Forum of Mathematics, Pi (2024), **Vol. 12**:e18 1–49
doi:10.1017/fmp.2024.18

CAMBRIDGE UNIVERSITY PRESS

RESEARCH ARTICLE

Zagier–Hoffman's Conjectures in Positive Characteristic

Bo-Hae Im[1], Hojin Kim[2], Khac Nhuan Le[3], Tuan Ngo Dac[4] and Lan Huong Pham[5]

[1]Dept. of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon, 34141, South Korea; E-mail: bhim@kaist.ac.kr.
[2]Dept. of Mathematical Sciences, KAIST, 291 Daehak-ro, Yuseong-gu, Daejeon, 34141, South Korea;
E-mail: hojinkim@kaist.ac.kr.
[3]Laboratoire de Mathématiques Nicolas Oresme (LMNO), UMR 6139, Normandie Université, Université de Caen Normandie - CNRS, BP 5186, Caen, 14000, France; E-mail: khac-nhuan.le@unicaen.fr.
[4]Laboratoire de Mathématiques Nicolas Oresme (LMNO), UMR 6139, Normandie Université, Université de Caen Normandie - CNRS, BP 5186, Caen, 14000, France; E-mail: tuan.ngodac@unicaen.fr (corresponding author).
[5]Institute of Mathematics, Vietnam Academy of Science and Technology, 18 Hoang Quoc Viet, Hanoi, 10307, Viet Nam;
E-mail: plhuong@math.ac.vn.

Received: 26 September 2023; **Revised:** 10 June 2024; **Accepted:** 6 July 2024

2020 Mathematics Subject Classification: *Primary* – 11M32; *Secondary* – 11G09, 11J93, 11M38, 11R58

Abstract

Multiples zeta values and alternating multiple zeta values in positive characteristic were introduced by Thakur and Harada as analogues of classical multiple zeta values of Euler and Euler sums. In this paper, we determine all linear relations between alternating multiple zeta values and settle the main goals of these theories. As a consequence, we completely establish Zagier–Hoffman's conjectures in positive characteristic formulated by Todd and Thakur which predict the dimension and an explicit basis of the span of multiple zeta values of Thakur of fixed weight.

Contents

1. Introduction

1.1. Classical setting

1.1.1. Multiple zeta values

Multiple zeta values of Euler (MZV's for short) are real positive numbers given by

$$\zeta(n_1, \dots, n_r) = \sum_{0<k_1<\dots<k_r} \frac{1}{k_1^{n_1} \dots k_r^{n_r}}, \quad \text{where } n_i \geq 1, n_r \geq 2.$$

Here, r is called the depth and $w = n_1 + \dots + n_r$ is called the weight of the presentation $\zeta(n_1, \dots, n_r)$. These values cover the special values $\zeta(n)$ for $n \geq 2$ of the Riemann zeta function and have been studied intensively, especially in the last three decades with important and deep connections to different branches of mathematics and physics, for example, arithmetic geometry, knot theory and higher energy physics. We refer the reader to [7, 43] for more details.

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The main goal of this theory is to understand all $\mathbb{Q}$-linear relations between MZV's. Goncharov [19, Conjecture 4.2] conjectures that all $\mathbb{Q}$-linear relations between MZV's can be derived from those between MZV's of the same weight. As the next step, precise conjectures formulated by Zagier [43] and Hoffman [23] predict the dimension and an explicit basis for the $\mathbb{Q}$-vector space $\mathcal{Z}_k$ spanned by MZV's of weight k for $k \in \mathbb{N}$.

Conjecture 1.1 (Zagier's conjecture). *We define a Fibonacci-like sequence of integers d_k as follows. Letting $d_0 = 1, d_1 = 0$ and $d_2 = 1$, we define $d_k = d_{k-2} + d_{k-3}$ for $k \geq 3$. Then for $k \in \mathbb{N}$, we have*

$$\dim_{\mathbb{Q}} \mathcal{Z}_k = d_k.$$

Conjecture 1.2 (Hoffman's conjecture). *The $\mathbb{Q}$-vector space $\mathcal{Z}_k$ is generated by the basis consisting of MZV's of weight k of the form $\zeta(n_1, \dots, n_r)$ with $n_i \in \{2, 3\}$.*

The algebraic part of these conjectures which concerns upper bounds for $\dim_{\mathbb{Q}} \mathcal{Z}_k$ was solved by Terasoma [34], Deligne–Goncharov [17] and Brown [5] using the theory of mixed Tate motives.

Theorem 1.3 (Deligne–Goncharov, Terasoma). *For $k \in \mathbb{N}$, we have $\dim_{\mathbb{Q}} \mathcal{Z}_k \leq d_k$.*

Theorem 1.4 (Brown). *The $\mathbb{Q}$-vector space $\mathcal{Z}_k$ is generated by MZV's of weight k of the form $\zeta(n_1, \dots, n_r)$ with $n_i \in \{2, 3\}$.*

Unfortunately, the transcendental part which concerns lower bounds for $\dim_{\mathbb{Q}} \mathcal{Z}_k$ is completely open. We refer the reader to [7, 16, 43] for more details and more exhaustive references.

1.1.2. Alternating multiple zeta values

There exists a variant of MZV's called the alternating multiple zeta values (AMZV's for short), also known as Euler sums. They are real numbers given by

$$\zeta\begin{pmatrix} \epsilon_1 & \dots & \epsilon_r \\ n_1 & \dots & n_r \end{pmatrix} = \sum_{0<k_1<\dots<k_r} \frac{\epsilon_1^{k_1} \dots \epsilon_r^{k_r}}{k_1^{n_1} \dots k_r^{n_r}},$$

where $\epsilon_i \in \{\pm 1\}$, $n_i \in \mathbb{N}$ and $(n_r, \epsilon_r) \neq (1, 1)$. Similar to MZV's, these values have been studied by Broadhurst, Deligne—Goncharov, Hoffman, Kaneko—Tsumura and many others because of the many connections in different contexts. We refer the reader to [21, 24, 44] for further references.

As before, it is expected that all $\mathbb{Q}$-linear relations between AMZV's can be derived from those between AMZV's of the same weight. In particular, it is natural to ask whether one could formulate conjectures similar to those of Zagier and Hoffman for AMZV's of fixed weight. By the work of Deligne–Goncharov [17], the sharp upper bounds are achieved:

Theorem 1.5 (Deligne–Goncharov). *For $k \in \mathbb{N}$, if we denote by $\mathcal{A}_k$ the $\mathbb{Q}$-vector space spanned by AMZV's of weight k, then $\dim_{\mathbb{Q}} \mathcal{A}_k \leq F_{k+1}$. Here, F_n is the n-th Fibonacci number defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$.*

The fact that the previous upper bounds would be sharp was also explained by Deligne in [15] (see also [17]) using a variant of a conjecture of Grothendieck. In the direction of extending Brown's theorem for AMZV's, there are several sets of generators for $\mathcal{A}_k$ (see, for example, [12, 15]). However, we mention that these generators are only linear combinations of AMZV's.

Finally, we know nothing about nontrivial lower bounds for $\dim_{\mathbb{Q}} \mathcal{A}_k$.

1.2. Function field setting

1.2.1. MZV's of Thakur and analogues of Zagier–Hoffman's conjectures

By analogy between number fields and function fields, based on the pioneering work of Carlitz [8], Thakur [35] defined analogues of multiple zeta values in positive characteristic. We now need to

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introduce some notations. Let $A = \mathbb{F}_q[\theta]$ be the polynomial ring in the variable θ over a finite field $\mathbb{F}_q$ of q elements of characteristic $p > 0$. We denote by A_+ the set of monic polynomials in A. Let $K = \mathbb{F}_q(\theta)$ be the fraction field of A equipped with the rational point ∞. Let K_∞ be the completion of K at ∞ and $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure $\overline{K}$ of K at ∞. We denote by v_∞ the discrete valuation on K corresponding to the place ∞ normalized such that $v_\infty(\theta) = -1$, and by $|\cdot|_\infty = q^{-v_\infty}$ the associated absolute value on K. The unique valuation of $\mathbb{C}_\infty$ which extends v_∞ will still be denoted by v_∞. Finally, we denote by $\overline{\mathbb{F}}_q$ the algebraic closure of $\mathbb{F}_q$ in $\overline{K}$.

Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of positive integers and $\mathbb{Z}^{\geq 0} = \{0, 1, 2, \dots\}$ be the set of nonnegative integers. In [8], Carlitz introduced the Carlitz zeta values $\zeta_A(n)$ for $n \in \mathbb{N}$ given by

$$\zeta_A(n) := \sum_{a \in A_+} \frac{1}{a^n} \in K_\infty$$

which are analogues of classical special zeta values in the function field setting. For any tuple of positive integers $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$, Thakur [35] defined the characteristic p multiple zeta value (MZV for short) $\zeta_A(\mathfrak{s})$ or $\zeta_A(s_1, \dots, s_r)$ by

$$\zeta_A(\mathfrak{s}) := \sum \frac{1}{a_1^{s_1} \dots a_r^{s_r}} \in K_\infty,$$

where the sum runs through the set of tuples $(a_1, \dots, a_r) \in A_+^r$ with $\deg a_1 > \cdots > \deg a_r$. We call r the depth of $\zeta_A(\mathfrak{s})$ and $w(\mathfrak{s}) = s_1 + \cdots + s_r$ the weight of $\zeta_A(\mathfrak{s})$. We note that Carlitz zeta values are exactly depth one MZV's. Thakur [36] showed that all the MZV's do not vanish. We refer the reader to [3, 4, 18, 28, 29, 33, 35, 37, 38, 39, 40, 42] for more details on these objects.

As in the classical setting, the main goal of the theory is to understand all linear relations over K between MZV's. We now state analogues of Zagier–Hoffman's conjectures in positive characteristic formulated by Thakur in [39, §8] and by Todd in [41]. For $w \in \mathbb{N}$, we denote by $\mathcal{Z}_w$ the K-vector space spanned by the MZV's of weight w. We denote by $\mathcal{T}_w$ the set of $\zeta_A(\mathfrak{s})$, where $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ of weight w with $1 \leq s_i \leq q$ for $1 \leq i \leq r-1$ and $s_r < q$.

Conjecture 1.6 (Zagier's conjecture in positive characteristic). *Letting*

$$d(w) = \begin{cases} 1 & \text{if } w = 0, \\ 2^{w-1} & \text{if } 1 \leq w \leq q-1, \\ 2^{w-1} - 1 & \text{if } w = q, \end{cases}$$

we put $d(w) = \sum_{i=1}^q d(w-i)$ for $w > q$. Then for any $w \in \mathbb{N}$, we have

$$\dim_K \mathcal{Z}_w = d(w).$$

Conjecture 1.7 (Hoffman's conjecture in positive characteristic). *A K-basis for $\mathcal{Z}_w$ is given by $\mathcal{T}_w$ consisting of $\zeta_A(s_1, \dots, s_r)$ of weight w with $s_i \leq q$ for $1 \leq i < r$, and $s_r < q$.*

In [31], one of the authors succeeded in proving the algebraic part of these conjectures (see [31, Theorem A]): For all $w \in \mathbb{N}$, we have

$$\dim_K \mathcal{Z}_w \leq d(w).$$

This part is based on shuffle relations for MZV's due to Chen and Thakur and some operations introduced by Todd. For the transcendental part, he used the Anderson–Brownawell–Papanikolas criterion in [2] and proved sharp lower bounds for small weights $w \leq 2q-2$ (see [31, Theorem D]). It has already been noted that it is very difficult to extend his method to general weights (see [31] for more details).

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1.2.2. AMZV's in positive characteristic

Recently, Harada [21] introduced the alternating multiple zeta values in positive characteristic (AMZV's) as follows. Letting $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^n$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_r) \in (\mathbb{F}_q^\times)^n$, we define

$$\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} := \sum \frac{\varepsilon_1^{\deg a_1} \dots \varepsilon_r^{\deg a_r}}{a_1^{s_1} \dots a_r^{s_r}} \in K_\infty,$$

where the sum runs through the set of tuples $(a_1, \dots, a_r) \in A_+^r$ with $\deg a_1 > \dots > \deg a_r$. The numbers r and $w(\mathfrak{s}) = s_1 + \dots + s_r$ are called the depth and the weight of $\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, respectively. We set $\zeta_A \begin{pmatrix} \emptyset \\ \emptyset \end{pmatrix} = 1$. Harada [21] extended basic properties of MZV's to AMZV's, that is, nonvanishing, shuffle relations, period interpretation and linear independence. Again, the main goal of this theory is to determine all linear relations over K between AMZV's. It is natural to ask whether the previous work on analogues of the Zagier–Hoffman conjectures can be extended to this setting. More precisely, if for $w \in \mathbb{N}$ we denote by $\mathcal{AZ}_w$ the K vector space spanned by the AMZV's of weight w, then we would like to determine the dimensions of $\mathcal{AZ}_w$ and show some nice bases of these vector spaces.

1.3. Main results

1.3.1. Statements of the main results

In this manuscript, we present complete answers to all the previous conjectures and problems raised in §1.2.

First, for all w we calculate the dimension of $\mathcal{AZ}_w$ and give an explicit basis in the spirit of Hoffman.

Theorem A. *We define a Fibonacci-like sequence $s(w)$ as follows. We put*

$$s(w) = \begin{cases} (q-1)q^{w-1} & \text{if } 1 \leq w < q, \\ (q-1)(q^{w-1}-1) & \text{if } w = q, \end{cases}$$

and for $w > q$, $s(w) = (q-1)\sum_{i=1}^{q-1} s(w-i) + s(w-q)$. Then for all $w \in \mathbb{N}$,

$$\dim_K \mathcal{AZ}_w = s(w).$$

Further, we can exhibit a Hoffman-like basis of $\mathcal{AZ}_w$.

Second, we give a proof of both Conjectures 1.6 and 1.7 which generalizes the previous work of the fourth author [31].

Theorem B. *For all $w \in \mathbb{N}$, $\mathcal{T}_w$ is a K-basis for $\mathcal{Z}_w$. In particular,*

$$\dim_K \mathcal{Z}_w = d(w).$$

We recall that analogues of Goncharov's conjectures in positive characteristic were proved in [9]. As a consequence, we give a framework for understanding all linear relations over K between MZV's and AMZV's and settle the main goals of these theories.

1.3.2. Ingredients of the proofs

Let us emphasize here that Theorem A is much harder than Theorem B and that it is not enough to work within the setting of AMZV's. On the one hand, although it is straightforward to extend the algebraic part for AMZV's following the same line in [31, §2 and §3], we only obtain a weak version of Brown's

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theorem in this setting. More precisely, we get a set of generators for $\mathcal{AZ}_w$ but it is too large to be a basis of this vector space. For small weights, we find ad hoc arguments to produce a smaller set of generators but it does not work for arbitrary weights (see §5.4). Roughly speaking, in [31, §2 and §3] we have an algorithm which moves forward so that we can express any AMZV as a linear combination of generators. But we lack some precise controls on the coefficients in these expressions so that we cannot go backward and change bases. On the other hand, the transcendental part for AMZV's shares the same difficulties with the case of MZV's as noted above.

In this paper, we use a completely new approach which is based on the study of alternating Carlitz multiple polylogarithms (ACMPL's for short) defined as follows. We put $\ell_0 := 1$ and $\ell_d := \prod_{i=1}^{d}(\theta - \theta^{q^i})$ for all $d \in \mathbb{N}$. For any tuple $\mathfrak{s} = (s_1, \ldots, s_r) \in \mathbb{N}^r$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_r) \in (\mathbb{F}_q^\times)^r$, we introduce the corresponding alternating Carlitz multiple polylogarithm by

$$\operatorname{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} := \sum_{d_1 > \cdots > d_r \geq 0} \frac{\varepsilon_1^{d_1} \ldots \varepsilon_r^{d_r}}{\ell_{d_1}^{s_1} \ldots \ell_{d_r}^{s_r}} \in K_\infty.$$

We also set $\operatorname{Li}\begin{pmatrix}\emptyset\\ \emptyset\end{pmatrix} = 1$.

The key result is to establish a nontrivial connection between AMZV's and ACMPL's which allows us to go back and forth between these objects (see Theorem 5.9). To do this, following [31, §2 and §3] we use stuffle relations to develop an algebraic theory for ACMPL's and obtain a weak version of Brown's theorem, that is, a set of generators for the K-vector space $\mathcal{AL}_w$ spanned by ACMPL's of weight w. We observe that this set of generators is exactly the same as that for AMZV's. Thus, $\mathcal{AL}_w = \mathcal{AZ}_w$, which provides a dictionary between AMZV's and ACMPL's.

