Can. J. Math., Vol. XXVII, No. 2, 1975, pp. 366–371

PARTIALLY BOUNDED SOLUTIONS OF LINEAR ORDINARY DIFFERENTIAL EQUATIONS

DAVID LOWELL LOVELADY

1. Introduction. Let R, R^+, and R^- be the intervals $(-\infty, \infty)$, $[0, \infty)$, and $(-\infty, 0]$ respectively. Let m be a positive integer, and let $\mathscr{A}$ be the algebra of all $m \times m$ matrices. Let A be a locally integrable function from R to $\mathscr{A}$. We propose to study the problems

$$\text{(NH)} \quad u'(t) = f(t) + A(t)u(t)$$

and

$$\text{(H)} \quad v'(t) = A(t)v(t)$$

in R^m. (H) and (NH) will denote whole-line problems, whereas $(\text{H})^+$, $(\text{NH})^+$, $(\text{H})^-$, and $(\text{NH})^-$ will denote corresponding semi-axis problems.

In [**1**] (see also [**2**, Theorem 1, p. 131]), W. A. Coppel obtained necessary and sufficient conditions for each bounded continuous f on R^+ to yield at least one bounded solution u of $(\text{NH})^+$. The present author [**3**] has determined that an analogous result holds for (NH).

If one attempts to apply these results to a higher order problem

$$(\text{NH})_n \quad u^{(n)}(t) = f(t) + A(t)u(t)$$

by converting to a first order problem over R^{mn}, one discovers that the results best fit the more general problem

$$u(t) = \sum_{k=1}^{n} t^{k-1}z_k + \sum_{k=1}^{n} \int_0^t \frac{(t-s)^{k-1}}{(k-1)!} f_k(s)ds + \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} A(s)u(s)ds$$

and yield boundedness not only of u but also of the intermediate derivatives $u', u'', \ldots, u^{(n-1)}$. There is, however, a generalization of the original problem which includes $(\text{NH})_n$ in a natural way.

Let each of S_1 and S_2 be a linear subspace of R^m, and consider the problem of finding conditions which ensure that if f is a bounded S_1-valued continuous function on R^+ then $(\text{NH})^+$ has a solution the projection of which into S_2 is bounded. It is clear that this problem not only includes the original problem,

Received September 13, 1973 and in revised form, March 11, 1974.

https://doi.org/10.4153/CJM-1975-044-1 Published online by Cambridge University Press

but also includes the aforementioned higher order problem. In § 2, we shall solve this problem for $(\mathrm{NH})^+$. In § 3, we shall use these results to obtain information on the solution space of (H), thus extending [**4**, Theorem 1]. We shall indicate in § 4 how this includes many of the results of [**5**], and how § 3 yields information on solution space structure for

$$(\mathrm{H})_n \quad v^{(n)}(t) = A(t)v(t).$$

2. The semi-axis problem. Let $\{z_1, \ldots, z_m\}$ be a basis for R^m, and if x is in R^m and $x = \sum_{k=1}^m a_k z_k$, let $|x| = \max\{|a_1|, \ldots, |a_m|\}$. Let $\|\ \|_0$ be the induced norm on $\mathscr{A}$. Let each of α and β be a continuous function from R to $(0, \infty)$. Let $\mathscr{B}_\alpha\mathscr{C}$ be the space of all continuous functions f from R to R^m such that there is a number b with $|f(t)| \leqq b\alpha(t)$ whenever t is in R. If f is in $\mathscr{B}_\alpha\mathscr{C}$ let

$$\|f\|_\alpha = \sup\{|f(t)|/\alpha(t) : t \text{ is in } R\}.$$

Let $\mathscr{B}_\alpha\mathscr{C}^+$ and $\mathscr{B}_\alpha\mathscr{C}^-$ be the corresponding semi-axis function spaces with norms $\|\ \|_\alpha^+$ and $\|\ \|_\alpha^-$ respectively. Define $\mathscr{B}_\beta\mathscr{C}$, $\mathscr{B}_\beta\mathscr{C}^+$, $\mathscr{B}_\beta\mathscr{C}^-$, $\|\ \|_\beta$, $\|\ \|_\beta^+$, and $\|\ \|_\beta^-$ analogously. Let S_1 and S_2 be as in § 1, and if i is in $\{1, 2\}$ let Q_i be a projection from R^m to S_i. Let M_1 be the subspace of R^m to which x belongs if and only if $Q_2 v$ is in $\mathscr{B}_\alpha\mathscr{C}^+$, where v is that solution of $(\mathrm{H})^+$ such that $v(0) = x$. Let M_2 be a subspace of R^m such that $R^m = M_1 \oplus M_2$, and let P_1 and P_2 be supplementary projections with ranges M_1 and M_2 respectively. Let Φ be the fundamental matrix for (H), i.e., Φ is that locally absolutely continuous function from R to $\mathscr{A}$ such that

$$\Phi(t) = I + \int_0^t A(s)\Phi(s)ds$$

whenever t is in R. Recall that each value of Φ is invertible. The following theorem is our main result.

