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# PRODUCT OF POLYNOMIAL VALUES BEING LARGE POWER

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*Abstract* Erdös and Selfridge first showed that the product of consecutive integers cannot be a perfect power. Later, this result was generalized to polynomial values by various authors. They demonstrated that the product of consecutive polynomial values cannot be the perfect power for a suitable polynomial. In this article, we consider a related problem to the product of consecutive integers. We consider all sequences of polynomial values from a given interval whose products are almost perfect powers. We study the size of these powers and give an asymptotic result. We also define a group theoretic invariant, which is a natural generalization of the Davenport constant. We provide a non-trivial upper bound of this group theoretic invariant.

*Keywords:* Davenport constant; smooth polynomial value; product of integers are almost perfect power

*2020 Mathematics subject classification:* Primary 11A05; 11N56; 11P70
Secondary 11B75

## 1. Introduction

In [10], Erdös and Selfridge solved a long standing conjecture by showing that the product of consecutive integers cannot be a perfect power. In particular, they proved that for a fixed non-negative integer $t$,

$$(n+d_1)(n+d_2)\cdots(n+d_k)=x^l, \tag{1}$$

where $1=d_1<d_2<\cdots<d_k\leq k+t$ and $l>1$, has only finite number of solutions. If $t=0$ then equation (1) has no solution. After Erdös and Selfridge's work, similar results were studied in an arithmetic progression. In [24], Saradha extended their result for an arithmetic progression. She proved that for integers $(n,d)=1$, $1\leq d\leq 6$, $k\geq 3$, $l\geq 2$, $n\geq 1$ and $y\geq 1$ the equation:

$$n(n+d)\cdots(n+(k-1)d)=y^l, \tag{2}$$

has no integral solution. Later, Saradha's result was improved by extending the ranges of $d, k$ and $l$. Ultimately, Bennett and Siksek gave a complete solution to equation (2) in [3]. Specifically, they showed that, for a large positive integer $k$, there are at most finitely many solutions to equation (2) in the positive integers $n, d, y$, and $l$, where $l \geq 2$ and $\gcd(n, d) = 1$.

A more general setting of this problem can be formulated as follows: Let $P \in \mathbb{Z}[x]$ with positive leading coefficients. Consider the equation

$$\prod_{k=1}^{m} P(k) = y^l. \tag{3}$$

The question is whether the Diophantine equation (3) has solution or not. Clearly, for an arbitrary polynomial $P(x)$ the problem is wide open. For some particular polynomials $P(x)$ the solutions of equation (3) are known. For example, in equations (1) and (1.2), the cases of linear polynomials $P(x) = x + n$ and $P(x) = ax + b$ are given. Also, solutions of equation (3) are known for some non-linear polynomials $P(x)$ when the power $l = 2$. Cilleruelo investigated the quadratic polynomial $P(x) = x^2 + 1$ in [6]. Following the result in [6] many authors studied the problem and gave solutions for the polynomials $4x^2 + 1$, $2x(x-1)+1$, $ax^2 + bx + c$, and $x^l + m^l$ for $m \in \mathbb{N}$ and $l \geq 2$.

In [9], Erdös, Malouf, Sellfridge, and Szekeres considered a related problem to equation (3). They investigated when the product of integers from a given interval has perfect power. They showed that in any interval of a certain length there are integers whose product is perfect power. This naturally raises the question: if at an interval there are sets of integers whose products are perfect powers, what is the maximal value of such a power? More specifically, the problem can be formulated as follows: Let $[1, N] \subset \mathbb{N}$ be a given interval. Consider all possible integers $x_1, x_2, x_3, \ldots, x_m, y$ from the above interval, and $l \in \mathbb{N}$ so that

$$x_1 x_2 x_3 \cdots x_m = y^l. \tag{4}$$

Let $L(N)$ be the maximum value of all $l$ satisfied equation (4). So what will be the supremum of $L(N)$ in terms of $N$? In [25], Skalba first considered this problem and gave upper and lower bounds for $L(N)$. Later Goudout [13] improves the upper bound of $L(N)$ given in [25]. If we combine the Skalba and Goudout results, then one has

$$L(N) = N \exp(-(\sqrt{2} + o(1))\sqrt{\log N \log\log N}),$$

as $N \to \infty$.