We then determine all K-linear relations between ACMPL's (see Theorem 4.6). The proof we give here, while using similar tools as in [31], differs in some crucial points and requires three new ingredients.

The first new ingredient is the construction of an appropriate Hoffman-like basis $\mathcal{AS}_w$ of $\mathcal{AL}_w$. In fact, our transcendental method dictates that we must find a complete system of bases $\mathcal{AS}_w$ of $\mathcal{AL}_w$ indexed by weights w with strong constraints as given in Theorem 3.4. The failure to find such a system of bases is the main obstacle to generalizing [31, Theorem D] (see §5.1 and [31, Remark 6.3] for more details).

The second new ingredient is formulating and proving (a strong version of) Brown's theorem for AMCPLs (see Theorem 2.11). As mentioned before, the method in [31] only gives a weak version of Brown's theorem for ACMPL's as the set of generators is not a basis. Roughly speaking, given any ACMPL's we can express it as a linear combination of generators. The fact that stuffle relations for ACMPL's are 'simpler' than shuffle relations for AMZV's gives more precise information about the coefficients of these expressions. Consequently, we show that a certain transition matrix is invertible and obtain Brown's theorem for ACMPL's. This completes the algebraic part for ACMPL's.

The last new ingredient is proving the transcendental part for ACMPL's in full generality, that is, the ACMPL's in $\mathcal{AS}_w$ are linearly independent over K (see Theorem 4.4). We emphasize that we do need the full strength of the algebraic part to prove the transcendental part. The proof follows the same line in [31, §4 and §5] which is formulated in a more general setting in §3. First, we have to consider not only linear relations between ACMPL's in $\mathcal{AS}_w$ but also those between ACMPL's in $\mathcal{AS}_w$ and the suitable power $\widetilde{\pi}^w$ of the Carlitz period $\widetilde{\pi}$. Second, starting from such a nontrivial relation we apply the Anderson–Brownawell–Papanikolas criterion in [2] and reduce to solve a system of σ-linear equations. While in [31, §4 and §5] this system does not have a nontrivial solution which allows us to conclude, our system has a unique solution for even w (i.e., $q-1$ divides w). This means that for such w up to a scalar there is a unique linear relation between ACMPL's in $\mathcal{AS}_w$ and $\widetilde{\pi}^w$. The last step consists of showing that in this unique relation, the coefficient of $\widetilde{\pi}^w$ is nonzero. Unexpectedly, this is a consequence of Brown's theorem for AMCPLs mentioned above.

1.3.3. Plan of the paper

We will briefly explain the organization of the manuscript.

- In §2, we recall the definition and basic properties of ACMPL's. We then develop an algebraic theory for these objects and obtain weak and strong Brown's theorems (see Proposition 2.10 and Theorem 2.11).
- In §3, we generalize some transcendental results in [31] and give statements in a more general setting (see Theorem 3.4).
- In §4, we prove transcendental results for ACMPL's and completely determine all linear relations between ACMPL's (see Theorems 4.4 and 4.6).
- Finally, in §5 we present two applications and prove the main results, that is, Theorems A and B. The first application is to prove the above connection between ACMPL's and AMZV's and then to determine all linear relations between AMZV's in positive characteristic (see §5.1). The second application is a proof of Zagier–Hoffman's conjectures in positive characteristic which generalizes the main results of [31] (see §5.3).

1.4. Remark

When our work was released in arXiv:2205.07165, Chieh-Yu Chang informed us that Chen, Mishiba and he were working towards a proof of Theorem B (e.g., the MZV version) by using a similar method, and their paper [10] is now available at arXiv:2205.09929.

2. Weak and strong Brown's theorems for ACMPL's

In this section, we first extend the work of [31] and develop an algebraic theory for ACMPL's. Then we prove a weak version of Brown's theorem for ACMPL's (see Theorem 2.10) which gives a set of generators for the K-vector space spanned by ACMPL's of weight w. The techniques of Sections 2.1–2.3 are similar to those of [31], and the reader may wish to skip the details.

Contrary to what happens in [31], it turns out that the previous set of generators is too large to be a basis. Consequently, in §2.4 we introduce another set of generators and prove a strong version of Brown's theorem for ACMPL's (see Theorem 2.11).

2.1. Analogues of power sums

2.1.1.

We recall and introduce some notation in [31]. A tuple $\mathfrak{s}$ is a sequence of the form $\mathfrak{s} = (s_1, \dots, s_n) \in \mathbb{N}^n$. We call $\operatorname{depth}(\mathfrak{s}) = n$ the depth and $w(\mathfrak{s}) = s_1 + \dots + s_n$ the weight of $\mathfrak{s}$. If $\mathfrak{s}$ is nonempty, we put $\mathfrak{s}_- := (s_2, \dots, s_n)$.

Let $\mathfrak{s}$ and $\mathfrak{t}$ be two tuples of positive integers. We set $s_i = 0$ (resp. $t_i = 0$) for all $i > \operatorname{depth}(\mathfrak{s})$ (resp. $i > \operatorname{depth}(\mathfrak{t})$). We say that $\mathfrak{s} \leq \mathfrak{t}$ if $s_1 + \dots + s_i \leq t_1 + \dots + t_i$ for all $i \in \mathbb{N}$, and $w(\mathfrak{s}) = w(\mathfrak{t})$. This defines a partial order on tuples of positive integers.

For $i \in \mathbb{N}$, we define $T_i(\mathfrak{s})$ to be the tuple $(s_1 + \dots + s_i, s_{i+1}, \dots, s_n)$. Further, for $i \in \mathbb{N}$, if $T_i(\mathfrak{s}) \leq T_i(\mathfrak{t})$, then $T_k(\mathfrak{s}) \leq T_k(\mathfrak{t})$ for all $k \geq i$.

Let $\mathfrak{s} = (s_1, \dots, s_n) \in \mathbb{N}^n$ be a tuple of positive integers. We denote by $0 \leq i \leq n$ the largest integer such that $s_j \leq q$ for all $1 \leq j \leq i$ and define the initial tuple $\operatorname{Init}(\mathfrak{s})$ of $\mathfrak{s}$ to be the tuple

$$\operatorname{Init}(\mathfrak{s}) := (s_1, \dots, s_i).$$

In particular, if $s_1 > q$, then $i = 0$ and $\operatorname{Init}(\mathfrak{s})$ is the empty tuple.

For two different tuples $\mathfrak{s}$ and $\mathfrak{t}$, we consider the lexicographic order for initial tuples and write $\operatorname{Init}(\mathfrak{t}) \preceq \operatorname{Init}(\mathfrak{s})$ (resp. $\operatorname{Init}(\mathfrak{t}) \prec \operatorname{Init}(\mathfrak{s})$, $\operatorname{Init}(\mathfrak{t}) \succeq \operatorname{Init}(\mathfrak{s})$ and $\operatorname{Init}(\mathfrak{t}) \succ \operatorname{Init}(\mathfrak{s})$).

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2.1.2.
Letting $\mathfrak{s} = (s_1, \dots, s_n) \in \mathbb{N}^n$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n) \in (\mathbb{F}_q^\times)^n$, we set $\mathfrak{s}_- := (s_2, \dots, s_n)$ and $\boldsymbol{\varepsilon}_- := (\varepsilon_2, \dots, \varepsilon_n)$. By definition, an array $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ is an array of the form

$$\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \cdots & \varepsilon_n \\ s_1 & \cdots & s_n \end{pmatrix}.$$

We call $\operatorname{depth}(\mathfrak{s}) = n$ the depth, $w(\mathfrak{s}) = s_1 + \cdots + s_n$ the weight and $\chi(\boldsymbol{\varepsilon}) = \varepsilon_1 \dots \varepsilon_n$ the character of $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$.

We say that $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} \leq \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$ if the following conditions are satisfied:

1. $\chi(\boldsymbol{\varepsilon}) = \chi(\boldsymbol{\epsilon})$,
2. $w(\mathfrak{s}) = w(\mathfrak{t})$,
3. $s_1 + \cdots + s_i \leq t_1 + \cdots + t_i$ for all $i \in \mathbb{N}$.

We note that this only defines a preorder on arrays.

Remark 2.1. We claim that if $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} \leq \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$, then $\operatorname{depth}(\mathfrak{s}) \geq \operatorname{depth}(\mathfrak{t})$. In fact, assume that $\operatorname{depth}(\mathfrak{s}) < \operatorname{depth}(\mathfrak{t})$. Thus,

$$w(\mathfrak{s}) = s_1 + \cdots + s_{\operatorname{depth}(\mathfrak{s})} \leq t_1 + \cdots + t_{\operatorname{depth}(\mathfrak{s})} < t_1 + \cdots + t_{\operatorname{depth}(\mathfrak{t})} = w(\mathfrak{t}),$$

which contradicts the condition $w(\mathfrak{s}) = w(\mathfrak{t})$.

2.1.3.
We recall the power sums and MZV's studied by Thakur [38]. For $d \in \mathbb{Z}$ and for $\mathfrak{s} = (s_1, \dots, s_n) \in \mathbb{N}^n$, we introduce

$$S_d(\mathfrak{s}) := \sum_{\substack{a_1, \dots, a_n \in A_+ \\ d = \deg a_1 > \cdots > \deg a_n \geq 0}} \frac{1}{a_1^{s_1} \dots a_n^{s_n}} \in K$$

and

$$S_{<d}(\mathfrak{s}) := \sum_{\substack{a_1, \dots, a_n \in A_+ \\ d > \deg a_1 > \cdots > \deg a_n \geq 0}} \frac{1}{a_1^{s_1} \dots a_n^{s_n}} \in K.$$

We define the multiple zeta value (MZV) by

$$\zeta_A(\mathfrak{s}) := \sum_{d \geq 0} S_d(\mathfrak{s}) = \sum_{d \geq 0} \sum_{\substack{a_1, \dots, a_n \in A_+ \\ d = \deg a_1 > \cdots > \deg a_n \geq 0}} \frac{1}{a_1^{s_1} \dots a_n^{s_n}} \in K_\infty.$$

We put $\zeta_A(\emptyset) = 1$. We call $\operatorname{depth}(\mathfrak{s}) = n$ the depth and $w(\mathfrak{s}) = s_1 + \cdots + s_n$ the weight of $\zeta_A(\mathfrak{s})$.

We also recall that $\ell_0 := 1$ and $\ell_d := \prod_{i=1}^d (\theta - \theta^{q^i})$ for all $d \in \mathbb{N}$. Letting $\mathfrak{s} = (s_1, \dots, s_n) \in \mathbb{N}^n$, for $d \in \mathbb{Z}$, we define analogues of power sums by

$$\operatorname{Si}_d(\mathfrak{s}) := \sum_{d = d_1 > \cdots > d_n \geq 0} \frac{1}{\ell_{d_1}^{s_1} \dots \ell_{d_n}^{s_n}} \in K,$$

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and

$$\mathrm{Si}_{<d}(\mathfrak{s}) := \sum_{d>d_1>\cdots>d_n\geq 0} \frac{1}{\ell_{d_1}^{s_1}\dots\ell_{d_n}^{s_n}} \in K.$$

We introduce the Carlitz multiple polylogarithm (CMPL for short) by

$$\mathrm{Li}(\mathfrak{s}) := \sum_{d\geq 0} \mathrm{Si}_d(\mathfrak{s}) = \sum_{d\geq 0} \sum_{d=d_1>\cdots>d_n\geq 0} \frac{1}{\ell_{d_1}^{s_1}\dots\ell_{d_n}^{s_n}} \in K_\infty.$$

We set $\mathrm{Li}(\emptyset) = 1$. We call $\mathrm{depth}(\mathfrak{s}) = n$ the depth and $w(\mathfrak{s}) = s_1 + \cdots + s_n$ the weight of $\mathrm{Li}(\mathfrak{s})$.

2.1.4.
Let $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \begin{pmatrix}\varepsilon_1 & \dots & \varepsilon_n\\ s_1 & \dots & s_n\end{pmatrix}$ be an array. For $d \in \mathbb{Z}$, we define

$$S_d\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} := \sum_{\substack{a_1,\dots,a_n\in A_+\\ d=\deg a_1>\cdots>\deg a_n\geq 0}} \frac{\varepsilon_1^{\deg a_1}\dots\varepsilon_n^{\deg a_n}}{a_1^{s_1}\dots a_n^{s_n}} \in K$$

and

$$S_{<d}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} := \sum_{\substack{a_1,\dots,a_n\in A_+\\ d>\deg a_1>\cdots>\deg a_n\geq 0}} \frac{\varepsilon_1^{\deg a_1}\dots\varepsilon_n^{\deg a_n}}{a_1^{s_1}\dots a_n^{s_n}} \in K.$$

We also introduce

$$\mathrm{Si}_d\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} := \sum_{d=d_1>\cdots>d_n\geq 0} \frac{\varepsilon_1^{d_1}\dots\varepsilon_n^{d_n}}{\ell_{d_1}^{s_1}\dots\ell_{d_n}^{s_n}} \in K,$$

and

$$\mathrm{Si}_{<d}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} := \sum_{d>d_1>\cdots>d_n\geq 0} \frac{\varepsilon_1^{d_1}\dots\varepsilon_n^{d_n}}{\ell_{d_1}^{s_1}\dots\ell_{d_n}^{s_n}} \in K.$$

One verifies easily the following formulas:

$$\mathrm{Si}_d\begin{pmatrix}\varepsilon\\ s\end{pmatrix} = \varepsilon^d\,\mathrm{Si}_d(s), \tag{2.1}$$

$$\mathrm{Si}_d\begin{pmatrix}1 & \dots & 1\\ s_1 & \dots & s_n\end{pmatrix} = \mathrm{Si}_d(s_1,\dots,s_n), \tag{2.2}$$

$$\mathrm{Si}_{<d}\begin{pmatrix}1 & \dots & 1\\ s_1 & \dots & s_n\end{pmatrix} = \mathrm{Si}_{<d}(s_1,\dots,s_n), \tag{2.3}$$

$$\mathrm{Si}_d\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \mathrm{Si}_d\begin{pmatrix}\varepsilon_1\\ s_1\end{pmatrix}\mathrm{Si}_{<d}\begin{pmatrix}\boldsymbol{\varepsilon}_-\\ \mathfrak{s}_-\end{pmatrix}. \tag{2.4}$$

Then we define the alternating Carlitz multiple polylogarithm (ACMPL for short) by

$$\mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} := \sum_{d\geq 0} \mathrm{Si}_d\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \sum_{d_1>\cdots>d_n\geq 0} \frac{\varepsilon_1^{d_1}\dots\varepsilon_n^{d_n}}{\ell_{d_1}^{s_1}\dots\ell_{d_n}^{s_n}} \in K_\infty.$$

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Recall that $\mathrm{Li}\begin{pmatrix}\emptyset\\ \emptyset\end{pmatrix} = 1$. We call $\mathrm{depth}(\mathfrak{s}) = n$ the depth, $w(\mathfrak{s}) = s_1 + \cdots + s_n$ the weight and $\chi(\boldsymbol{\varepsilon}) = \varepsilon_1 \ldots \varepsilon_n$ the character of $\mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$.

Lemma 2.2. *For all* $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$ *as above such that* $s_i \leq q$ *for all* i*, we have*

$$S_d\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \mathrm{Si}_d\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} \quad \text{for all } d \in \mathbb{Z}.$$

Therefore,

$$\zeta_A\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}.$$

Proof. We denote by $\mathcal{J}$ the set of all arrays $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \begin{pmatrix}\varepsilon_1 & \ldots & \varepsilon_n\\ s_1 & \ldots & s_n\end{pmatrix}$ for some n such that $s_1, \ldots, s_n \leq q$.

