, 1, 1, \dots, 1\}$ with 1 repeated now $(m + \mu_l - l)$ times. Then*

$$|\lambda_\mu| < |\lambda_\nu|.$$

Proof We first discard common terms in the ratio $|\lambda_\nu/\lambda_\mu|$. The ratio can then be written as

$$\left|\frac{\lambda_\nu}{\lambda_\mu}\right| = \prod_{i=m+1}^{m+\mu_l-1} (1-\alpha)^{(i-1)} \cdot \left| \frac{\alpha^{(2-l)}}{\alpha^{(\mu_l - l+1)}} \cdot \prod_{i=m+1}^{m+\mu_l-1} \alpha^{(2-i)} \right|,$$

and using (3) on the three terms $\alpha^{(2-i)}$, $\alpha^{(2-l)}$ and $\alpha^{(\mu_l-l+1)}$, this equals

$$\frac{|(1-\alpha)^{(l-1-\mu_l)}|}{(1-\alpha)^{(l-2)}} \cdot \prod_{i=m+1}^{m+\mu_l-1} \frac{(1-\alpha)^{(i-1)}}{(1-\alpha)^{(i-2)}} = \frac{|(1-\alpha)^{(l-1-\mu_l)}|}{(1-\alpha)^{(l-2)}} \cdot \frac{(1-\alpha)^{(m+\mu_l-2)}}{(1-\alpha)^{(m-1)}}.$$

Among the rising factorials in this last form, only the sign of $(\mu_l - l+1)$ (and therefore of $\alpha^{(\mu_l-l+1)}$) is not known. This is why it is the only term that still carries an absolute value. Using the definition $(1-\alpha)^{(n)} = \Gamma(n+1-\alpha)/\Gamma(1-\alpha)$, we can show that both ratios of the above expression are the products of $(\mu_l - 1)$ monomials of the forms $(i + 1 - \alpha)$ for some $i \in \mathbb{Z}$. (Since $\alpha \in (0, 1)$, none of these monomials vanishes.) These $(\mu_l - 1)$ monomials of the first ratio appear in the denominator and those of the second in the numerator. Therefore

$$\Big|\frac{\lambda_\nu}{\lambda_\mu}\Big| = \prod_{i=1}^{\mu_l-1} \frac{(i+m-1-\alpha)}{|i-l+1+\alpha|}.$$

We now show that every term in this product is larger than 1. This will indeed be the case if $i + m - 1 - \alpha > |i - l + 1 + \alpha|$ or, equivalently: $m + l > 2(1 + \alpha)$ and $2i+(m-l) > 0$ for $i = 1, \dots, \mu_l - 1$. Since both m and l are greater or equal to 2 and $2 > 1 + \alpha$, the first inequality is satisfied. As for the second, the integer i is positive and $(m - l)$ non-negative. The lemma is proved. ∎

It follows that $|\lambda_\mu| < |\lambda_{\{1,1,\dots,1\}}|$ for all partitions μ satisfying the hypotheses of the lemma. Actually all partitions satisfy them except $\mu = \{n\}$ (because $m = 1 \not\geq 2$), $\mu = \{n-1, 1\}$ (because $l = 1 \not\geq 2$) and $\{1, 1, \dots, 1\}$. These three eigenvalues are, by (5),

$$\begin{aligned} \lambda_{\{n\}} &= \alpha^{(n)}, \\ \lambda_{\{n-1,1\}} &= -(1-\alpha)^{(1)}\alpha^{(n-1)}, \\ \lambda_{\{1,1,\dots,1\}} &= (-1)^{n-1}\alpha^{(1)}(1-\alpha)^{(n-1)}, \end{aligned}$$

and

$$\Big|\frac{\lambda_{\{n-1,1\}}}{\lambda_{\{1,1,\dots,1\}}}\Big| \begin{cases} \leq 1 & 0 < \alpha \leq \frac{1}{2}, \\ \geq 1 & \frac{1}{2} \leq \alpha < 1. \end{cases}$$

The ratio of the two largest eigenvalues $\lambda_0 = \lambda_{\{n\}} > |\lambda_1|$ is therefore

$$\Big|\frac{\lambda_0}{\lambda_1}\Big| = \begin{cases} \Big|\frac{\lambda_{\{n\}}}{\lambda_{\{1,1,\dots,1\}}}\Big| \sim \frac{\Gamma(1-\alpha)}{\Gamma(1+\alpha)} n^{2\alpha} & 0 < \alpha \leq \frac{1}{2}, \\ \Big|\frac{\lambda_{\{n\}}}{\lambda_{\{n-1,1\}}}\Big| \sim \frac{n}{1-\alpha} & \frac{1}{2} \leq \alpha < 1. \end{cases}$$

The fact that $\lambda_0 = \lambda_{\{n\}}$ is non-degenerate can be seen either from the fact that $d_{\{n\}} = 1$ or from Frobenius' theorem, as all matrix elements of $R_n(\alpha)$ are strictly positive. This ends the proof of (b) of Proposition 1.

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