In this article, we will consider sequences $\{x_i\}$'s with certain restrictions. In particular, we will take all sequences $\{x_i\}$'s those are some polynomial values, but the product of $\{x_i\}$'s are perfect powers. We can rewrite this problem in the following way: Let $P(x) \in \mathbb{Z}[x]$ be a polynomial with a positive leading coefficient. Then for any given interval $[1, N] \cap \mathbb{N}$ we consider all possible products of the form:

$$P(x_1)P(x_2)\cdots P(x_m) = y^{\omega}, \tag{5}$$

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where $P(x_i)$'s are taken from the interval $[1, N] \cap \mathbb{N}$. Let $\omega_P(N)$ be the maximum value of all $\omega$ satisfies equation (5). What are the infimum and supremum of $\omega_P(N)$? In general finding such bounds for a general polynomial $P(x)$ are difficult.

Here we will obtain such bounds for some specific polynomials. First, we consider the following product:

$$P(x_1)P(x_2)\cdots P(x_m) = by^{\omega}, \tag{6}$$

for integers $P(x_i), y, b$ in $[1, N]$ and $\gcd(b, y) = 1$.

Let $\omega_P(N)$ be the maximum value of all $\omega$ satisfies equation (6). Our first result is an asymptotic of $\omega_P(N)$ when $P(x)$ is a linear polynomial generate integers those are in arithmetic progression.

**Theorem 1.1.** *Let $a$ and $q$ be two integers with $(a, q) = 1$ and $P(x) = ax + q$. Let $\omega_P(N)$ be the largest possible integer so that*

$$P(x_1)P(x_2)\cdots P(x_k) = by^{\omega_P(N)},$$

*when $P(x_i)$, $b$, and $y$ are integer in $[1, N]$, and $\gcd(b, y) = 1$. Then uniformly for,*

$$\log q \le c\sqrt{\log N \log\log N},$$

*we have*

$$\omega_P(N) \sim \frac{N}{q\exp((\sqrt{2} + o(1))\sqrt{\log N \log\log N})},$$

*as $N \to \infty$.*

In our next result, we give bounds for $\omega_P(N)$ when $P(x)$ is not a linear polynomial but some other suitable polynomials.

**Theorem 1.2.** *Let $\omega_P(N)$ be the largest possible integer so that*

$$P(x_1)P(x_2)\cdots P(x_k) = by^{\omega_P(N)},$$

*for $P(x_i), b$, and $y$ are in teger in $[1, N]$ and $\gcd(b, y) = 1$.*

(1) *Let $a$ and $q$ be two fixed positive integer and $P(x) = x(ax + q)$. Then for any $\epsilon > 0$, there exist constants $C_1$ and $C_2$ depends on $\epsilon$, $a$, and $b$ such that*

$$\frac{C_1 N^{1-\epsilon}}{\log N} \le \omega_P(N) \le C_2 N^{1-\epsilon}.$$

(2) *Let $P(x) = x^2 + 1$. Then there exist constants $C_1$ and $C_2$ depends on $\epsilon$ such that*

$$\frac{C_1 N^{\frac{30}{179}-\epsilon}}{\log N} \le \omega_P(N) \le C_2 N^{\frac{179}{328}-\epsilon}.$$

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(3) *Let $P(x) = (x+1)(x+2)\cdots(x+l)$ for some positive integer $l$. Then for sufficiently large $N$ there exist a constants $C_1$ and $C_2$ depends on $\epsilon$ and $l$ such that*

$$\frac{C_1 N^{1-\exp(-1/l)-\epsilon}}{\log N} \leq \omega_P(N) \leq C_2 N^{1+\exp(-1/l)-\epsilon},$$

*provided none of the $P(x_i)$ has any common factors.*

## 2. Preliminaries

### 2.1. Davenport constant

Our argument depends on the Davenport constant, which is a group-theoretic invariant. Let $G$ be a finite abelian multiplicative group with the identity element 1. Then the Davenport constant $D(G)$ of the group $G$ is the minimal integer such that every sequence of length $D(G)$ from $G$ has a sub-sequence whose product is equal to the identity elements of the group. A trivial bound of the Davenport constant is $D(G) \leq |G|$. Due to its various implications, from the decomposition of irreducible integers in the ideal class group (see [8]) to the proof of the infinitude of Carmichael numbers (see [1]), a great deal of work has been done on obtaining the best possible bound of the Davenport constant. Let $\mathbb{M}_n$ denote the cyclic group of order $n$. Then any finite abelian group $G$ can be written as $G = \mathbb{M}_{n_1} \times \mathbb{M}_{n_2} \times \cdots \times \mathbb{M}_{n_d}$ where $n_1, n_2, \ldots, n_d$ are unique integers with $n_1 \geq 2$ and $n_i \mid n_{i+1}$ for $1 \leq i \leq d$. Here, the integers $d$ and $n_d$ are the rank and the exponent of the group $G$, respectively. If $G = \mathbb{M}_n$ is a cyclic group, then $D(G) = n$. This can be seen by just considering the sequence $(a, a, \cdots, a)$ where $a$ is a generator of $G$. In [21, 22], Olson proved that if $G$ is a finite $p$-group then