The second statement follows at once from the first statement. We prove the first statement by induction on $\mathrm{depth}(\mathfrak{s})$. For $\mathrm{depth}(\mathfrak{s}) = 1$, we let $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \begin{pmatrix}\varepsilon\\ s\end{pmatrix}$ with $s \leq q$. It follows from special cases of power sums in [37, §3.3] that for all $d \in \mathbb{Z}$, $S_d\begin{pmatrix}\varepsilon\\ s\end{pmatrix} = \frac{\varepsilon^d}{\ell_d^s} = \mathrm{Si}_d\begin{pmatrix}\varepsilon\\ s\end{pmatrix}$. Suppose that the first statement holds for all arrays $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} \in \mathcal{J}$ with $\mathrm{depth}(\mathfrak{s}) = n - 1$ and for all $d \in \mathbb{Z}$. Let $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \begin{pmatrix}\varepsilon_1 & \ldots & \varepsilon_n\\ s_1 & \ldots & s_n\end{pmatrix}$ be an element of $\mathcal{J}$. Note that if $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} \in \mathcal{J}$, then $\begin{pmatrix}\boldsymbol{\varepsilon}_-\\ \mathfrak{s}_-\end{pmatrix} \in \mathcal{J}$. It follows from induction hypothesis and the fact $s_1 \leq q$ that for all $d \in \mathbb{Z}$

$$S_d\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = S_d\begin{pmatrix}\varepsilon_1\\ s_1\end{pmatrix} S_{<d}\begin{pmatrix}\boldsymbol{\varepsilon}_-\\ \mathfrak{s}_-\end{pmatrix} = \mathrm{Si}_d\begin{pmatrix}\varepsilon_1\\ s_1\end{pmatrix} \mathrm{Si}_{<d}\begin{pmatrix}\boldsymbol{\varepsilon}_-\\ \mathfrak{s}_-\end{pmatrix} = \mathrm{Si}_d\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}.$$

This proves the lemma. □

2.1.5.

Let $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}, \begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}$ be two arrays. We recall $s_i = 0$ and $\varepsilon_i = 1$ for all $i > \mathrm{depth}(\mathfrak{s})$; $t_i = 0$ and $\epsilon_i = 1$ for all $i > \mathrm{depth}(\mathfrak{t})$. We define the following operation

$$\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} + \begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix} := \begin{pmatrix}\boldsymbol{\varepsilon\epsilon}\\ \mathfrak{s} + \mathfrak{t}\end{pmatrix},$$

where $\boldsymbol{\varepsilon\epsilon}$ and $\mathfrak{s} + \mathfrak{t}$ are defined by component multiplication and component addition, respectively.

We now consider some formulas related to analogues of power sums. It is easily seen that

$$\mathrm{Si}_d\begin{pmatrix}\varepsilon\\ s\end{pmatrix} \mathrm{Si}_d\begin{pmatrix}\epsilon\\ t\end{pmatrix} = \mathrm{Si}_d\begin{pmatrix}\varepsilon\epsilon\\ s + t\end{pmatrix}, \tag{2.5}$$

hence, for $\mathfrak{t} = (t_1, \ldots, t_n)$,

$$\mathrm{Si}_d\begin{pmatrix}\varepsilon\\ s\end{pmatrix} \mathrm{Si}_d\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix} = \mathrm{Si}_d\begin{pmatrix}\varepsilon\epsilon_1 & \boldsymbol{\epsilon}_-\\ s + t_1 & \mathfrak{t}_-\end{pmatrix}. \tag{2.6}$$

More generally, we deduce the following proposition which will be used frequently later.

Proposition 2.3. *Let* $\binom{\boldsymbol{\varepsilon}}{\mathfrak{s}}, \binom{\boldsymbol{\epsilon}}{\mathfrak{t}}$ *be two arrays. Then we have the following:*

1. *There exist* $f_i \in \mathbb{F}_q$ *and arrays* $\binom{\boldsymbol{\mu}_i}{\mathfrak{u}_i}$ *with* $\binom{\boldsymbol{\mu}_i}{\mathfrak{u}_i} \le \binom{\boldsymbol{\varepsilon}}{\mathfrak{s}} + \binom{\boldsymbol{\epsilon}}{\mathfrak{t}}$ *and* $\mathrm{depth}(\mathfrak{u}_i) \le \mathrm{depth}(\mathfrak{s}) + \mathrm{depth}(\mathfrak{t})$ *for all i such that*
$$\mathrm{Si}_d\binom{\boldsymbol{\varepsilon}}{\mathfrak{s}}\mathrm{Si}_d\binom{\boldsymbol{\epsilon}}{\mathfrak{t}} = \sum_i f_i\,\mathrm{Si}_d\binom{\boldsymbol{\mu}_i}{\mathfrak{u}_i} \quad \textit{for all } d \in \mathbb{Z}.$$
2. *There exist* $f_i' \in \mathbb{F}_q$ *and arrays* $\binom{\boldsymbol{\mu}_i'}{\mathfrak{u}_i'}$ *with* $\binom{\boldsymbol{\mu}_i'}{\mathfrak{u}_i'} \le \binom{\boldsymbol{\varepsilon}}{\mathfrak{s}} + \binom{\boldsymbol{\epsilon}}{\mathfrak{t}}$ *and* $\mathrm{depth}(\mathfrak{u}_i') \le \mathrm{depth}(\mathfrak{s}) + \mathrm{depth}(\mathfrak{t})$ *for all i such that*
$$\mathrm{Si}_{<d}\binom{\boldsymbol{\varepsilon}}{\mathfrak{s}}\mathrm{Si}_{<d}\binom{\boldsymbol{\epsilon}}{\mathfrak{t}} = \sum_i f_i'\,\mathrm{Si}_{<d}\binom{\boldsymbol{\mu}_i'}{\mathfrak{u}_i'} \quad \textit{for all } d \in \mathbb{Z}.$$
3. *There exist* $f_i'' \in \mathbb{F}_q$ *and arrays* $\binom{\boldsymbol{\mu}_i''}{\mathfrak{u}_i''}$ *with* $\binom{\boldsymbol{\mu}_i''}{\mathfrak{u}_i''} \le \binom{\boldsymbol{\varepsilon}}{\mathfrak{s}} + \binom{\boldsymbol{\epsilon}}{\mathfrak{t}}$ *and* $\mathrm{depth}(\mathfrak{u}_i'') \le \mathrm{depth}(\mathfrak{s}) + \mathrm{depth}(\mathfrak{t})$ *for all i such that*
$$\mathrm{Si}_d\binom{\boldsymbol{\varepsilon}}{\mathfrak{s}}\mathrm{Si}_{<d}\binom{\boldsymbol{\epsilon}}{\mathfrak{t}} = \sum_i f_i''\,\mathrm{Si}_d\binom{\boldsymbol{\mu}_i''}{\mathfrak{u}_i''} \quad \textit{for all } d \in \mathbb{Z}.$$

Proof. The proof follows the same line as in [31, Proposition 2.1]. We omit the proof here and refer the reader to [25, Proposition 1.3] for more details. □

We denote by $\mathcal{AL}$ (resp. $\mathcal{L}$) the K-vector space generated by the ACMPL's (resp. by the CMPL's) and by $\mathcal{AL}_w$ (resp. $\mathcal{L}_w$) the K-vector space generated by the ACMPL's of weight w (resp. by the CMPL's of weight w). It follows from Proposition 2.3 that $\mathcal{AL}$ is a K-algebra. By considering only arrays with trivial characters, Proposition 2.3 implies that $\mathcal{L}$ is also a K-algebra.

2.2. Operators $\mathcal{B}^*$, $\mathcal{C}$ and $\mathcal{BC}$

In this section, we extend operators $\mathcal{B}^*$ and $\mathcal{C}$ of Todd [41] and the operator $\mathcal{BC}$ of Ngo Dac [31] in the case of ACMPL's.

Definition 2.4. A binary relation is a K-linear combination of the form
$$\sum_i a_i\,\mathrm{Si}_d\binom{\boldsymbol{\varepsilon}_i}{\mathfrak{s}_i} + \sum_i b_i\,\mathrm{Si}_{d+1}\binom{\boldsymbol{\epsilon}_i}{\mathfrak{t}_i} = 0 \quad \text{for all } d \in \mathbb{Z},$$
where $a_i, b_i \in K$ and $\binom{\boldsymbol{\varepsilon}_i}{\mathfrak{s}_i}, \binom{\boldsymbol{\epsilon}_i}{\mathfrak{t}_i}$ are arrays of the same weight.

A binary relation is called a fixed relation if $b_i = 0$ for all i.

We denote by $\mathfrak{BR}_w$ the set of all binary relations of weight w. One verifies at once that $\mathfrak{BR}_w$ is a K-vector space. It follows from the fundamental relation in [37, §3.4.6] and Lemma 2.2, an important example of binary relations
$$R_\varepsilon: \quad \mathrm{Si}_d\binom{\varepsilon}{q} + \varepsilon^{-1}D_1\,\mathrm{Si}_{d+1}\begin{pmatrix}\varepsilon & 1\\ 1 & q-1\end{pmatrix} = 0,$$
where $D_1 = \theta^q - \theta \in A$.

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For later definitions, let $R \in \mathfrak{BR}_w$ be a binary relation of the form

$$R(d): \quad \sum_i a_i \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_{d+1} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} = 0, \tag{2.7}$$

where $a_i, b_i \in K$ and $\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix}, \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix}$ are arrays of the same weight. We now define some operators on K-vector spaces of binary relations.

2.2.1. Operators $\mathcal{B}^*$

Let $\begin{pmatrix} \sigma \\ v \end{pmatrix}$ be an array. We define an operator

$$\mathcal{B}^*_{\sigma,v} : \mathfrak{BR}_w \longrightarrow \mathfrak{BR}_{w+v}$$

as follows: For each $R \in \mathfrak{BR}_w$ given as in Equation (2.7), the image $\mathcal{B}^*_{\sigma,v}(R) = \operatorname{Si}_d \begin{pmatrix} \sigma \\ v \end{pmatrix} \sum_{j<d} R(j)$ is a fixed relation of the form

$$\begin{aligned} 0 &= \operatorname{Si}_d \begin{pmatrix} \sigma \\ v \end{pmatrix} \left(\sum_i a_i \operatorname{Si}_{<d} \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_{<d+1} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} \right) \\ &= \sum_i a_i \operatorname{Si}_d \begin{pmatrix} \sigma \\ v \end{pmatrix} \operatorname{Si}_{<d} \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \sigma \\ v \end{pmatrix} \operatorname{Si}_{<d} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \sigma \\ v \end{pmatrix} \operatorname{Si}_d \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} \\ &= \sum_i a_i \operatorname{Si}_d \begin{pmatrix} \sigma & \boldsymbol{\varepsilon}_i \\ v & \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \sigma & \boldsymbol{\epsilon}_i \\ v & \mathfrak{t}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \sigma \epsilon_{i1} & \boldsymbol{\epsilon}_{i-} \\ v + t_{i1} & \mathfrak{t}_{i-} \end{pmatrix}. \end{aligned}$$

The last equality follows from Equation (2.6).

Let $\begin{pmatrix} \Sigma \\ V \end{pmatrix} = \begin{pmatrix} \sigma_1 & \dots & \sigma_n \\ v_1 & \dots & v_n \end{pmatrix}$ be an array. We define an operator $\mathcal{B}^*_{\Sigma,V}(R)$ by

$$\mathcal{B}^*_{\Sigma,V}(R) := \mathcal{B}^*_{\sigma_1,v_1} \circ \cdots \circ \mathcal{B}^*_{\sigma_n,v_n}(R).$$

Lemma 2.5. *Let* $\begin{pmatrix} \Sigma \\ V \end{pmatrix} = \begin{pmatrix} \sigma_1 & \dots & \sigma_n \\ v_1 & \dots & v_n \end{pmatrix}$ *be an array. Then* $\mathcal{B}^*_{\Sigma,V}(R)$ *is of the form*

$$\sum_i a_i \operatorname{Si}_d \begin{pmatrix} \Sigma & \boldsymbol{\varepsilon}_i \\ V & \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \Sigma & \boldsymbol{\epsilon}_i \\ V & \mathfrak{t}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \sigma_1 & \dots & \sigma_{n-1} & \sigma_n \epsilon_{i1} & \boldsymbol{\epsilon}_{i-} \\ v_1 & \dots & v_{n-1} & v_n + t_{i1} & \mathfrak{t}_{i-} \end{pmatrix} = 0.$$

Proof. From the definition and Equation (2.6), we have $\mathcal{B}^*_{\sigma_n,v_n}(R)$ is of the form

$$\sum_i a_i \operatorname{Si}_d \begin{pmatrix} \sigma_n & \boldsymbol{\varepsilon}_i \\ v_n & \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \sigma_n & \boldsymbol{\epsilon}_i \\ v_n & \mathfrak{t}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \sigma_n \epsilon_{i1} & \boldsymbol{\epsilon}_{i-} \\ v_n + t_{i1} & \mathfrak{t}_{i-} \end{pmatrix} = 0.$$

Apply the operator $\mathcal{B}^*_{\sigma_1,v_1} \circ \cdots \circ \mathcal{B}^*_{\sigma_{n-1},v_{n-1}}$ to $\mathcal{B}^*_{\sigma_n,v_n}(R)$, the result then follows from the definition. □

2.2.2. Operators $\mathcal{C}$

Let $\begin{pmatrix} \Sigma \\ V \end{pmatrix}$ be an array of weight v. We define an operator

$$\mathcal{C}_{\Sigma,V}(R) : \mathfrak{BR}_w \longrightarrow \mathfrak{BR}_{w+v}$$

as follows: For each $R \in \mathfrak{BR}_w$ given as in Equation (2.7), the image $\mathcal{C}_{\Sigma,V}(R) = R(d)\,\mathrm{Si}_{<d+1}\begin{pmatrix}\Sigma\\ V\end{pmatrix}$ is a binary relation of the form

$$\begin{aligned}
0 &= \left(\sum_i a_i \,\mathrm{Si}_d\begin{pmatrix}\boldsymbol{\varepsilon}_i\\ \mathfrak{s}_i\end{pmatrix} + \sum_i b_i \,\mathrm{Si}_{d+1}\begin{pmatrix}\boldsymbol{\epsilon}_i\\ \mathfrak{t}_i\end{pmatrix}\right)\mathrm{Si}_{<d+1}\begin{pmatrix}\Sigma\\ V\end{pmatrix}\\
&= \sum_i a_i \,\mathrm{Si}_d\begin{pmatrix}\boldsymbol{\varepsilon}_i\\ \mathfrak{s}_i\end{pmatrix}\mathrm{Si}_d\begin{pmatrix}\Sigma\\ V\end{pmatrix} + \sum_i a_i \,\mathrm{Si}_d\begin{pmatrix}\boldsymbol{\varepsilon}_i\\ \mathfrak{s}_i\end{pmatrix}\mathrm{Si}_{<d}\begin{pmatrix}\Sigma\\ V\end{pmatrix} + \sum_i b_i \,\mathrm{Si}_{d+1}\begin{pmatrix}\boldsymbol{\epsilon}_i\\ \mathfrak{t}_i\end{pmatrix}\mathrm{Si}_{<d+1}\begin{pmatrix}\Sigma\\ V\end{pmatrix}\\
&= \sum_i c_i \,\mathrm{Si}_d\begin{pmatrix}\boldsymbol{\mu}_i\\ \mathfrak{u}_i\end{pmatrix} + \sum_i c_i' \,\mathrm{Si}_{d+1}\begin{pmatrix}\boldsymbol{\mu}_i'\\ \mathfrak{u}_i'\end{pmatrix}.
\end{aligned}$$

The last equality follows from Proposition 2.3.

In particular, the following proposition gives the form of $\mathcal{C}_{\Sigma,V}(R_\varepsilon)$.

Proposition 2.6. *Let $\begin{pmatrix}\Sigma\\ V\end{pmatrix}$ be an array with $V = (v_1, V_-)$ and $\Sigma = (\sigma_1, \Sigma_-)$. Then $\mathcal{C}_{\Sigma,V}(R_\varepsilon)$ is of the form*

$$\mathrm{Si}_d\begin{pmatrix}\varepsilon\sigma_1 & \Sigma_-\\ q+v_1 & V_-\end{pmatrix} + \mathrm{Si}_d\begin{pmatrix}\varepsilon & \Sigma\\ q & V\end{pmatrix} + \sum_i b_i \,\mathrm{Si}_{d+1}\begin{pmatrix}\varepsilon & \boldsymbol{\epsilon}_i\\ 1 & \mathfrak{t}_i\end{pmatrix} = 0,$$

where $b_i \in A$ are divisible by D_1 and $\begin{pmatrix}\boldsymbol{\epsilon}_i\\ \mathfrak{t}_i\end{pmatrix}$ are arrays satisfying $\begin{pmatrix}\boldsymbol{\epsilon}_i\\ \mathfrak{t}_i\end{pmatrix} \le \begin{pmatrix}1\\ q-1\end{pmatrix} + \begin{pmatrix}\Sigma\\ V\end{pmatrix}$ for all i.