THEOREM 1. *The following are equivalent:*

(i) *If f is in $\mathscr{B}_\beta\mathscr{C}^+$ and $Q_1 f = f$ then there is a solution u of* $(\mathrm{NH})^+$ *such that $Q_2 u$ is in $\mathscr{B}_\alpha\mathscr{C}^+$.*

$$(2) \quad (ii) \quad \int_0^\infty \|(I - Q_2)P_2\Phi(s)^{-1}Q_1\|\beta(s)ds < \infty$$

and there is a number K such that

$$(3) \quad \int_0^t \|Q_2\Phi(t)P_1\Phi(s)^{-1}Q_1\|\beta(s)ds + \int_t^\infty \|Q_2\Phi(t)P_2\Phi(s)^{-1}Q_1\|\beta(s)ds \leqq K\alpha(t)$$

whenever t is in R^+.

https://doi.org/10.4153/CJM-1975-044-1 Published online by Cambridge University Press

Note that statement (ii) holds with respect to one norm on $\mathscr{A}$ if and only if it holds with respect to every norm on $\mathscr{A}$. Thus we see that our *a priori* specification of the norm on R^m, and hence on $\mathscr{A}$, is more a matter of convenience than of necessity. In the case $Q_2 = I$, inequality (2) is trivially satisfied and hence does not appear in [**2**, Theorem 1, p. 131]. When auxiliary conditions similar to (2) were given in [**5**, Theorems 1 and 5], it appeared that there was an essential difference between first order cases and higher order cases. Theorem 1 now makes it clear that all of these cases are part of a common phenomenon. This will be explored more fully in § 4.

Proof of Theorem 1. First suppose that (ii) is true. Now (3) says that

$$\int_0^\infty ||Q_2P_2\Phi(s)^{-1}Q_1||\beta(s)ds \leqq K\alpha(0),$$

so (2) and (3) together say

$$\int_0^\infty ||P_2\Phi(s)^{-1}Q_1||\beta(s)ds < \infty.$$

Conclusion (i) is clearly equivalent to showing that if f is *any* member of $\mathscr{B}_\beta\mathscr{C}^+$ then there is a solution u of

(4) $u'(t) = Q_1f(t) + A(t)u(t)$

such that Q_2u is in $\mathscr{B}_\beta\mathscr{C}^+$. Let f be in $\mathscr{B}_\beta\mathscr{C}^+$. Let u from R^+ to R^m be given by

$$u(t) = \int_0^t \Phi(t)P_1\Phi(s)^{-1}Q_1f(s)ds - \int_t^\infty \Phi(t)P_2\Phi(s)^{-1}Q_1f(s)ds.$$

The above remarks assure us that the improper integrals exist and that u is differentiable. Clearly u satisfies (4) on R^+. Also, if t is in R^+,

$$\begin{aligned}|Q_2u(t)| &= \left|\int_0^t Q_2\Phi(t)P_1\Phi(s)^{-1}Q_1f(s)ds - \int_t^\infty Q_2\Phi(t)P_2\Phi(s)^{-1}Q_1f(s)ds\right| \\ &\leqq ||f||_\beta^+ \int_0^t ||Q_2\Phi(t)P_1\Phi(s)^{-1}Q_1||\beta(s)ds \\ &\qquad + ||f||_\beta^+ \int_t^\infty ||Q_2\Phi(t)P_2\Phi(s)^{-1}Q_1||\beta(s)ds \\ &\leqq ||f||_\beta^+K\alpha(t),\end{aligned}$$

so Q_2u is in $\mathscr{B}_\alpha\mathscr{C}^+$, and (i) is proved.

Now suppose that (i) is true. Let $\mathscr{D}$ be the linear space to which u belongs if and only if u is locally absolutely continuous, Q_2u is in $\mathscr{B}_\alpha\mathscr{C}^+$, $u(0)$ is in M_2, and there is an S_1-valued member $\hat{u}$ of $\mathscr{B}_\beta\mathscr{C}^+$ such that $\hat{u}(t) = u'(t) - A(t)u(t)$ for almost all t in R^+. If u is in $\mathscr{D}$, let $||u||_D = ||Q_2u||_\alpha^+ + |u(0)| + ||\hat{u}||_\beta^+$. Suppose that $\{u_n\}_{n=1}^\infty$ is a $\mathscr{D}$-valued sequence, and is a Cauchy sequence with respect to $||\ \ ||_D$. Find that z in M_2 and that S_1-valued member w of