$$D(G) = (n_1 + n_2 + \cdots + n_d) - d + 1, \tag{7}$$

and $D(G) = n_1 + n_2 - 1$ when the rank of $G$ is 2. It is still unknown whether the equality equation (7) holds for any finite abelian group of rank greater than 2. Boas [27] and Gao [12] showed that equality equation (7) holds for a wide class of finite abelian groups of rank 3. In particular, finding the right size of Davenport constant is still an open problem. In [20], Narkiewicz conjectured that $D(G) \leq (n_1 + n_2 + \cdots + n_d)$. The best upper bound of $D(G)$ is

$$D(G) \leq \exp(G)\left(1 + \log\frac{|G|}{\exp(G)}\right), \tag{8}$$

which is due to Van Emde Boas and Kruyswijk [26], Meshulam [19], and Alford, Granville and Pomerance [1]. Here $\exp(G)$ is the the exponent of the group $G$. Various generalizations of the Davenport constant are studied in the literature.

Now we will define a generalization of the Davenport constant.

**Definition 2.1.** *Let $A$ be a subgroup of $G$ and $e$ be the identity element of $G$. We define the $A$-relative Davenport Constant of $G$ by the least positive integer $\ell$ such that*

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*every sequence* $(\bar{x}_1, \cdots, \bar{x}_\ell)$ *of* $G/A$ *of length* $\ell$ *has a non-trivial sub-sequence* $(\bar{x}_{i_1}, \cdots, \bar{x}_{i_r})$ *such that* $\prod_{j=1}^{r} \bar{x}_{i_j} = \bar{e}$.

*In the rest of the paper, we will use the notation* $D^{(A)}(G)$ *for the A-relative Davenport constant. Note that,* $D^{(A)}(G) = D(G)$ *when* $A = \{e\}$.

In the next lemma, we will give a non-trivial upper bound of $D^{(A)}(G)$.

**Lemma 2.2.** *Let* $G$ *be a finite abelian group and* $A$ *be a subgroup of* $G$. *Then one has*

$$D^{(A)}(G) \leq \exp(G)\left(1 + \log \frac{|G|}{\exp(G)} - \frac{D(A) - 1}{\exp(G)}\right).$$

**Proof.** It is enough to prove for a non-trivial subgroup $A$ of $G$. Let $e$ be the identity element. We will show that

$$D^{(A)}(G) \leq D(G) - D(A) + 1. \tag{9}$$

Let $(a_1, a_2, \cdots, a_m)$ be a sequence from $A$ of length $D(A) - 1$ such that there is no sub-sequence whose product is $e$. To arrive a contradiction, we consider $D^{(A)}(G) > D(G) - D(A) + 1$. Note that, by the definition $D^{(A)}(G) \leq D(G)$. Let $(x_1, x_2, \cdots, x_l)$ be a sequence of length $D^{(A)}(G) - 1$ such that there is no sub-sequence whose product is in $A$. Next, we consider the sequence $(a_1, a_2, \cdots, a_m, x_1, x_2, \cdots, x_l)$. Since this sequence has length at least $D(G)$ then there exists a sub-sequence $(a_{s_1}, a_{s_2}, \cdots, a_{s_q}, x_{r_1}, x_{r_2}, \cdots, x_{r_t})$ such that

$$a_{s_1} a_{s_2} \ldots a_{s_q} x_{r_1} x_{r_2} \ldots x_{r_t} = e.$$

This shows $x_{r_1} x_{r_2} \ldots x_{r_t} \in A$. This gives us a contradiction. Combining equation (9) with equation (8) we have the result of the lemma. $\square$

### 2.2. Smooth polynomial values

Consider the set of $y$-smooth integers

$$S(x, y) = \{n \leq x : p^+(n) \leq y\},$$

where $p^+(n)$ denotes the largest prime factor of $n$. It is well known that the cardinality of the set $S(x, y)$, which is denoted by $\Psi(x, y)$, is

$$\Psi(x, y) = (1 + o(1))\, \rho(u) x,$$

where $u = \frac{\log x}{\log y}$ and $\rho$ is the Dickman-de Brujin function satisfies the following differential equation:

$$u\rho'(u) + \rho(u - 1) = 0.$$

We need the following asymptotic results. The most important special case of smooth number estimate is (see [16, p. 270])

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**Lemma 2.3.** *Let* $L(x) = \exp(\sqrt{\log x \log\log x})$. *Then*

$$\psi(x, L(x)^c) = \frac{x}{L(x)^{1/2c+o(1)}}$$

*as* $x \to \infty$.