Proof. We see that $\mathcal{C}_{\Sigma,V}(R_\varepsilon)$ is of the form

$$\mathrm{Si}_d\begin{pmatrix}\varepsilon\\ q\end{pmatrix}\mathrm{Si}_d\begin{pmatrix}\Sigma\\ V\end{pmatrix} + \mathrm{Si}_d\begin{pmatrix}\varepsilon\\ q\end{pmatrix}\mathrm{Si}_{<d}\begin{pmatrix}\Sigma\\ V\end{pmatrix} + \varepsilon^{-1}D_1\,\mathrm{Si}_{d+1}\begin{pmatrix}\varepsilon & 1\\ 1 & q-1\end{pmatrix}\mathrm{Si}_{<d+1}\begin{pmatrix}\Sigma\\ V\end{pmatrix} = 0.$$

It follows from Equation (2.6) and Proposition 2.3 that

$$\begin{aligned}
\mathrm{Si}_d\begin{pmatrix}\varepsilon\\ q\end{pmatrix}\mathrm{Si}_d\begin{pmatrix}\Sigma\\ V\end{pmatrix} + \mathrm{Si}_d\begin{pmatrix}\varepsilon\\ q\end{pmatrix}\mathrm{Si}_{<d}\begin{pmatrix}\Sigma\\ V\end{pmatrix} &= \mathrm{Si}_d\begin{pmatrix}\varepsilon\sigma_1 & \Sigma_-\\ q+v_1 & V_-\end{pmatrix} + \mathrm{Si}_d\begin{pmatrix}\varepsilon & \Sigma\\ q & V\end{pmatrix},\\
\varepsilon^{-1}D_1\,\mathrm{Si}_{d+1}\begin{pmatrix}\varepsilon & 1\\ 1 & q-1\end{pmatrix}\mathrm{Si}_{<d+1}\begin{pmatrix}\Sigma\\ V\end{pmatrix} &= \sum_i b_i \,\mathrm{Si}_{d+1}\begin{pmatrix}\varepsilon & \boldsymbol{\epsilon}_i\\ 1 & \mathfrak{t}_i\end{pmatrix},
\end{aligned}$$

where $b_i \in A$ are divisible by D_1 and $\begin{pmatrix}\boldsymbol{\epsilon}_i\\ \mathfrak{t}_i\end{pmatrix}$ are arrays satisfying $\begin{pmatrix}\boldsymbol{\epsilon}_i\\ \mathfrak{t}_i\end{pmatrix} \le \begin{pmatrix}1\\ q-1\end{pmatrix} + \begin{pmatrix}\Sigma\\ V\end{pmatrix}$ for all i. This proves the proposition. □

2.2.3. Operators $\mathcal{BC}$

Let $\varepsilon \in \mathbb{F}_q^\times$. We define an operator

$$\mathcal{BC}_{\varepsilon,q} : \mathfrak{BR}_w \longrightarrow \mathfrak{BR}_{w+q}$$

as follows: For each $R \in \mathfrak{BR}_w$ given as in Equation (2.7), the image $\mathcal{BC}_{\varepsilon,q}(R)$ is a binary relation given by

$$\mathcal{BC}_{\varepsilon,q}(R) = \mathcal{B}^*_{\varepsilon,q}(R) - \sum_i b_i \mathcal{C}_{\boldsymbol{\epsilon}_i,\mathfrak{t}_i}(R_\varepsilon).$$

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Let us clarify the definition of $\mathcal{BC}_{\varepsilon,q}$. We know that $\mathcal{B}^*_{\varepsilon,q}(R)$ is of the form

$$\sum_i a_i \operatorname{Si}_d \begin{pmatrix} \varepsilon & \boldsymbol{\varepsilon}_i \\ q & \mathfrak{s}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \varepsilon & \boldsymbol{\epsilon}_i \\ q & \mathfrak{t}_i \end{pmatrix} + \sum_i b_i \operatorname{Si}_d \begin{pmatrix} \varepsilon\epsilon_{i1} & \boldsymbol{\epsilon}_{i-} \\ q + t_{i1} & \mathfrak{t}_{i-} \end{pmatrix} = 0.$$

Moreover, $\mathcal{C}_{\boldsymbol{\epsilon}_i,\mathfrak{t}_i}(R_\varepsilon)$ is of the form

$$\operatorname{Si}_d \begin{pmatrix} \varepsilon & \boldsymbol{\epsilon}_i \\ q & \mathfrak{t}_i \end{pmatrix} + \operatorname{Si}_d \begin{pmatrix} \varepsilon\epsilon_{i1} & \boldsymbol{\epsilon}_{i-} \\ q + t_{i1} & \mathfrak{t}_{i-} \end{pmatrix} + \varepsilon^{-1} D_1 \operatorname{Si}_{d+1} \begin{pmatrix} \varepsilon \\ 1 \end{pmatrix} \operatorname{Si}_{<d+1} \begin{pmatrix} 1 \\ q-1 \end{pmatrix} \operatorname{Si}_{<d+1} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} = 0.$$

Combining with Proposition 2.3, Part 2, we have that $\mathcal{BC}_{\varepsilon,q}(R)$ is of the form

$$\sum_i a_i \operatorname{Si}_d \begin{pmatrix} \varepsilon & \boldsymbol{\varepsilon}_i \\ q & \mathfrak{s}_i \end{pmatrix} + \sum_{i,j} b_{ij} \operatorname{Si}_{d+1} \begin{pmatrix} \varepsilon & \boldsymbol{\epsilon}_{ij} \\ 1 & \mathfrak{t}_{ij} \end{pmatrix} = 0,$$

where $b_{ij} \in K$ and $\begin{pmatrix} \boldsymbol{\epsilon}_{ij} \\ \mathfrak{t}_{ij} \end{pmatrix}$ are arrays satisfying $\begin{pmatrix} \boldsymbol{\epsilon}_{ij} \\ \mathfrak{t}_{ij} \end{pmatrix} \leq \begin{pmatrix} 1 \\ q-1 \end{pmatrix} + \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix}$ for all j.

2.3. A weak version of Brown's theorem for ACMPL's

2.3.1. Preparatory results

Proposition 2.7. 1) *Let* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_n \\ s_1 & \dots & s_n \end{pmatrix}$ *be an array such that* $\operatorname{Init}(\mathfrak{s}) = (s_1, \dots, s_{k-1})$ *for some* $1 \leq k \leq n$, *and let* ε *be an element in* $\mathbb{F}_q^\times$. *Then* $\operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ *can be decomposed as follows:*

$$\operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \underbrace{-\operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon}' \\ \mathfrak{s}' \end{pmatrix}}_{\text{type 1}} + \underbrace{\sum_i b_i \operatorname{Li} \begin{pmatrix} \boldsymbol{\epsilon}_i' \\ \mathfrak{t}_i' \end{pmatrix}}_{\text{type 2}} + \underbrace{\sum_i c_i \operatorname{Li} \begin{pmatrix} \boldsymbol{\mu}_i \\ \mathfrak{u}_i \end{pmatrix}}_{\text{type 3}},$$

where $b_i, c_i \in A$ *are divisible by* D_1 *such that for all* i, *the following properties are satisfied:*

- *For all arrays* $\begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$ *appearing on the right-hand side,*

$$\operatorname{depth}(\mathfrak{t}) \geq \operatorname{depth}(\mathfrak{s}) \quad \text{and} \quad T_k(\mathfrak{t}) \leq T_k(\mathfrak{s}).$$

- *For the array* $\begin{pmatrix} \boldsymbol{\varepsilon}' \\ \mathfrak{s}' \end{pmatrix}$ *of type* 1 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, *we have*

$$\begin{pmatrix} \boldsymbol{\varepsilon}' \\ \mathfrak{s}' \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_{k-1} & \varepsilon & \varepsilon^{-1}\varepsilon_k & \varepsilon_{k+1} & \dots & \varepsilon_n \\ s_1 & \dots & s_{k-1} & q & s_k - q & s_{k+1} & \dots & s_n \end{pmatrix}.$$

  Moreover, for all $k \leq \ell \leq n$,

$$s_1' + \dots + s_\ell' < s_1 + \dots + s_\ell.$$

- *For the array* $\begin{pmatrix} \boldsymbol{\epsilon}' \\ \mathfrak{t}' \end{pmatrix}$ *of type* 2 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, *for all* $k \leq \ell \leq n$,

$$t_1' + \dots + t_\ell' < s_1 + \dots + s_\ell.$$

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- *For the array* $\begin{pmatrix} \boldsymbol{\mu} \\ \mathfrak{u} \end{pmatrix}$ *of type* 3 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, *we have* $\operatorname{Init}(\mathfrak{s}) \prec \operatorname{Init}(\mathfrak{u})$.

2) *Let* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_k \\ s_1 & \dots & s_k \end{pmatrix}$ *be an array such that* $\operatorname{Init}(\mathfrak{s}) = \mathfrak{s}$ *and* $s_k = q$. *Then* $\operatorname{Li}\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ *can be decomposed as follows:*

$$\operatorname{Li}\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \underbrace{\sum_i b_i \operatorname{Li}\begin{pmatrix} \boldsymbol{\epsilon}'_i \\ \mathfrak{t}'_i \end{pmatrix}}_{\text{type 2}} + \underbrace{\sum_i c_i \operatorname{Li}\begin{pmatrix} \boldsymbol{\mu}_i \\ \mathfrak{u}_i \end{pmatrix}}_{\text{type 3}},$$

where $b_i, c_i \in A$ *divisible by* D_1 *such that for all i, the following properties are satisfied:*

- *For all arrays* $\begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$ *appearing on the right-hand side,*

$$\operatorname{depth}(\mathfrak{t}) \geq \operatorname{depth}(\mathfrak{s}) \quad \textit{and} \quad T_k(\mathfrak{t}) \leq T_k(\mathfrak{s}).$$

- *For the array* $\begin{pmatrix} \boldsymbol{\epsilon}' \\ \mathfrak{t}' \end{pmatrix}$ *of type* 2 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$,

$$t'_1 + \dots + t'_k < s_1 + \dots + s_k.$$

- *For the array* $\begin{pmatrix} \boldsymbol{\mu} \\ \mathfrak{u} \end{pmatrix}$ *of type* 3 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, *we have* $\operatorname{Init}(\mathfrak{s}) \prec \operatorname{Init}(\mathfrak{u})$.

Proof. The proof follows the same line as in [31, Proposition 2.12 and 2.13]. We outline the proof here and refer the reader to [25] for more details. For Part 1, since $\operatorname{Init}(\mathfrak{s}) = (s_1, \dots, s_{k-1})$, we get $s_k > q$. Set $\begin{pmatrix} \Sigma \\ V \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1}\varepsilon_k & \varepsilon_{k+1} & \dots & \varepsilon_n \\ s_k - q & s_{k+1} & \dots & s_n \end{pmatrix}$. By Proposition 2.6, $\mathcal{C}_{\Sigma,V}(R_\varepsilon)$ is of the form

$$\operatorname{Si}_d\begin{pmatrix} \varepsilon_k & \dots & \varepsilon_n \\ s_k & \dots & s_n \end{pmatrix} + \operatorname{Si}_d\begin{pmatrix} \varepsilon & \varepsilon^{-1}\varepsilon_k & \varepsilon_{k+1} & \dots & \varepsilon_n \\ q & s_k - q & s_{k+1} & \dots & s_n \end{pmatrix} + \sum_i b_i \operatorname{Si}_{d+1}\begin{pmatrix} \varepsilon & \boldsymbol{\epsilon}_i \\ 1 & \mathfrak{t}_i \end{pmatrix} = 0, \tag{2.8}$$

where $b_i \in A$ divisible by D_1 and $\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix}$ are arrays satisfying for all i,

$$\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} \leq \begin{pmatrix} 1 \\ q-1 \end{pmatrix} + \begin{pmatrix} \Sigma \\ V \end{pmatrix} = \begin{pmatrix} \varepsilon^{-1}\varepsilon_k & \varepsilon_{k+1} & \dots & \varepsilon_n \\ s_k - 1 & s_{k+1} & \dots & s_n \end{pmatrix}.$$

For $m \in \mathbb{N}$, we denote by $q^{\{m\}}$ the sequence of length m with all terms equal to q. We agree by convention that $q^{\{0\}}$ is the empty sequence. Setting $s_0 = 0$, we may assume that there exists a maximal index j with $0 \leq j \leq k-1$ such that $s_j < q$, hence $\operatorname{Init}(\mathfrak{s}) = (s_1, \dots, s_j, q^{\{k-j-1\}})$. Then the operator $\mathcal{BC}_{\varepsilon_{j+1},q} \circ \dots \circ \mathcal{BC}_{\varepsilon_{k-1},q}$ applied to the relation (2.8) gives

$$\begin{aligned} &\operatorname{Si}_d\begin{pmatrix} \varepsilon_{j+1} & \dots & \varepsilon_{k-1} & \varepsilon_k & \dots & \varepsilon_n \\ q & \dots & q & s_k & \dots & s_n \end{pmatrix} + \operatorname{Si}_d\begin{pmatrix} \varepsilon_{j+1} & \dots & \varepsilon_{k-1} & \varepsilon & \varepsilon^{-1}\varepsilon_k & \epsilon_{k+1} & \dots & \epsilon_n \\ q & \dots & q & q & s_k - q & s_{k+1} & \dots & s_n \end{pmatrix} \\ &\qquad + \sum_i b_{i_1\dots i_{k-j}} \operatorname{Si}_{d+1}\begin{pmatrix} \varepsilon_{j+1} & \boldsymbol{\epsilon}_{i_1\dots i_{k-j}} \\ 1 & \mathfrak{t}_{i_1\dots i_{k-j}} \end{pmatrix} = 0, \end{aligned}$$

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where $b_{i_1\dots i_{k-j}} \in A$ are divisible by D_1 and

$$\begin{pmatrix} \boldsymbol{\epsilon}_{i_1\dots i_{k-j}} \\ \mathfrak{t}_{i_1\dots i_{k-j}} \end{pmatrix} \leq \begin{pmatrix} \varepsilon_{j+2} & \dots & \varepsilon_{k-1} & \varepsilon & \varepsilon^{-1}\varepsilon_k & \varepsilon_{k+1} & \dots & \varepsilon_n \\ q & \dots & q & q & s_k-1 & s_{k+1} & \dots & s_n \end{pmatrix}. \tag{2.9}$$

Set $\begin{pmatrix} \Sigma' \\ V' \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_j \\ s_1 & \dots & s_j \end{pmatrix}$. Applying $\mathcal{B}^*_{\Sigma',V'}$ to the above relation and using Lemma 2.5, we can deduce that

$$\begin{aligned} \operatorname{Li}\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} &= -\operatorname{Li}\begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_{k-1} & \varepsilon & \varepsilon^{-1}\varepsilon_k & \varepsilon_{k+1} & \dots & \varepsilon_n \\ s_1 & \dots & s_{k-1} & q & s_k-q & s_{k+1} & \dots & s_n \end{pmatrix} \\ &\quad - \sum_i b_{i_1\dots i_{k-j}} \operatorname{Li}\begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_j & \varepsilon_{j+1} & \boldsymbol{\epsilon}_{i_1\dots i_{k-j}} \\ s_1 & \dots & s_j & 1 & \mathfrak{t}_{i_1\dots i_{k-j}} \end{pmatrix} \\ &\quad - \sum_i b_{i_1\dots i_{k-j}} \operatorname{Li}\begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_{j-1} & \varepsilon_j\varepsilon_{j+1} & \boldsymbol{\epsilon}_{i_1\dots i_{k-j}} \\ s_1 & \dots & s_{j-1} & s_j+1 & \mathfrak{t}_{i_1\dots i_{k-j}} \end{pmatrix}. \end{aligned} \tag{2.10}$$

The first term, the second term and the third term on the right-hand side of Equation (2.10) are referred to as type 1, type 2 and type 3, respectively. From Equation (2.9) and Remark 2.1, one verifies that the arrays of type 1, type 2 and type 3 satisfy the desired conditions. We have proved Part 1.