$\mathscr{B}_\beta\mathscr{C}^+$ such that $|u_n(0) - z| \to 0$ and $||\hat{u}_n - w||_\beta{}^+ \to 0$ as $n \to \infty$. Now, if t is in R^+ and n is a positive integer,

$$u_n(t) = \Phi(t)u_n(0) + \int_0^t \Phi(t)\Phi(s)^{-1}\hat{u}_n(s)ds,$$

so there is a continuous function u_0 from R^+ to R^m such that $u_n(t) \to u_0(t)$ uniformly on compact subsets of R^+. Since $\{Q_2u_n\}_{n=1}^\infty$ has pointwise limit Q_2u_0, and is a Cauchy sequence with respect to $|| \quad ||_\alpha{}^+$, we see that Q_2u_0 is in $\mathscr{B}_\alpha\mathscr{C}^+$. Thus, u_0 is in $\mathscr{D}$ and $||u_n - u_0||_D \to 0$ as $n \to \infty$. Clearly now, $\mathscr{D}$ is a Banach space with respect to $|| \quad ||_D$.

Let $\mathscr{E}$ be that closed linear subspace of $\mathscr{B}_\beta\mathscr{C}^+$ consisting of all S_1-valued members of $\mathscr{B}_\beta\mathscr{C}^+$. Let T be the linear transformation from $\mathscr{D}$ to $\mathscr{E}$ given by $Tu = \hat{u}$. Clearly T is continuous, and T is onto by hypothesis. Suppose that u is in $\mathscr{D}$ and $Tu = 0$. Now Q_2u is in $\mathscr{B}_\alpha\mathscr{C}^+$, u satisfies (H)$^+$, and $u(0)$ is in M_2. Thus $u = 0$, so T is one-to-one. Hence [**6**, Theorem 4.1, p. 63], T^{-1} is continuous and there is a number L such that

(5) $\quad ||u||_D \leqq L||u||_\beta{}^+$

whenever u is in $\mathscr{D}$.

If f is in $\mathscr{E}$ let u_f be that solution of (NH)$^+$ such that Q_2u_f is in $\mathscr{B}_\alpha\mathscr{C}^+$ and $P_1u_f(0) = 0$. Now (5) says that

$$||u_f||_D \leqq L||f||_\beta{}^+$$

whenever f is in $\mathscr{E}$. But $|u_f(0)| \leqq ||u_f||_D$ and $||Q_2u_f||_\alpha{}^+ \leqq ||u_f||_D$, so

(6) $\quad |u_f(0)| \leqq L||f||_\beta{}^+$

and

(7) $\quad ||Q_2u_f||_\alpha{}^+ \leqq L||f||_\beta{}^+$

whenever f is in $\mathscr{E}$.

If f is in $\mathscr{E}$ and has compact support; let w_f be given by

$$w_f(t) = \int_0^t \Phi(t)P_1\Phi(s)^{-1}Q_1f(s)ds - \int_t^\infty \Phi(t)P_2\Phi(s)^{-1}Q_1f(s)ds.$$

Routine computations show that $w_f = u_f$. Now the formula given for w_f, the inequalities (6) and (7), and an argument similar to that of [**2**, pp. 133–134], show that

(8) $\quad \int_0^\infty ||P_2\Phi(s)^{-1}Q_1||\beta(s)ds < \infty$

and that (3) holds with $K = mL$. If s is in R then

$$||(I - Q_2)P_2\Phi(s)^{-1}Q_1|| \leqq ||P_2\Phi(s)^{-1}Q_1|| + ||Q_2P_2\Phi(s)^{-1}Q_1||,$$

so (8) and (3) imply (2). This completes the proof.

3. Solution space structure on the whole line. In [**4**, Theorem 1] it was shown that if (NH) has a bounded solution on R whenever f is a bounded continuous function on R, then every solution v of (H) is if the form $v = v_{-1} + v_0 + v_1$, where each of v_{-1}, v_0, and v_1 satisfies (H), v_{-1} is bounded on R^+, v_0 is bounded on R, and v_1 is bounded on R^-. The corresponding result in our present situation is not quite so tidy, but it does give additional understanding of [**4**, Theorem 1].

We take S_1, S_2, Q_1, and Q_2 as before. Let M_0 be the subspace of R^m to which x belongs if and only if $Q_2 v$ is in $\mathscr{B}_\alpha \mathscr{C}$, where v satisfies (H) and $v(0) = x$. Let M_{-1} be determined by the requirement that $M_{-1} \oplus M_0$ is the subspace of initial points for solutions v of (H)$^+$ with $Q_2 v$ in $\mathscr{B}_\alpha \mathscr{C}^+$. Let M_1 be similarly determined by problem (H)$^-$. (Note that M_1 here is *not* as in § 2.) Let M_∞ be determined by the requirement that

$$R^m = M_0 \oplus M_{-1} \oplus M_1 \oplus M_\infty.$$

Let P_0, P_1, P_{-1}, and P_∞ be supplementary projections with ranges M_0, M_1, M_{-1}, and M_∞ respectively.