One generalization of $y$-smooth number is polynomial values having prime factor no greater than $y$. Consider the polynomial ring $\mathbb{Z}[x]$. Let $P(x) \in \mathbb{Z}[x]$ and define the set

$$S_P(x, y) = \{n \le x : p^+ (P(n)) \le y\}.$$

Let $\Psi_P(x, y)$ denote the cardinality of the set $S_P(x, y)$. For a linear polynomial $P(x) = ax + q$ Chowla and Vijayaraghavan [5] and Buchstab [4] gave an estimate of $\Psi_P(x, y)$ for a fixed $f$ and $u$. Later, Ramaswami [23] gave an uniform version of Buchstab's results. Fouvry and Tenenbaum [11] and Granville [14, 15] made significant improvement of Ramaswami's uniform result. The following result can be found in [11, 14, 15]. See also Hildebrand and Tenenbaum [18, Sec. 6]

**Lemma 2.4.** *Let* $(a, q) = 1$ *and* $P(x) = ax + q$. *Then*

$$\Psi_P(x, y) = \frac{x}{qu^{u+o(u)}},$$

*for* $x \ge 3$, $1 \le u \le e^{c\sqrt{\log y}}$, *and* $q \le e^{c\sqrt{\log y}}$.

For degree 2 polynomial Balog ans Ruzsa [2] gave bounds of $\Psi_P(x, y)$.

**Lemma 2.5.** *Let* $a, b \in \mathbb{Z}$ *and* $P(x) = x(ax + b)$. *Then for all* $\alpha > 0$

$$\Psi_P(x, x^\alpha) \asymp_{P,\alpha} x.$$

For degree 2 irreducible polynomial we have little weaker result. Dartyge [7] showed that

**Lemma 2.6.** *Let* $P(x) = (x^2 + 1)$ *and* $\alpha > \frac{149}{179}$. *Then*

$$\Psi_P(x, x^\alpha) \asymp_{\alpha,P} x$$

*holds for all large* $x$.

Now, if $P(x) \in \mathbb{Z}$ is a completely reducible polynomial of any degree then Hildebrand [17] computed bounds of $\Psi_P(x, y)$. In particular, Hildebrand [17] proved the following. A set $A \subset \mathbb{N}$ is said to be stable if for each fixed $t \in \mathbb{N}$, $n \in A \Rightarrow tn \in A$. Define the lower asymptotic density of the set $A$ by

$$d(A) = \liminf_{x\to\infty} \frac{1}{x} \#\{n \le x : n \in A\}.$$

Then

**Lemma 2.7.** *Let $k \geq 2$ be an integer and $\alpha_k = \frac{k-2}{k-1}$. Then any stable set $A \subset \mathbb{N}$ with $d(A) > \alpha_k$ satisfies*

$$d(\{n : n+i \in A, i = 0, 1, 2, \ldots, k\}) > 0.$$

In particular if we take

$$A = \{n : p^+(n) > y\},$$

then for $P(x) = (x+1)(x+2)\cdots(x+k)$ and $\alpha > e^{-\frac{1}{k-1}}$ one has

$$\Psi_P(x, x^\alpha) \asymp_{\alpha,P} x \tag{10}$$

for all large $x$.