The proof of Part 2 follows the same arguments as that of Part 1. We first begin with the relation R_{ε_k}. Next, we apply $\mathcal{BC}_{\varepsilon_{j+1},q} \circ \dots \circ \mathcal{BC}_{\varepsilon_{k-1},q}$ to R_{ε_k} and then apply $\mathcal{B}^*_{\Sigma',V'}$. We can deduce that

$$\begin{aligned} \operatorname{Li}\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} &= - \sum_i b_{i_1\dots i_{k-j}} \operatorname{Li}\begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_j & \varepsilon_{j+1} & \boldsymbol{\epsilon}_{i_1\dots i_{k-j}} \\ s_1 & \dots & s_j & 1 & \mathfrak{t}_{i_1\dots i_{k-j}} \end{pmatrix} \\ &\quad - \sum_i b_{i_1\dots i_{k-j}} \operatorname{Li}\begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_{j-1} & \varepsilon_j\varepsilon_{j+1} & \boldsymbol{\epsilon}_{i_1\dots i_{k-j}} \\ s_1 & \dots & s_{j-1} & s_j+1 & \mathfrak{t}_{i_1\dots i_{k-j}} \end{pmatrix}, \end{aligned} \tag{2.11}$$

where $b_{i_1\dots i_{k-j}} \in A$ are divisible by D_1 and

$$\begin{pmatrix} \boldsymbol{\epsilon}_{i_1\dots i_{k-j}} \\ \mathfrak{t}_{i_1\dots i_{k-j}} \end{pmatrix} \leq \begin{pmatrix} \varepsilon_{j+2} & \dots & \varepsilon_k & 1 \\ q & \dots & q & q-1 \end{pmatrix}. \tag{2.12}$$

The first term and the second term on the right-hand side of Equation (2.11) are referred to as type 2 and type 3, respectively. From Equation (2.12) and Remark 2.1, one verifies that the arrays of type 2 and type 3 satisfy the desired conditions. We finish the proof. □

We recall the following definition of [31] (see [31, Definition 3.1]):

Definition 2.8. Let $k \in \mathbb{N}$ and $\mathfrak{s}$ be a tuple of positive integers. We say that $\mathfrak{s}$ is k-admissible if it satisfies the following two conditions:

1) $s_1, \dots, s_k \leq q$.
2) $\mathfrak{s}$ is not of the form $(s_1, \dots, s_r)$ with $r \leq k$, $s_1, \dots, s_{r-1} \leq q$, and $s_r = q$.

Here, we recall $s_i = 0$ for $i > \operatorname{depth}(\mathfrak{s})$. By convention the empty array $\begin{pmatrix} \emptyset \\ \emptyset \end{pmatrix}$ is always k-admissible.

An array is k-admissible if the corresponding tuple is k-admissible.

Proposition 2.9. *For all $k \in \mathbb{N}$ and for all arrays $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, $\operatorname{Li}\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ can be expressed as a K-linear combination of $\operatorname{Li}\begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$'s of the same weight such that $\mathfrak{t}$ is k-admissible.*

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Proof. The proof follows the same line as that of [31, Proposition 3.2]. We outline the proof here and refer the reader to [25] for more details. We consider the following statement: (H_k) For all arrays $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$, we can express $\mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$ as a K-linear combination of $\mathrm{Li}\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}$'s of the same weight such that $\mathfrak{t}$ is k-admissible.

We will show that (H_k) holds for all $k \in \mathbb{N}$ by induction on k. For $k = 1$, we consider all the cases for the first component s_1 of $\mathfrak{s}$. If $s_1 \leq q$, then either $\mathfrak{s}$ is 1-admissible, or $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \begin{pmatrix}\varepsilon\\ q\end{pmatrix}$. We deduce from the relation R_ε that (H_1) holds for the case $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \begin{pmatrix}\varepsilon\\ q\end{pmatrix}$. If $s_1 > q$, we assume that $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \begin{pmatrix}\varepsilon_1 & \cdots & \varepsilon_n\\ s_1 & \cdots & s_n\end{pmatrix}$. Set $\begin{pmatrix}\Sigma\\ V\end{pmatrix} = \begin{pmatrix}\varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n\\ s_1 - q & s_2 & \cdots & s_n\end{pmatrix}$. Applying $C_{\Sigma,V}$ to the relation R_1 and using Proposition 2.6, we can deduce that

$$\mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = -\mathrm{Li}\begin{pmatrix}1 & \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n\\ q & s_1 - q & s_2 & \cdots & s_n\end{pmatrix} - \sum_i b_i \,\mathrm{Li}\begin{pmatrix}1 & \boldsymbol{\epsilon}_i\\ 1 & \mathfrak{t}_i\end{pmatrix},$$

where $b_i \in K$ for all i. It proves that (H_1) holds.

We next assume that (H_{k-1}) holds. We need to show that (H_k) holds. By using the induction hypothesis of (H_{k-1}), we can restrict our attention to the array $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \begin{pmatrix}\varepsilon_1 & \cdots & \varepsilon_n\\ s_1 & \cdots & s_n\end{pmatrix}$, where $\mathfrak{s}$ is not k-admissible and $\mathrm{depth}(\mathfrak{s}) \geq k$. We will prove that (H_k) holds for the array $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$ by induction on $s_1 + \cdots + s_k$. The case $s_1 + \cdots + s_k = 1$ is a simple check. Assume that (H_k) holds when $s_1 + \cdots + s_k < s$. We need to show that (H_k) holds when $s_1 + \cdots + s_k = s$. To do so, we give the following algorithm:

Algorithm: We begin with an array $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$ where $\mathfrak{s}$ is not k-admissible, $\mathrm{depth}(\mathfrak{s}) \geq k$ and $s_1 + \cdots + s_k = s$.

Step 1: From Proposition 2.7, we can decompose $\mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$ as follows:

$$\mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \underbrace{-\mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}'\\ \mathfrak{s}'\end{pmatrix}}_{\text{type 1}} + \underbrace{\sum_i b_i \,\mathrm{Li}\begin{pmatrix}\boldsymbol{\epsilon}_i'\\ \mathfrak{t}_i'\end{pmatrix}}_{\text{type 2}} + \underbrace{\sum_i c_i \,\mathrm{Li}\begin{pmatrix}\boldsymbol{\mu}_i\\ \mathfrak{u}_i\end{pmatrix}}_{\text{type 3}}, \tag{2.13}$$

where $b_i, c_i \in A$. The term of type 1 disappears when $\mathrm{Init}(\mathfrak{s}) = \mathfrak{s}$ and $s_n = q$.

Step 2: For all arrays $\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}$ appearing on the right-hand side of Equation (2.13), if $\mathfrak{t}$ is either k-admissible or $\mathfrak{t}$ satisfies the condition $t_1 + \cdots + t_k < s$, then we deduce from the induction hypothesis that (H_k) holds for the array $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$, and hence we stop the algorithm. Otherwise, there exists an array $\begin{pmatrix}\boldsymbol{\varepsilon}_1\\ \mathfrak{s}_1\end{pmatrix}$ where $\mathfrak{s}_1$ is not k-admissible, $\mathrm{depth}(\mathfrak{s}_1) \geq k$ and $s_{11} + \cdots + s_{1k} = s$. For such an array, we repeat the algorithm for $\begin{pmatrix}\boldsymbol{\varepsilon}_1\\ \mathfrak{s}_1\end{pmatrix}$.

It remains to show that the above algorithm stops after a finite number of steps. Indeed, assume that the above algorithm does not stop. Then there exists a sequence of arrays $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\varepsilon}_0\\ \mathfrak{s}_0\end{pmatrix}, \begin{pmatrix}\boldsymbol{\varepsilon}_1\\ \mathfrak{s}_1\end{pmatrix}, \begin{pmatrix}\boldsymbol{\varepsilon}_2\\ \mathfrak{s}_2\end{pmatrix}, \ldots$ such that $\mathfrak{s}_i$ is not k-admissible and $\mathrm{depth}(\mathfrak{s}_i) \geq k$ for all $i \geq 0$. Using Proposition 2.7, one verifies that $\begin{pmatrix}\boldsymbol{\varepsilon}_{i+1}\\ \mathfrak{s}_{i+1}\end{pmatrix}$ is of type 3 with respect to $\begin{pmatrix}\boldsymbol{\varepsilon}_i\\ \mathfrak{s}_i\end{pmatrix}$ for all $i \geq 0$, hence we obtain an infinite sequence $\mathrm{Init}(\mathfrak{s}_0) \prec \mathrm{Init}(\mathfrak{s}_1) \prec \mathrm{Init}(\mathfrak{s}_2) \prec \cdots$. For all $i \geq 0$, since $\mathfrak{s}_i$ is not k-admissible and $\mathrm{depth}(\mathfrak{s}_i) \geq k$, we have $\mathrm{depth}(\mathrm{Init}(\mathfrak{s}_i)) \leq k$, hence $\mathrm{Init}(\mathfrak{s}_i) \preceq q^{\{k\}}$. This shows that $\mathrm{Init}(\mathfrak{s}_i) = \mathrm{Init}(\mathfrak{s}_{i+1})$ for all i sufficiently large, which is a contradiction. □

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2.3.2. A set of generators $\mathcal{AT}_w$ for ACMPL's

We recall that $\mathcal{AL}_w$ is the K-vector space generated by ACMPL's of weight w. We denote by $\mathcal{AT}_w$ the set of all ACMPL's $\mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} = \mathrm{Li}\begin{pmatrix}\varepsilon_1 & \dots & \varepsilon_n\\ s_1 & \dots & s_n\end{pmatrix}$ of weight w such that $s_1, \dots, s_{n-1} \leq q$ and $s_n < q$.

We put $t(w) = |\mathcal{AT}_w|$. Then one verifies that

$$t(w) = \begin{cases} (q-1)q^{w-1} & \text{if } 1 \leq w < q,\\ (q-1)(q^{w-1}-1) & \text{if } w = q,\end{cases}$$

and for $w > q$, $t(w) = (q-1)\sum_{i=1}^{q} t(w-i)$.

We are ready to state a weak version of Brown's theorem for ACMPL's.

Proposition 2.10. *The set of all elements* $\mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$ *such that* $\mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} \in \mathcal{AT}_w$ *forms a set of generators for* $\mathcal{AL}_w$.

Proof. The result follows immediately from Proposition 2.9 in the case of $k = w$. □

2.4. A strong version of Brown's theorem for ACMPL's

2.4.1. Another set of generators $\mathcal{AS}_w$ for ACMPL's

We consider the set $\mathcal{J}_w$ consisting of positive tuples $\mathfrak{s} = (s_1, \dots, s_n)$ of weight w such that $s_1, \dots, s_{n-1} \leq q$ and $s_n < q$, together with the set $\mathcal{J}'_w$ consisting of positive tuples $\mathfrak{s} = (s_1, \dots, s_n)$ of weight w such that $q \nmid s_i$ for all i. Then there is a bijection

$$\iota \colon \mathcal{J}'_w \longrightarrow \mathcal{J}_w$$

given as follows: For each tuple $\mathfrak{s} = (s_1, \dots, s_n) \in \mathcal{J}'_w$, since $q \nmid s_i$, we can write $s_i = h_i q + r_i$, where $0 < r_i < q$ and $h_i \in \mathbb{Z}^{\geq 0}$. The image $\iota(\mathfrak{s})$ is the tuple

$$\iota(\mathfrak{s}) = (\underbrace{q, \dots, q}_{h_1 \text{ times}}, r_1, \dots, \underbrace{q, \dots, q}_{h_n \text{ times}}, r_n).$$

Let $\mathcal{AS}_w$ denote the set of ACMPL's $\mathrm{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$ such that $\mathfrak{s} \in \mathcal{J}'_w$. We note that in general, $\mathcal{AS}_w$ is strictly smaller than $\mathcal{AT}_w$. The only exceptions are when $q = 2$ or $w \leq q$.

2.4.2. Cardinality of $\mathcal{AS}_w$.

We now compute $s(w) = |\mathcal{AS}_w|$. To do so, we denote by $\mathcal{AJ}_w$ the set consisting of arrays $\begin{pmatrix}\varepsilon_1 & \dots & \varepsilon_n\\ s_1 & \dots & s_n\end{pmatrix}$ of weight w such that $q \nmid s_i$ for all i and by $\mathcal{AJ}^1_w$ the set consisting of arrays $\begin{pmatrix}\varepsilon_1 & \dots & \varepsilon_n\\ s_1 & \dots & s_n\end{pmatrix}$ of weight w such that $s_1, \dots, s_{n-1} \leq q$, $s_n < q$ and $\varepsilon_i = 1$ whenever $s_i = q$ for $1 \leq i \leq n$. We construct a map

$$\varphi \colon \mathcal{AJ}_w \longrightarrow \mathcal{AJ}^1_w$$

as follows: For each array $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_n \\ s_1 & \dots & s_n \end{pmatrix} \in \mathcal{AJ}_w$, since $q \nmid s_i$, we can write $s_i = (h_i - 1)q + r_i$, where $0 < r_i < q$ and $h_i \in \mathbb{N}$. The image $\varphi \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ is the array

$$\varphi \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \Bigg(\underbrace{\begin{pmatrix} 1 & \dots & 1 \\ q & \dots & q \end{pmatrix}}_{h_1 - 1 \text{ times}} \begin{pmatrix} \varepsilon_1 \\ r_1 \end{pmatrix} \dots \underbrace{\begin{pmatrix} 1 & \dots & 1 \\ q & \dots & q \end{pmatrix}}_{h_n - 1 \text{ times}} \begin{pmatrix} \varepsilon_n \\ r_n \end{pmatrix} \Bigg).$$

It is easily seen that φ is a bijection, hence $|\mathcal{AS}_w| = |\mathcal{AJ}_w| = |\mathcal{AJ}_w^1|$. Thus, $s(w) = |\mathcal{AJ}_w^1|$. One verifies that

$$s(w) = \begin{cases} (q-1)q^{w-1} & \text{if } 1 \leq w < q, \\ (q-1)(q^{w-1} - 1) & \text{if } w = q, \end{cases}$$

and for $w > q$,

$$s(w) = (q-1) \sum_{i=1}^{q-1} s(w-i) + s(w-q).$$

2.4.3.
We state a strong version of Brown's theorem for ACMPL's.

Theorem 2.11. *The set $\mathcal{AS}_w$ forms a set of generators for $\mathcal{AL}_w$. In particular,*

$$\dim_K \mathcal{AL}_w \leq s(w).$$

Proof. We recall that $\mathcal{AT}_w$ is the set of all ACMPL's $\operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ with $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AJ}_w$.

Let $\operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \operatorname{Li} \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_n \\ s_1 & \dots & s_n \end{pmatrix} \in \mathcal{AT}_w$. Then $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AJ}_w$, which implies $s_1, \dots, s_{n-1} \leq q$ and $s_n < q$. We express $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ in the following form

$$\Bigg(\underbrace{\begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_{h_1-1} \\ q & \dots & q \end{pmatrix}}_{h_1 - 1 \text{ times}} \begin{pmatrix} \varepsilon_{h_1} \\ r_1 \end{pmatrix} \dots \underbrace{\begin{pmatrix} \varepsilon_{h_1+\dots+h_{\ell-1}+1} & \dots & \varepsilon_{h_1+\dots+h_{\ell-1}+(h_\ell-1)} \\ q & \dots & q \end{pmatrix}}_{h_\ell - 1 \text{ times}} \begin{pmatrix} \varepsilon_{h_1+\dots+h_{\ell-1}+h_\ell} \\ r_\ell \end{pmatrix} \Bigg),$$

where $h_1, \dots, h_\ell \geq 1$, $h_1 + \dots + h_\ell = n$ and $0 < r_1, \dots, r_\ell < q$. Then we set

$$\begin{pmatrix} \boldsymbol{\varepsilon}' \\ \mathfrak{s}' \end{pmatrix} = \begin{pmatrix} \varepsilon_1' & \dots & \varepsilon_\ell' \\ s_1' & \dots & s_\ell' \end{pmatrix},$$

where $\varepsilon_i' = \varepsilon_{h_1+\dots+h_{i-1}+1} \cdots \varepsilon_{h_1+\dots+h_{i-1}+h_i}$ and $s_i' = (h_i - 1)q + r_i$ for $1 \leq i \leq \ell$. We note that $\iota(\mathfrak{s}') = \mathfrak{s}$.