THEOREM 2. *Suppose that if f is an S_1-valued member of $\mathscr{B}_\beta \mathscr{C}$ then there is a solution u of (NH) such that $Q_2 u$ is in $\mathscr{B}_\alpha \mathscr{C}$. Then*

$$S_1 \subseteq \bigcap_{(t \in R)} \Phi(t)[M_{-1} \oplus M_0 \oplus M_1].$$

Note that if $S_1 = R^m$ then $P_\infty = 0$ and we get an analogue of [**4**, Theorem 1]. Also, the extent to which "$P_\infty = 0$" may fail is determined by the size of S_1 and is independent of the size of S_2.

Indication of proof. It can be shown, using techniques almost identical to those of [**4**, Proof of Theorem 1], that our hypotheses imply that

$$\int_{-\infty}^{\infty} P_\infty \Phi(s)^{-1} Q_1 f(s) ds = 0$$

whenever f is in $\mathscr{B}_\beta \mathscr{C}$. Thus $P_\infty \Phi(t)^{-1} Q_1 = 0$ whenever t is in R. Now if (t, x, y) is in $R \times R^m \times R^m$ and $y = \Phi(t)^{-1} Q_1 x$, then $P_\infty y = 0$, so y is in $M_{-1} \oplus M_0 \oplus M_1$. Thus $Q_1 x$ is in $\Phi(t)[M_{-1} \oplus M_0 \oplus M_1]$, the conclusion follows, and the proof is complete.

4. Higher order equations. Let n be a positive integer and consider the problems $(\text{NH})_n$, $(\text{H})_n$, $(\text{NH})_n^+$, and $(\text{H})_n^+$. If we write $(\text{NH})_n^+$ as a first order problem over R^{mn}, then Theorem 1 includes [**5**, Theorem 5] with Q_1 and Q_2 given by $Q_1(x_0, \ldots, x_{n-1}, x_n) = (0, \ldots, 0, x_n)$ and $Q_2(x_1, x_2, \ldots, x_n) = (x_1, 0, \ldots, 0)$. In this case, (2) implies that if k is an integer in $[1, n-1]$ then the mapping described by $f \to u_f^{(k)}$ is continuous considered as a function from $\mathscr{B}_\beta \mathscr{C}^+$ to $\mathscr{C}[R^+, R^m]$ with compact-open topology. Thus we see another point of view from which (2) can be considered superfluous in the case $n = 1$, $S_1 = S_2 = R^m$.

It does not follow from Theorem 2 that the hypothesis "if f is in $\mathscr{B}_\beta\mathscr{C}$ then there is a solution u of $(\mathrm{NH})_n$ in $\mathscr{B}_\alpha\mathscr{C}$" gives a decomposition of the solution space of $(\mathrm{H})_n$. The comments following Theorem 2, however, indicate that a stronger hypothesis will yield such a decomposition. We state our result without proof.

THEOREM 3. *Suppose that if* $(f_1, \ldots, f_n)$ *is in* $\mathscr{B}_\beta\mathscr{C}^n$ *then there is a subset* $\{z_1, \ldots, z_n\}$ *of* R^m *such that the solution* u *of* (1) *is in* $\mathscr{B}_\alpha\mathscr{C}$. *Then, if* v *satisfies* $(\mathrm{H})_n$, v *is of the form* $v_{-1} + v_0 + v_1$ *where each of* v_{-1}, v_0, *and* v_1 *satisfies* $(\mathrm{H})_n$, v_0 *is in* $\mathscr{B}_\alpha\mathscr{C}$, *the restriction of* v_{-1} *to* R^+ *is in* $\mathscr{B}_\alpha\mathscr{C}^+$, *and the restriction of* v_1 *to* R^- *is in* $\mathscr{B}_\alpha\mathscr{C}^-$.

REFERENCES

1. W. A. Coppel, *On the stability of ordinary differential equations*, J. London Math. Soc. *38* (1963), 255–260.

2. ——— *Stability and asymptotic behavior of differential equations* (D. C. Heath & Co., Boston, 1965).

3. D. L. Lovelady, *Bounded solutions of whole-line differential equations*, Bull. Amer. Math. Soc. *79* (1973), 752–753.

4. ——— *Boundedness and ordinary differential equations on the real line*, J. London Math. Soc. *7* (1973), 597–603.

5. ——— *Relative boundedness and second order differential equations*, Tôhoku Math. J. *25* (1973), 365–374.

6. M. Schechter, *Principles of functional analysis* (Academic Press, New York, 1971).

Florida State University,
Tallahassee, Florida