## 3. Proof of Theorem 1.1

### 3.1. Lower bound

Let $\mathbb{Q}_+$ be the set of all positive rational numbers and

$$\mathbb{Q}_+^\omega = \{q^\omega : q \in \mathbb{Q}_+\}$$

for some positive integer $\omega$ which will be chosen later. Then $\mathbb{Q}_+/\mathbb{Q}_+^\omega$ form a multiplicative group. Let $y$ be a fixed positive integer and $\{p_1, p_2, \cdots, p_t\}$ are primes in $[1, y]$. Clearly, $t = \pi(y)$. Let us denote $\overline{p_i}$ be the image of the prime $p_i$ in the quotient group $\mathbb{Q}_+/\mathbb{Q}_+^\omega$ and $G$ be the finite abelian subgroup of $\mathbb{Q}_+/\mathbb{Q}_+^\omega$ generated by the elements $\{\overline{p_1}, \overline{p_2}, \cdots, \overline{p_t}\}$. Hence,

$$G \cong \underbrace{\mathcal{C}_\omega \times \mathcal{C}_\omega \times \cdots \times \mathcal{C}_\omega}_{t\text{times}}., \tag{11}$$

where $\mathcal{C}_\omega$ is the cyclic group of the order $\omega > 2$. Let $S$ be the set of all $y$-smooth integer from $[1, N]$ and $S_P = S \cap \{P(n) : 1 \leq n \leq N\}$, where $P(x) = ax + q$. Let us consider $S_P = \{P(n_1), P(n_2), \cdots, P(n_s)\}$. Note that $\overline{P(n_i)} \in G$. Now we choose $\omega$ such that

$$(\omega - 1)y \log N \leq s. \tag{12}$$

Clearly for large $N$, one has $s \geq \omega\pi(y)\log(\omega) - 1 = \omega t \log(\omega) - 1 \geq D^{(A)}(G)$ for some non-trivial subgroup $A$ of $G$. Then by Lemma 2.2, there exists a subgroup $A$ of $G$ such that

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$$\overline{P(n_{r_1})} \cdot \overline{P(n_{r_2})} \cdots \overline{P(n_{r_k})} = \overline{b}$$

for some $\overline{b} \in A$ and hence

$$P(n_{r_1}) \cdot P(n_{r_2}) \cdots P(n_{r_k}) \in b\mathbb{Q}_+^{\omega}.$$

Therefore

$$P(n_{r_1}) \cdot P(n_{r_2}) \cdots P(n_{r_k}) = bl^{\omega},$$

for some integer $l$ with $\gcd(b, l) = 1$. Now, form the right side of equation (12), one has

$$\log \omega \geq \log s - \log y - \log\log N.$$

From the lemma 2.4 we can choose $y = \exp(\sqrt{\log N \log\log N/2})$ and hence $\log s = \log N - \log q - (u + o(u)) \log u$. Therefore

$$\begin{aligned}
\log \omega &\geq \log N - \log q - (u + o(u)) \log u - \sqrt{\log N \log\log N/2} - \log\log N \\
&\geq \log N - \log q - \frac{\log N}{\log y}(\log\log N - \log\log y) - \sqrt{\log N \log\log N/2} \\
&\quad - \log\log N + o(u) \log u \\
&\geq \log N - \log q - \sqrt{2 \log N \log\log N} + o(\sqrt{\log N \log\log N}).
\end{aligned}$$

In the penultimate step above we have used the definition

$$u = \frac{\log N}{\log y}.$$

Hence, we obtain

$$\omega \geq \frac{N}{q \exp((\sqrt{2} + o(1))\sqrt{\log N \log\log N})}.$$

### 3.2. Upper bound

Let us consider

$$P(x_1)P(x_2) \cdots P(x_m) = bl^{\omega},$$

and $p$ be the largest prime factor of $l^{\omega}$. Clearly, $p^{\omega}$ is a factor of $p^{\nu_p(bl^{\omega})}$. Here $\nu_p(x)$ is the $p$-adic valuation of the integer $x$. Next we consider the set

$$A = \{P(n) : P(n) \leq N, P(n) \text{ is } p\text{-smooth}, p \mid P(n)\}.$$

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Hence $|A| \leq \psi_P(N/p, p)$. One can check if $k$ is the largest value for which $p^k \leq N$ then $k \leq \log N/\log p$. Put all these information together with Lemma 2.4 we have

$$\begin{aligned} \omega \leq \nu_p(bl^\omega) &\leq \psi_P\left(\frac{N}{p}, p\right)\frac{\log N}{\log p} \\ &\leq \frac{1}{q}L(p), \end{aligned} \tag{13}$$

where $L(p) = \frac{N}{p}v^{-v+o(v)}\frac{\log N}{\log p}$ and $v = \frac{\log N/p}{\log p}$. Now we will maximize the function $L(p)$. Note that

$$\begin{aligned} \log L(p) &= \log N - \log p \\ &\quad + \left(1 - \frac{\log N}{\log p}\right)(\log\log N - \log\log p + \log(1 - \log p/\log N)) + o(v\log v). \end{aligned} \tag{14}$$