From Proposition 2.7 and Proposition 2.10, we can decompose $\operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon}' \\ \mathfrak{s}' \end{pmatrix}$ as follows:

$$\operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon}' \\ \mathfrak{s}' \end{pmatrix} = \sum a_{\boldsymbol{\epsilon}, \mathfrak{t}}^{\boldsymbol{\varepsilon}', \mathfrak{s}'} \operatorname{Li} \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix},$$

where $\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}$ ranges over all elements of $\mathcal{AJ}_w$ and $a_{\boldsymbol{\epsilon},\mathfrak{t}}^{\boldsymbol{\varepsilon}',\mathfrak{s}'} \in A$ satisfying

$$a_{\boldsymbol{\epsilon},\mathfrak{t}}^{\boldsymbol{\varepsilon}',\mathfrak{s}'} \equiv \begin{cases} \pm 1 \pmod{D_1} & \text{if } \begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix} = \begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}, \\ 0 \pmod{D_1} & \text{otherwise.} \end{cases}$$

Note that $\operatorname{Li}\begin{pmatrix}\boldsymbol{\varepsilon}'\\ \mathfrak{s}'\end{pmatrix} \in \mathcal{AS}_w$. Thus, the transition matrix from the set consisting of such $\operatorname{Li}\begin{pmatrix}\boldsymbol{\varepsilon}'\\ \mathfrak{s}'\end{pmatrix}$ as above (we allow repeated elements) to the set consisting of $\operatorname{Li}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$ with $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix} \in \mathcal{AJ}_w$ is invertible. It then follows again from Proposition 2.10 that $\mathcal{AS}_w$ is a set of generators for $\mathcal{AL}_w$, as desired. □

3. Dual *t*-motives and linear independence

We continue with the notation given in the Introduction. Further, letting t be another independent variable, we denote by $\mathbb{T}$ the Tate algebra in the variable t with coefficients in $\mathbb{C}_\infty$ equipped with the Gauss norm $\|.\|_\infty$ and by $\mathbb{L}$ the fraction field of $\mathbb{T}$.

We denote by $\mathcal{E}$ the ring of series $\sum_{n\geq 0} a_n t^n \in \overline{K}[[t]]$ such that $\lim_{n\to+\infty} \sqrt[n]{|a_n|_\infty} = 0$ and $[K_\infty(a_0, a_1, \ldots) : K_\infty] < \infty$. Then any $f \in \mathcal{E}$ is an entire function.

For $a \in A = \mathbb{F}_q[\theta]$, we set $a(t) := a|_{\theta=t} \in \mathbb{F}_q[t]$.

3.1. Dual t-motives

We recall the notion of dual t-motives due to Anderson (see [6, §4] and [22, §5] for more details). We refer the reader to [1] for the related notion of t-motives.

For $i \in \mathbb{Z}$, we consider the i-fold twisting of $\mathbb{C}_\infty((t))$ defined by

$$\mathbb{C}_\infty((t)) \to \mathbb{C}_\infty((t))$$
$$f = \sum_j a_j t^j \mapsto f^{(i)} := \sum_j a_j^{q^i} t^j.$$

We extend i-fold twisting to matrices with entries in $\mathbb{C}_\infty((t))$ by twisting entrywise.

Let $\overline{K}[t,\sigma]$ be the noncommutative $\overline{K}[t]$-algebra generated by the new variable σ subject to the relation $\sigma f = f^{(-1)}\sigma$ for all $f \in \overline{K}[t]$.

Definition 3.1. An effective dual t-motive is a $\overline{K}[t,\sigma]$-module $\mathcal{M}'$ which is free and finitely generated over $\overline{K}[t]$ such that for $\ell \gg 0$ we have

$$(t-\theta)^\ell(\mathcal{M}'/\sigma\mathcal{M}') = \{0\}.$$

We mention that effective dual t-motives are called Frobenius modules in [11, 14, 21, 27]. Note that Hartl and Juschka [22, §4] introduced a more general notion of dual t-motives. In particular, effective dual t-motives are always dual t-motives.

Throughout this paper, we will always work with effective dual t-motives. Therefore, we will sometimes drop the word 'effective' where there is no confusion.

Let $\mathcal{M}$ and $\mathcal{M}'$ be two effective dual t-motives. Then a morphism of effective dual t-motives $\mathcal{M} \to \mathcal{M}'$ is just a homomorphism of left $\overline{K}[t,\sigma]$-modules. We denote by $\mathcal{F}$ the category of effective dual t-motives equipped with the trivial object $\mathbf{1}$.

We say that an object $\mathcal{M}$ of $\mathcal{F}$ is given by a matrix $\Phi \in \operatorname{Mat}_r(\overline{K}[t])$ if $\mathcal{M}$ is a $\overline{K}[t]$-module free of rank r and the action of σ is represented by the matrix Φ on a given $\overline{K}[t]$-basis for $\mathcal{M}$. We say that

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an object $\mathcal{M}$ of $\mathcal{F}$ is uniformizable or rigid analytically trivial if there exists a matrix $\Psi \in \mathrm{GL}_r(\mathbb{T})$ satisfying $\Psi^{(-1)} = \Phi\Psi$. The matrix Ψ is called a rigid analytic trivialization of $\mathcal{M}$.

We now recall the Anderson–Brownawell–Papanikolas criterion which is crucial in the sequel (see [2, Theorem 3.1.1]).

Theorem 3.2 (Anderson–Brownawell–Papanikolas). *Let* $\Phi \in \mathrm{Mat}_\ell(\overline{K}[t])$ *be a matrix such that* $\det \Phi = c(t-\theta)^s$ *for some* $c \in \overline{K}^\times$ *and* $s \in \mathbb{Z}^{\geq 0}$. *Let* $\psi \in \mathrm{Mat}_{\ell \times 1}(\mathcal{E})$ *be a vector satisfying* $\psi^{(-1)} = \Phi\psi$ *and* $\rho \in \mathrm{Mat}_{1\times\ell}(\overline{K})$ *such that* $\rho\psi(\theta) = 0$. *Then there exists a vector* $P \in \mathrm{Mat}_{1\times\ell}(\overline{K}[t])$ *such that*

$$P\psi = 0 \quad \textit{and} \quad P(\theta) = \rho.$$

3.2. *Some constructions of dual t-motives*

3.2.1. General case

We briefly review some constructions of dual t-motives introduced in [11] (see also [9, 14, 21]). Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be a tuple of positive integers and $\mathfrak{Q} = (Q_1, \dots, Q_r) \in \overline{K}[t]^r$ satisfying the condition

$$(\|Q_1\|_\infty / |\theta|_\infty^{\frac{qs_1}{q-1}})^{q^{i_1}} \dots (\|Q_r\|_\infty / |\theta|_\infty^{\frac{qs_r}{q-1}})^{q^{i_r}} \to 0 \tag{3.1}$$

as $0 \leq i_r < \dots < i_1$ and $i_1 \to \infty$.

We consider the dual t-motives $\mathcal{M}_{\mathfrak{s},\mathfrak{Q}}$ and $\mathcal{M}'_{\mathfrak{s},\mathfrak{Q}}$ attached to $(\mathfrak{s}, \mathfrak{Q})$ given by the matrices

$$\Phi_{\mathfrak{s},\mathfrak{Q}} = \begin{pmatrix} (t-\theta)^{s_1+\dots+s_r} & 0 & 0 & \dots & 0 \\ Q_1^{(-1)}(t-\theta)^{s_1+\dots+s_r} & (t-\theta)^{s_2+\dots+s_r} & 0 & \dots & 0 \\ 0 & Q_2^{(-1)}(t-\theta)^{s_2+\dots+s_r} & \ddots & & \vdots \\ \vdots & & \ddots & (t-\theta)^{s_r} & 0 \\ 0 & \dots & 0 & Q_r^{(-1)}(t-\theta)^{s_r} & 1 \end{pmatrix}$$
$$\in \mathrm{Mat}_{r+1}(\overline{K}[t]),$$

and $\Phi'_{\mathfrak{s},\mathfrak{Q}} \in \mathrm{Mat}_r(\overline{K}[t])$ is the upper left $r \times r$ submatrix of $\Phi_{\mathfrak{s},\mathfrak{Q}}$.

Throughout this paper, we work with the Carlitz period $\widetilde{\pi}$ which is a fundamental period of the Carlitz module (see [20, 35]). We fix a choice of $(q-1)$st root of $(-\theta)$ and set

$$\Omega(t) := (-\theta)^{-q/(q-1)} \prod_{i \geq 1} \left(1 - \frac{t}{\theta^{q^i}}\right) \in \mathbb{T}^\times$$

so that

$$\Omega^{(-1)} = (t-\theta)\Omega \quad \text{and} \quad \frac{1}{\Omega(\theta)} = \widetilde{\pi}.$$

Given $(\mathfrak{s}, \mathfrak{Q})$ as above, Chang introduced the following series (see [9, Lemma 5.3.1] and also [11, Eq. (2.3.2)])

$$\mathfrak{L}(\mathfrak{s}; \mathfrak{Q}) = \mathfrak{L}(s_1, \dots, s_r; Q_1, \dots, Q_r) := \sum_{i_1 > \dots > i_r \geq 0} (\Omega^{s_r} Q_r)^{(i_r)} \dots (\Omega^{s_1} Q_1)^{(i_1)}. \tag{3.2}$$

It is proved that $\mathfrak{L}(\mathfrak{s}, \mathfrak{Q}) \in \mathcal{E}$ (see [9, Lemma 5.3.1]). Here, we recall that $\mathcal{E}$ denotes the ring of series $\sum_{n\geq 0} a_n t^n \in \overline{K}[[t]]$ such that $\lim_{n\to+\infty} \sqrt[n]{|a_n|_\infty} = 0$ and $[K_\infty(a_0, a_1, \dots) : K_\infty] < \infty$. In the sequel, we

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will use the following crucial property of this series (see [9, Lemma 5.3.5] and [11, Proposition 2.3.3]): For all $j \in \mathbb{Z}^{\geq 0}$, we have

$$\mathfrak{L}(\mathfrak{s};\mathfrak{Q})\left(\theta^{q^j}\right) = (\mathfrak{L}(\mathfrak{s};\mathfrak{Q})(\theta))^{q^j}. \tag{3.3}$$

Then the matrix given by

$$\Psi_{\mathfrak{s},\mathfrak{Q}} = \begin{pmatrix} \Omega^{s_1+\cdots+s_r} & 0 & 0 & \dots & 0 \\ \mathfrak{L}(s_1;Q_1)\Omega^{s_2+\cdots+s_r} & \Omega^{s_2+\cdots+s_r} & 0 & \dots & 0 \\ \vdots & \mathfrak{L}(s_2;Q_2)\Omega^{s_3+\cdots+s_r} & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \mathfrak{L}(s_1,\dots,s_{r-1};Q_1,\dots,Q_{r-1})\Omega^{s_r} & \mathfrak{L}(s_2,\dots,s_{r-1};Q_2,\dots,Q_{r-1})\Omega^{s_r} & \dots & \Omega^{s_r} & 0 \\ \mathfrak{L}(s_1,\dots,s_r;Q_1,\dots,Q_r) & \mathfrak{L}(s_2,\dots,s_r;Q_2,\dots,Q_r) & \dots & \mathfrak{L}(s_r;Q_r) & 1 \end{pmatrix}$$
$$\in \mathrm{GL}_{r+1}(\mathbb{T})$$

satisfies

$$\Psi^{(-1)}_{\mathfrak{s},\mathfrak{Q}} = \Phi_{\mathfrak{s},\mathfrak{Q}}\Psi_{\mathfrak{s},\mathfrak{Q}}.$$

Thus, $\Psi_{\mathfrak{s},\mathfrak{Q}}$ is a rigid analytic trivialization associated to the dual t-motive $\mathcal{M}_{\mathfrak{s},\mathfrak{Q}}$.

We also denote by $\Psi'_{\mathfrak{s},\mathfrak{Q}}$ the upper $r \times r$ submatrix of $\Psi_{\mathfrak{s},\mathfrak{Q}}$. It is clear that $\Psi'_{\mathfrak{s}}$ is a rigid analytic trivialization associated to the dual t-motive $\mathcal{M}'_{\mathfrak{s},\mathfrak{Q}}$.

Further, combined with Equation (3.3), the above construction of dual t-motives implies that $\widetilde{\pi}^w\mathfrak{L}(\mathfrak{s};\mathfrak{Q})(\theta)$, where $w = s_1 + \cdots + s_r$ has the MZ (multizeta) property in the sense of [9, Definition 3.4.1]. By [9, Proposition 4.3.1], we get

Proposition 3.3. *Let $(\mathfrak{s}_i;\mathfrak{Q}_i)$ as before for $1 \leq i \leq m$. We suppose that all the tuples of positive integers $\mathfrak{s}_i$ have the same weight, say w. Then the following assertions are equivalent:*

i) $\mathfrak{L}(\mathfrak{s}_1;\mathfrak{Q}_1)(\theta),\dots,\mathfrak{L}(\mathfrak{s}_m;\mathfrak{Q}_m)(\theta)$ *are K-linearly independent.*
ii) $\mathfrak{L}(\mathfrak{s}_1;\mathfrak{Q}_1)(\theta),\dots,\mathfrak{L}(\mathfrak{s}_m;\mathfrak{Q}_m)(\theta)$ *are $\overline{K}$-linearly independent.*

We end this section by mentioning that Chang [9] also proved analogue of Goncharov's conjecture in this setting.

3.2.2. Dual t-motives connected to MZV's and AMZV's

Following Anderson and Thakur [4], we introduce dual t-motives connected to MZV's and AMZV's. We briefly review Anderson–Thakur polynomials introduced in [3]. For $k \geq 0$, we set $[k] := \theta^{q^k} - \theta$ and $D_k := \prod_{\ell=1}^{k}[\ell]^{q^{k-\ell}}$. For $n \in \mathbb{N}$, we write $n - 1 = \sum_{j\geq 0} n_j q^j$ with $0 \leq n_j \leq q-1$ and define

$$\Gamma_n := \prod_{j\geq 0} D_j^{n_j}.$$

We set $\gamma_0(t) := 1$ and $\gamma_j(t) := \prod_{\ell=1}^{j}(\theta^{q^j} - t^{q^\ell})$ for $j \geq 1$. Then Anderson–Thakur polynomials $\alpha_n(t) \in A[t]$ are given by the generating series

$$\sum_{n\geq 1}\frac{\alpha_n(t)}{\Gamma_n}x^n := x\left(1 - \sum_{j\geq 0}\frac{\gamma_j(t)}{D_j}x^{q^j}\right)^{-1}.$$

Finally, we define $H_n(t)$ by switching θ and t

$$H_n(t) = \alpha_n(t)\big|_{t=\theta,\ \theta=t}.$$

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By [3, Eq. (3.7.3)], we get

$$\deg_\theta H_n \leq \frac{(n-1)q}{q-1} < \frac{nq}{q-1}. \tag{3.4}$$

Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be a tuple and $\epsilon = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$. Recall that $\overline{\mathbb{F}}_q$ denotes the algebraic closure of $\mathbb{F}_q$ in $\overline{K}$. For all $1 \leq i \leq r$, we fix a fixed $(q-1)$-th root $\gamma_i \in \overline{\mathbb{F}}_q$ of $\epsilon_i \in \mathbb{F}_q^\times$ and set $Q_{s_i,\epsilon_i} := \gamma_i H_{s_i}$. Then we set $\mathfrak{Q}_{\mathfrak{s},\epsilon} := (Q_{s_1,\epsilon_1}, \dots, Q_{s_r,\epsilon_r})$ and put $\mathfrak{L}(\mathfrak{s};\boldsymbol{\epsilon}) := \mathfrak{L}(\mathfrak{s};\mathfrak{Q}_{\mathfrak{s},\epsilon})$. By Equation (3.4), we know that $\|H_n\|_\infty < |\theta|_\infty^{\frac{nq}{q-1}}$ for all $n \in \mathbb{N}$, thus $\mathfrak{Q}_{\mathfrak{s},\epsilon}$ satisfies condition (3.1). Thus,w e can define the dual t-motives $\mathcal{M}_{\mathfrak{s},\epsilon} = \mathcal{M}_{\mathfrak{s},\mathfrak{Q}_{\mathfrak{s},\epsilon}}$ and $\mathcal{M}'_{\mathfrak{s},\epsilon} = \mathcal{M}'_{\mathfrak{s},\mathfrak{Q}_{\mathfrak{s},\epsilon}}$ attached to $\mathfrak{s}$ whose matrices and rigid analytic trivializations will be denoted by $(\Phi_{\mathfrak{s},\epsilon}, \Psi_{\mathfrak{s},\epsilon})$ and $(\Phi'_{\mathfrak{s},\epsilon}, \Psi'_{\mathfrak{s},\epsilon})$, respectively. These dual t-motives are connected to MZV's and AMZV's by the following result (see [14, Proposition 2.12] for more details):

$$\mathfrak{L}(\mathfrak{s};\boldsymbol{\epsilon})(\theta) = \frac{\gamma_1 \dots \gamma_r \Gamma_{s_1} \dots \Gamma_{s_r} \zeta_A \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{s} \end{pmatrix}}{\widetilde{\pi}^{w(\mathfrak{s})}}. \tag{3.5}$$

By a result of Thakur [37], one can show (see [21, Theorem 2.1]) that $\zeta_A \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{s} \end{pmatrix} \neq 0$. Thus, $\mathfrak{L}(\mathfrak{s};\boldsymbol{\epsilon})(\theta) \neq 0$.