To maximize equation (14) one needs to choose $p$ so that $\log p$ and

$$\frac{\log N}{\log p}\left(\log\log N - \log\log p\right),$$

are of same size. Let us set

$$p = \exp\left(\left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log N \log\log N}\right).$$

Therefore

$$\begin{aligned} \log p + \frac{\log N}{\log p}(\log\log N - \log\log p) &= \left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\log N\log\log N} \\ &\quad + \frac{\log N}{2\log p}\log\log N + O\left(\sqrt{\log N\log\log N}\right) \\ &= \left(\sqrt{2} + o(1)\right)\sqrt{\log N\log\log N}. \end{aligned} \tag{15}$$

Hence from equations (14) and (15) we have

$$\max_p L(p) = N\exp\left(-\left(\sqrt{2} + o(1)\right)\sqrt{\log N\log\log N}\right). \tag{16}$$

Substituting equation (16) in equation (13) will give the required upper bound.

## 4. Proof of Theorem 1.2

Proof of Theorem 1.2 is similar to the proof of Theorem 1.1.

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**Lower bounds:** Consider the set $S$ be the set of all $y$-smooth integers in $[1, N]$. Let us define $S_P = S \cap \{P(n) : 1 \leq n \leq N\}$ for a polynomial $P(x)$. Let us denote $S_P = \{P(n_1), P(n_2), \cdots, P(n_s)\}$. Clearly $\overline{P(n_i)} \in G$, where $G$ is defined in equation (11). Similar to the previous section we consider

$$(\omega - 1)y \log N \leq s \leq \omega y \log N. \tag{17}$$

Hence $s \geq \omega\pi(y)\log(\omega) - 1 = \omega t \log(\omega) - 1 \geq D^{(A)}(G)$ for some non-trivial subgroup $A$ of $G$. Then by Lemma 2.2 we have

$$\overline{P(n_{r_1})} \cdot \overline{P(n_{r_2})} \cdots \overline{P(n_{r_k})} = \overline{b},$$

where $\overline{b} \in A$ Therefore there exists an integer $l$ with $\gcd(b, l) = 1$ such that

$$P(n_{r_1}) \cdot P(n_{r_2}) \cdots P(n_{r_k}) = bl^{\omega}.$$

From the inequality equation (17) we have

$$\log \omega \geq \log s - \log y - \log\log N.$$

(1) Let us consider the polynomial $P(x) = x(ax + q)$. By Lemma 2.5 we have that for all $\alpha > 0$, $s \geq C_\alpha N$ when $y = N^\alpha$. Therefore,

$$w \geq C_\alpha \frac{N^{1-\alpha}}{\log N}.$$

(2) Next we consider the polynomial $P(x) = x^2 + 1$. By Lemma 2.6 we have for all $\alpha > 149/179$, $s \geq C_\alpha N$ when $y = N^\alpha$. Set $\alpha = \frac{149}{179} + \epsilon$. Therefore,

$$w \geq C_\epsilon \frac{N^{\frac{30}{179} - \epsilon}}{\log N}.$$

(3) Lastly, we consider the polynomial of degree $l$ defined by $P(x) = (x+1)(x+2)\ldots(x+l)$. From the equation (10) and for $\alpha > e^{-\frac{1}{l-1}}$ we have $s \geq C_\alpha N$ when $y = N^\alpha$. Take $\alpha = e^{-\frac{1}{l-1}}\epsilon$ and one has

$$w \geq C_\epsilon \frac{N^{1-\exp(-1/l)-\epsilon}}{\log N}.$$

**Upper bounds:** From the inequality equation (13) we find

$$\omega \leq \psi_P\left(\frac{N}{p}, p\right) \frac{\log N}{\log p} \tag{18}$$

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for any polynomial $P$. From Lemmas 2.5, 2.6, and equation (10) one finds that the right side of equation (12) maximizes when $p = N^{\frac{\alpha}{\alpha+1}}$ for a suitable $\alpha$. Hence

$$w \leq C_\alpha N^{\frac{1}{\alpha+1}}.$$

Now we choose any $\alpha > 0$, $\alpha = \frac{149}{179} + \epsilon$, and $\alpha = e^{-\frac{1}{l-1}} + \epsilon$ respectively for $P(x) = x(ax+q)$, $P(x) = x^2 + 1$, and $P(x) = (x+1)(x+2)\ldots(x+l)$. This will give the desired upper bounds and which completes the proof of the theorem.

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