3.2.3. Dual t-motives connected to CMPL's and ACMPL's

We keep the notation as above. Let $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ be a tuple and $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_r) \in (\mathbb{F}_q^\times)^r$. For all $1 \leq i \leq r$, we have a fixed $(q-1)$-th root γ_i of $\epsilon_i \in \mathbb{F}_q^\times$ and set $Q'_{s_i,\epsilon_i} := \gamma_i$. Then we set $\mathfrak{Q}'_{\mathfrak{s},\epsilon} := (Q'_{s_1,\epsilon_1}, \dots, Q'_{s_r,\epsilon_r})$ and put

$$\mathfrak{Li}(\mathfrak{s};\boldsymbol{\epsilon}) = \mathfrak{L}(\mathfrak{s};\mathfrak{Q}'_{\mathfrak{s},\epsilon}) = \sum_{i_1 > \dots > i_r \geq 0} (\gamma_{i_r} \Omega^{s_r})^{(i_r)} \dots (\gamma_{i_1} \Omega^{s_1})^{(i_1)}. \tag{3.6}$$

Thus, we can define the dual t-motives $\mathcal{N}_{\mathfrak{s},\epsilon} = \mathcal{N}_{\mathfrak{s},\mathfrak{Q}'_{\mathfrak{s},\epsilon}}$ and $\mathcal{N}'_{\mathfrak{s},\epsilon} = \mathcal{N}'_{\mathfrak{s},\mathfrak{Q}'_{\mathfrak{s},\epsilon}}$ attached to $(\mathfrak{s},\boldsymbol{\epsilon})$. These dual t-motives are connected to CMPL's and ACMPL's by the following result (see [9, Lemma 5.3.5] and [11, Prop. 2.3.3]):

$$\mathfrak{Li}(\mathfrak{s};\boldsymbol{\epsilon})(\theta) = \frac{\gamma_1 \dots \gamma_r \operatorname{Li} \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{s} \end{pmatrix}}{\widetilde{\pi}^{w(\mathfrak{s})}}. \tag{3.7}$$

3.3. A result for linear independence

3.3.1. Setup

Let $w \in \mathbb{N}$ be a positive integer. Let $\{(\mathfrak{s}_i;\mathfrak{Q}_i)\}_{1 \leq i \leq n}$ be a collection of pairs satisfying condition (3.1) such that $\mathfrak{s}_i$ always has weight w. We write $\mathfrak{s}_i = (s_{i1}, \dots, s_{i\ell_i}) \in \mathbb{N}^{\ell_i}$ and $\mathfrak{Q}_i = (Q_{i1}, \dots, Q_{i\ell_i}) \in (\mathbb{F}_q^\times)^{\ell_i}$ so that $s_{i1} + \dots + s_{i\ell_i} = w$. We introduce the set of tuples

$$I(\mathfrak{s}_i;\mathfrak{Q}_i) := \{\emptyset, (s_{i1};Q_{i1}), \dots, (s_{i1}, \dots, s_{i(\ell_i-1)};Q_{i1}, \dots, Q_{i(\ell_i-1)})\},$$

and set

$$I := \cup_i I(\mathfrak{s}_i;\mathfrak{Q}_i).$$

3.3.2. Linear independence

We are now ready to state the main result of this section.

Theorem 3.4. *We keep the above notation. We suppose further that* $\{(\mathfrak{s}_i;\mathfrak{Q}_i)\}_{1\le i\le n}$ *satisfies the following conditions:*

(LW) *For any weight* $w' < w$*, the values* $\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta)$ *with* $(\mathfrak{t};\mathfrak{Q}) \in I$ *and* $w(\mathfrak{t}) = w'$ *are all K-linearly independent. In particular,* $\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta)$ *is always nonzero.*

(LD) *There exist* $a \in A$ *and* $a_i \in A$ *for* $1 \le i \le n$ *which are not all zero such that*

$$a + \sum_{i=1}^{n} a_i \mathfrak{L}(\mathfrak{s}_i;\mathfrak{Q}_i)(\theta) = 0.$$

For all $(\mathfrak{t};\mathfrak{Q}) \in I$*, we set the following series in t*

$$f_{\mathfrak{t};\mathfrak{Q}} := \sum_i a_i(t)\mathfrak{L}(s_{i(k+1)},\dots,s_{i\ell_i};Q_{i(k+1)},\dots,Q_{i\ell_i}), \tag{3.8}$$

where the sum runs through the set of indices i such that $(\mathfrak{t};\mathfrak{Q}) = (s_{i1},\dots,s_{ik};Q_{i1},\dots,Q_{ik})$ *for some* $0 \le k \le \ell_i - 1$.

Then for all $(\mathfrak{t};\mathfrak{Q}) \in I$*,* $f_{\mathfrak{t};\mathfrak{Q}}(\theta)$ *belongs to K.*

Remark 3.5. 1) Here, we note that LW stands for Lower Weights and LD for Linear Dependence.

2) With the above notation, we have

$$f_{\emptyset} = \sum_i a_i(t)\mathfrak{L}(\mathfrak{s}_i;\mathfrak{Q}_i).$$

2) In fact, we improve [31, Theorem B] in two directions. First, we remove the restriction to Anderson–Thakur polynomials and tuples $\mathfrak{s}_i$. Second, and more importantly, we allow an additional term a, which is crucial in the sequel. More precisely, in the case of MZV's, while [31, Theorem B] investigates linear relations between MZV's of weight w, Theorem 3.4 investigates linear relations between MZV's of weight w and suitable powers $\widetilde{\pi}^w$ of the Carlitz period.

Proof. The proof will be divided into two steps.

Step 1. We first construct a dual t-motive to which we will apply the Anderson–Brownawell–Papanikolas criterion. We recall $a_i(t) := a_i|_{\theta=t} \in \mathbb{F}_q[t]$.

For each pair $(\mathfrak{s}_i;\mathfrak{Q}_i)$, we have attached to it a matrix $\Phi_{\mathfrak{s}_i,\mathfrak{Q}_i}$. For $\mathfrak{s}_i = (s_{i1},\dots,s_{i\ell_i}) \in \mathbb{N}^{\ell_i}$ and $\mathfrak{Q}_i = (Q_{i1},\dots,Q_{i\ell_i}) \in (\mathbb{F}_q^\times)^{\ell_i}$ we recall

$$I(\mathfrak{s}_i;\mathfrak{Q}_i) = \{\emptyset,(s_{i1};Q_{i1}),\dots,(s_{i1},\dots,s_{i(\ell_i-1)};Q_{i1},\dots,Q_{(\ell_i-1)})\},$$

and $I := \cup_i I(\mathfrak{s}_i;\mathfrak{Q}_i)$.

We now construct a new matrix Φ' indexed by elements of I, say

$$\Phi' = \left(\Phi'_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')}\right)_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')\in I} \in \mathrm{Mat}_{|I|}(\overline{K}[t]).$$

For the row which corresponds to the empty pair $\emptyset$, we put

$$\Phi'_{\emptyset,(\mathfrak{t}';\mathfrak{Q}')} = \begin{cases} (t-\theta)^w & \text{if } (\mathfrak{t}';\mathfrak{Q}') = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

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For the row indexed by $(\mathfrak{t};\mathfrak{Q}) = (s_{i1},\dots,s_{ij};Q_{i1},\dots,Q_{ij})$ for some i and $1 \le j \le \ell_i - 1$, we put

$$\Phi'_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')} = \begin{cases} (t-\theta)^{w-w(\mathfrak{t}')} & \text{if } (\mathfrak{t}';\mathfrak{Q}') = (\mathfrak{t};\mathfrak{Q}), \\ Q_{ij}^{(-1)}(t-\theta)^{w-w(\mathfrak{t}')} & \text{if } (\mathfrak{t}';\mathfrak{Q}') = (s_{i1},\dots,s_{i(j-1)};Q_{i1},\dots,Q_{i(j-1)}), \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\Phi'_{\mathfrak{s}_i,\mathfrak{Q}_i} = \left(\Phi'_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')}\right)_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')\in I(\mathfrak{s}_i;\mathfrak{Q}_i)}$ for all i.

We define $\Phi \in \mathrm{Mat}_{|I|+1}(\overline{K}[t])$ by

$$\Phi = \begin{pmatrix} \Phi' & 0 \\ \mathbf{v} & 1 \end{pmatrix} \in \mathrm{Mat}_{|I|+1}(\overline{K}[t]), \quad \mathbf{v} = (v_{\mathfrak{t},\mathfrak{Q}})_{(\mathfrak{t};\mathfrak{Q})\in I} \in \mathrm{Mat}_{1\times|I|}(\overline{K}[t]),$$

where

$$v_{\mathfrak{t};\mathfrak{Q}} = \sum_i a_i(t) Q_{i\ell_i}^{(-1)}(t-\theta)^{w-w(\mathfrak{t})}.$$

Here, the sum runs through the set of indices i such that $(\mathfrak{t};\mathfrak{Q}) = (s_{i1},\dots,s_{i(\ell_i-1)};Q_{i1},\dots,Q_{i(\ell_i-1)})$, and the empty sum is defined to be zero.

We now introduce a rigid analytic trivialization matrix Ψ for Φ. We define $\Psi' = \left(\Psi'_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')}\right)_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')\in I} \in \mathrm{GL}_{|I|}(\mathbb{T})$ as follows. For the row which corresponds to the empty pair $\emptyset$, we define

$$\Psi'_{\emptyset,(\mathfrak{t}';\mathfrak{Q}')} = \begin{cases} \Omega^w & \text{if } (\mathfrak{t}';\mathfrak{Q}') = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

For the row indexed by $(\mathfrak{t};\mathfrak{Q}) = (s_{i1},\dots,s_{ij};Q_{i1},\dots,Q_{ij})$ for some i and $1 \le j \le \ell_i - 1$, we put

$$\Psi'_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')} = \begin{cases} \mathfrak{L}(\mathfrak{t};\mathfrak{Q})\Omega^{w-w(\mathfrak{t})} & \text{if } (\mathfrak{t}';\mathfrak{Q}') = \emptyset, \\ \mathfrak{L}(s_{i(k+1)},\dots,s_{ij}; \\ \quad Q_{i(k+1)},\dots,Q_{ij})\Omega^{w-w(\mathfrak{t})} & \text{if } (\mathfrak{t}';\mathfrak{Q}') = (s_{i1},\dots,s_{ik};Q_{i1},\dots,Q_{ik}) \text{ for some } 1 \le k \le j, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\Psi'_{\mathfrak{s}_i,\mathfrak{Q}_i} = \left(\Psi'_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')}\right)_{(\mathfrak{t};\mathfrak{Q}),(\mathfrak{t}';\mathfrak{Q}')\in I(\mathfrak{s}_i;\mathfrak{Q}_i)}$ for all i.

We define $\Psi \in \mathrm{GL}_{|I|+1}(\mathbb{T})$ by

$$\Psi = \begin{pmatrix} \Psi' & 0 \\ \mathbf{f} & 1 \end{pmatrix} \in \mathrm{GL}_{|I|+1}(\mathbb{T}), \quad \mathbf{f} = (f_{\mathfrak{t};\mathfrak{Q}})_{\mathfrak{t}\in I} \in \mathrm{Mat}_{1\times|I|}(\mathbb{T}).$$

Here, we recall (see Equation (3.8))

$$f_{\mathfrak{t};\mathfrak{Q}} = \sum_i a_i(t)\mathfrak{L}(s_{i(k+1)},\dots,s_{i\ell_i};Q_{i(k+1)},\dots,Q_{i\ell_i}),$$

where the sum runs through the set of indices i such that $(\mathfrak{t};\mathfrak{Q}) = (s_{i1},\dots,s_{ik};Q_{i1},\dots,Q_{ik})$ for some $0 \le k \le \ell_i - 1$. In particular, $f_\emptyset = \sum_i a_i(t)\mathfrak{L}(\mathfrak{s}_i;\mathfrak{Q}_i)$.

By construction and by §3.2, we get $\Psi^{(-1)} = \Phi\Psi$, that means Ψ is a rigid analytic trivialization for Φ.

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Step 2. Next, we apply the Anderson–Brownawell–Papanikolas criterion (see Theorem 3.2) to prove Theorem 3.4.

In fact, we define

$$\widetilde{\Phi} = \begin{pmatrix} 1 & 0 \\ 0 & \Phi \end{pmatrix} \in \mathrm{Mat}_{|I|+2}(\overline{K}[t])$$

and consider the vector constructed from the first column vector of Ψ

$$\widetilde{\psi} = \begin{pmatrix} 1 \\ \Psi'_{(\mathfrak{t};\mathfrak{Q}),\emptyset} \\ f_\emptyset \end{pmatrix}_{(\mathfrak{t};\mathfrak{Q})\in I}.$$

Then we have $\widetilde{\psi}^{(-1)} = \widetilde{\Phi}\widetilde{\psi}$.

We also observe that for all $(\mathfrak{t};\mathfrak{Q}) \in I$ we have $\Psi'_{(\mathfrak{t};\mathfrak{Q}),\emptyset} = \mathfrak{L}(\mathfrak{t};\mathfrak{Q})\Omega^{w-w(\mathfrak{t})}$. Further,

$$a + f_\emptyset(\theta) = a + \sum_i a_i \mathfrak{L}(\mathfrak{s}_i;\mathfrak{Q}_i)(\theta) = 0.$$

By Theorem 3.2 with $\rho = (a, 0, \dots, 0, 1)$, we deduce that there exists $\mathbf{h} = (g_0, g_{\mathfrak{t},\mathfrak{Q}}, g) \in \mathrm{Mat}_{1\times(|I|+2)}(\overline{K}[t])$ such that $\mathbf{h}\psi = 0$ and that $g_{\mathfrak{t},\mathfrak{Q}}(\theta) = 0$ for $(\mathfrak{t},\mathfrak{Q}) \in I$, $g_0(\theta) = a$ and $g(\theta) = 1 \neq 0$. If we put $\mathbf{g} := (1/g)\mathbf{h} \in \mathrm{Mat}_{1\times(|I|+2)}(\overline{K}(t))$, then all the entries of $\mathbf{g}$ are regular at $t = \theta$.

Now, we have

$$(\mathbf{g} - \mathbf{g}^{(-1)}\widetilde{\Phi})\widetilde{\psi} = \mathbf{g}\widetilde{\psi} - (\mathbf{g}\widetilde{\psi})^{(-1)} = 0. \tag{3.9}$$

We write $\mathbf{g} - \mathbf{g}^{(-1)}\widetilde{\Phi} = (B_0, B_{\mathfrak{t},\mathfrak{Q}}, 0)_{\mathfrak{t}\in I}$. We claim that $B_0 = 0$ and $B_{\mathfrak{t},\mathfrak{Q}} = 0$ for all $(\mathfrak{t};\mathfrak{Q}) \in I$. In fact, expanding Equation (3.9) we obtain

$$B_0 + \sum_{\mathfrak{t}\in I} B_{\mathfrak{t},\mathfrak{Q}} \mathfrak{L}(\mathfrak{t};\mathfrak{Q})\Omega^{w-w(\mathfrak{t})} = 0. \tag{3.10}$$

By Equation (3.3), we see that for $(\mathfrak{t};\mathfrak{Q}) \in I$ and $j \in \mathbb{N}$,

$$\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta^{q^j}) = (\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta))^{q^j}$$

which is nonzero by Condition (LW).

First, as the function Ω has a simple zero at $t = \theta^{q^k}$ for $k \in \mathbb{N}$, specializing Equation (3.10) at $t = \theta^{q^j}$ yields $B_0(\theta^{q^j}) = 0$ for $j \geq 1$. Since B_0 belongs to $\overline{K}(t)$, it follows that $B_0 = 0$.

Next, we put $w_0 := \max_{(\mathfrak{t};\mathfrak{Q})\in I} w(\mathfrak{t})$ and denote by $I(w_0)$ the set of $(\mathfrak{t};\mathfrak{Q}) \in I$ such that $w(\mathfrak{t}) = w_0$. Then dividing Equation (3.10) by Ω^{w-w_0} yields

$$\sum_{(\mathfrak{t};\mathfrak{Q})\in I} B_{\mathfrak{t},\mathfrak{Q}} \mathfrak{L}(\mathfrak{t};\mathfrak{Q})\Omega^{w_0-w(\mathfrak{t})} = \sum_{(\mathfrak{t};\mathfrak{Q})\in I(w_0)} B_{\mathfrak{t},\mathfrak{Q}} \mathfrak{L}(\mathfrak{t};\mathfrak{Q}) + \sum_{(\mathfrak{t};\mathfrak{Q})\in I\setminus I(w_0)} B_{\mathfrak{t},\mathfrak{Q}} \mathfrak{L}(\mathfrak{t};\mathfrak{Q})\Omega^{w_0-w(\mathfrak{t})} = 0. \tag{3.11}$$

Since each $B_{\mathfrak{t},\mathfrak{Q}}$ belongs to $\overline{K}(t)$, they are defined at $t = \theta^{q^j}$ for $j \gg 1$. Note that the function Ω has a simple zero at $t = \theta^{q^k}$ for $k \in \mathbb{N}$. Specializing Equation (3.11) at $t = \theta^{q^j}$ and using Equation (3.3) yields

$$\sum_{(\mathfrak{t};\mathfrak{Q})\in I(w_0)} B_{\mathfrak{t},\mathfrak{Q}}(\theta^{q^j})(\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta))^{q^j} = 0$$

for $j \gg 1$.

We claim that $B_{\mathfrak{t},\mathfrak{Q}}(\theta^{q^j}) = 0$ for $j \gg 1$ and for all $(\mathfrak{t};\mathfrak{Q}) \in I(w_0)$. Otherwise, we get a nontrivial $\overline{K}$-linear relation between $\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta)$ with $(\mathfrak{t};\mathfrak{Q}) \in I$ of weight w_0. By Proposition 3.3, we deduce a nontrivial K-linear relation between $\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta)$ with $(\mathfrak{t};\mathfrak{Q}) \in I(w_0)$, which contradicts with Condition (LW). Now, we know that $B_{\mathfrak{t},\mathfrak{Q}}(\theta^{q^j}) = 0$ for $j \gg 1$ and for all $(\mathfrak{t};\mathfrak{Q}) \in I(w_0)$. Since each $B_{\mathfrak{t},\mathfrak{Q}}$ belongs to $\overline{K}(t)$, it follows that $B_{\mathfrak{t},\mathfrak{Q}} = 0$ for all $(\mathfrak{t};\mathfrak{Q}) \in I(w_0)$.

Next, we put $w_1 := \max_{(\mathfrak{t};\mathfrak{Q}) \in I \setminus I(w_0)} w(\mathfrak{t})$ and denote by $I(w_1)$ the set of $(\mathfrak{t};\mathfrak{Q}) \in I$ such that $w(\mathfrak{t}) = w_1$. Dividing Equation (3.10) by Ω^{w-w_1} and specializing at $t = \theta^{q^j}$ yields

$$\sum_{(\mathfrak{t};\mathfrak{Q}) \in I(w_1)} B_{\mathfrak{t},\mathfrak{Q}}(\theta^{q^j})(\mathfrak{L}(\mathfrak{t};\mathfrak{Q})(\theta))^{q^j} = 0$$

for $j \gg 1$. Since $w_1 < w$, by Proposition 3.3 and Condition (LW) again we deduce that $B_{\mathfrak{t},\mathfrak{Q}}(\theta^{q^j}) = 0$ for $j \gg 1$ and for all $(\mathfrak{t};\mathfrak{Q}) \in I(w_1)$. Since each $B_{\mathfrak{t},\mathfrak{Q}}$ belongs to $\overline{K}(t)$, it follows that $B_{\mathfrak{t},\mathfrak{Q}} = 0$ for all $(\mathfrak{t};\mathfrak{Q}) \in I(w_1)$. Repeating the previous arguments, we deduce that $B_{\mathfrak{t},\mathfrak{Q}} = 0$ for all $(\mathfrak{t};\mathfrak{Q}) \in I$ as required.

We have proved that $\mathbf{g} - \mathbf{g}^{(-1)}\widetilde{\Phi} = 0$. Thus,

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ g_0/g & (g_{\mathfrak{t},\mathfrak{Q}}/g)_{(\mathfrak{t};\mathfrak{Q})\in I} & 1 \end{pmatrix}^{(-1)} \begin{pmatrix} 1 & 0 \\ 0 & \Phi \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \Phi' & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \mathrm{Id} & 0 \\ g_0/g & (g_{\mathfrak{t},\mathfrak{Q}}/g)_{(\mathfrak{t};\mathfrak{Q})\in I} & 1 \end{pmatrix}.$$

By [11, Prop. 2.2.1], we see that the common denominator b of g_0/g and $g_{\mathfrak{t},\mathfrak{Q}}/g$ for $(\mathfrak{t},\mathfrak{Q}) \in I$ belongs to $\mathbb{F}_q[t] \setminus \{0\}$. If we put $\delta_0 = bg_0/g$ and $\delta_{\mathfrak{t},\mathfrak{Q}} = bg_{\mathfrak{t},\mathfrak{Q}}/g$ for $(\mathfrak{t},\mathfrak{Q}) \in I$ which belong to $\overline{K}[t]$ and $\delta := (\delta_{\mathfrak{t},\mathfrak{Q}})_{\mathfrak{t}\in I} \in \mathrm{Mat}_{1\times|I|}(\overline{K}[t])$, then $\delta_0^{(-1)} = \delta_0$ and

$$\begin{pmatrix} \mathrm{Id} & 0 \\ \delta & 1 \end{pmatrix}^{(-1)} \begin{pmatrix} \Phi' & 0 \\ b\mathbf{v} & 1 \end{pmatrix} = \begin{pmatrix} \Phi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ \delta & 1 \end{pmatrix}. \tag{3.12}$$

If we put $X := \begin{pmatrix} \mathrm{Id} & 0 \\ \delta & 1 \end{pmatrix} \begin{pmatrix} \Psi' & 0 \\ b\mathbf{f} & 1 \end{pmatrix}$, then $X^{(-1)} = \begin{pmatrix} \Phi' & 0 \\ 0 & 1 \end{pmatrix} X$. By [32, §4.1.6], there exist $\nu_{\mathfrak{t},\mathfrak{Q}} \in \mathbb{F}_q(t)$ for $(\mathfrak{t},\mathfrak{Q}) \in I$ such that if we set $\nu = (\nu_{\mathfrak{t},\mathfrak{Q}})_{(\mathfrak{t},\mathfrak{Q})\in I} \in \mathrm{Mat}_{1\times|I|}(\mathbb{F}_q(t))$,

$$X = \begin{pmatrix} \Psi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ \nu & 1 \end{pmatrix}.$$

Thus, the equation $\begin{pmatrix} \mathrm{Id} & 0 \\ \delta & 1 \end{pmatrix} \begin{pmatrix} \Psi' & 0 \\ b\mathbf{f} & 1 \end{pmatrix} = \begin{pmatrix} \Psi' & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{Id} & 0 \\ \nu & 1 \end{pmatrix}$ implies

$$\delta\Psi' + b\mathbf{f} = \nu. \tag{3.13}$$

The left-hand side belongs to $\mathbb{T}$, so does the right-hand side. Thus, $\nu = (\nu_{\mathfrak{t},\mathfrak{Q}})_{(\mathfrak{t},\mathfrak{Q})\in I} \in \mathrm{Mat}_{1\times|I|}(\mathbb{F}_q[t])$. For any $j \in \mathbb{N}$, by specializing Equation (3.13) at $t = \theta^{q^j}$ and using Equation (3.3) and the fact that Ω has a simple zero at $t = \theta^{q^j}$ we deduce that

$$\mathbf{f}(\theta) = \nu(\theta)/b(\theta).$$

Thus, for all $(\mathfrak{t},\mathfrak{Q}) \in I$, $f_{\mathfrak{t};\mathfrak{Q}}(\theta)$ given as in Equation (3.8) belongs to K. □

4. Linear relations between ACMPL's

In this section, we use freely the notation of §2 and §3.2.3.

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4.1. Preliminaries

We begin this section by proving several auxiliary lemmas which will be useful in the sequel. We recall that $\overline{\mathbb{F}}_q$ denotes the algebraic closure of $\mathbb{F}_q$ in $\overline{K}$.

Lemma 4.1. *Let $\epsilon_i \in \mathbb{F}_q^\times$ be different elements. We denote by $\gamma_i \in \overline{\mathbb{F}}_q$ a $(q-1)$-th root of ϵ_i. Then γ_i are all $\mathbb{F}_q$-linearly independent.*

Proof. We know that $\mathbb{F}_q^\times$ is cyclic as a multiplicative group. Let ϵ be a generating element of $\mathbb{F}_q^\times$ so that $\mathbb{F}_q^\times = \langle \epsilon \rangle$. Let γ be the associated $(q-1)$-th root of ϵ. Then for all $1 \le i \le q-1$ it follows that γ^i is a $(q-1)$-th root of ϵ^i. Thus, it suffices to show that the polynomial $P(X) = X^{q-1} - \epsilon$ is irreducible in $\mathbb{F}_q[X]$. Suppose that this is not the case, write $P(X) = P_1(X)P_2(X)$ with $1 \le \deg P_1 < q-1$. Since the roots of $P(X)$ are of the form $\alpha\gamma$ with $\alpha \in \mathbb{F}_q^\times$, those of $P_1(X)$ are also of this form. Looking at the constant term of $P_1(X)$, we deduce that $\gamma^{\deg P_1} \in \mathbb{F}_q^\times$. If we put $m = \gcd(\deg P_1, q-1)$, then $1 \le m < q-1$ and $\gamma^m \in \mathbb{F}_q^\times$. Letting $\beta := \gamma^m \in \mathbb{F}_q^\times$, we get $\beta^{\frac{q-1}{m}} = \gamma^{q-1} = \epsilon$. Since $1 \le m < q-1$, we get a contradiction with the fact that $\mathbb{F}_q^\times = \langle \epsilon \rangle$. The proof is finished. □

Lemma 4.2. *Let* $\operatorname{Li}\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AL}_w$ *and $a_i \in K$ satisfying*

$$\sum_i a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0.$$

For $\epsilon \in \mathbb{F}_q^\times$, we denote by $I(\epsilon) = \{i : \chi(\boldsymbol{\epsilon}_i) = \epsilon\}$ the set of indices whose corresponding character equals ϵ. Then for all $\epsilon \in \mathbb{F}_q^\times$,

$$\sum_{i \in I(\epsilon)} a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0.$$

Proof. We keep the notation of Lemma 4.1. Suppose that we have a relation

$$\sum_i \gamma_i a_i = 0$$

with $a_i \in K_\infty$. By Lemma 4.1 and the fact that $K_\infty = \mathbb{F}_q((1/\theta))$, we deduce that $a_i = 0$ for all i.

By Equation (3.7), the relation $\sum_i a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0$ is equivalent to the following one

$$\sum_i a_i \gamma_{i1} \dots \gamma_{i\ell_i} \operatorname{Li}\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{s}_i \end{pmatrix} = 0.$$

By the previous discussion, for all $\epsilon \in \mathbb{F}_q^\times$,

$$\sum_{i \in I(\epsilon)} a_i \gamma_{i1} \dots \gamma_{i\ell_i} \operatorname{Li}\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{s}_i \end{pmatrix} = 0.$$

By Equation (3.7), again we deduce the desired relation

$$\sum_{i \in I(\epsilon)} a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0.$$

□

Lemma 4.3. *Let $m \in \mathbb{N}$, $\varepsilon \in \mathbb{F}_q^\times$, $\delta \in \overline{K}[t]$ and $F(t,\theta) \in \overline{\mathbb{F}}_q[t,\theta]$ (resp. $F(t,\theta) \in \mathbb{F}_q[t,\theta]$) satisfying*

$$\varepsilon\delta = \delta^{(-1)}(t-\theta)^m + F^{(-1)}(t,\theta).$$

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Then $\delta \in \overline{\mathbb{F}}_q[t,\theta]$ *(resp.* $\delta \in \mathbb{F}_q[t,\theta]$*) and*

$$\deg_\theta \delta \le \max\left\{\frac{qm}{q-1}, \frac{\deg_\theta F(t,\theta)}{q}\right\}.$$

Proof. The proof follows the same line as that of [27, Theorem 2] where it is shown that if $F(t,\theta) \in \mathbb{F}_q[t,\theta]$ and $\varepsilon = 1$, then $\delta \in \mathbb{F}_q[t,\theta]$. We write down the proof for the case $F(t,\theta) \in \overline{\mathbb{F}}_q[t,\theta]$ for the convenience of the reader.

By twisting once the equality $\varepsilon\delta = \delta^{(-1)}(t-\theta)^m + F^{(-1)}(t,\theta)$ and the fact that $\varepsilon^q = \varepsilon$, we get

$$\varepsilon\delta^{(1)} = \delta(t-\theta^q)^m + F(t,\theta).$$

We put $n = \deg_t \delta$ and express

$$\delta = a_n t^n + \cdots + a_1 t + a_0 \in \overline{K}[t]$$

with $a_0, \ldots, a_n \in \overline{K}$. For $i < 0$, we put $a_i = 0$.

Since $\deg_t \delta^{(1)} = \deg_t \delta = n < \delta(t-\theta^q)^m = n+m$, it follows that $\deg_t F(t,\theta) = n+m$. Thus, we write $F(t,\theta) = b_{n+m}t^{n+m} + \cdots + b_1 t + b_0$ with $b_0, \ldots, b_{n+m} \in \overline{\mathbb{F}}_q[\theta]$. Plugging into the previous equation, we obtain

$$\varepsilon(a_n^q t^n + \cdots + a_0^q) = (a_n t^n + \cdots + a_0)(t-\theta^q)^m + b_{n+m}t^{n+m} + \cdots + b_0.$$

Comparing the coefficients t^j for $n+1 \le j \le n+m$ yields

$$a_{j-m} + \sum_{i=j-m+1}^{n} \binom{m}{j-i}(-\theta^q)^{m-j+i} a_i + b_j = 0.$$

Since $b_j \in \overline{\mathbb{F}}_q[\theta]$ for all $n+1 \le j \le n+m$, we can show by descending induction that $a_j \in \overline{\mathbb{F}}_q[\theta]$ for all $n+1-m \le j \le n$.

If $n+1-m \le 0$, then we are done. Otherwise, comparing the coefficients t^j for $m \le j \le n$ yields

$$a_{j-m} + \sum_{i=j-m+1}^{n} \binom{m}{j-i}(-\theta^q)^{m-j+i} a_i + b_j - \varepsilon a_j^q = 0.$$

Since $b_j \in \overline{\mathbb{F}}_q[\theta]$ for all $m \le j \le n$ and $a_j \in \overline{\mathbb{F}}_q[\theta]$ for all $n+1-m \le j \le n$, we can show by descending induction that $a_j \in \overline{\mathbb{F}}_q[\theta]$ for all $0 \le j \le n-m$. We conclude that $\delta \in \overline{\mathbb{F}}_q[t,\theta]$.

We now show that $\deg_\theta \delta \le \max\{\frac{qm}{q-1}, \frac{\deg_\theta F(t,\theta)}{q}\}$. Otherwise, suppose that $\deg_\theta \delta > \max\{\frac{qm}{q-1}, \frac{\deg_\theta F(t,\theta)}{q}\}$. Then $\deg_\theta \delta^{(1)} = q \deg_\theta \delta$. It implies that $\deg_\theta \delta^{(1)} > \deg_\theta(\delta(t-\theta^q)^m) = \deg_\theta \delta + qm$ and $\deg_\theta \delta^{(1)} > \deg_\theta F(t,\theta)$. Hence, we get

$$\deg_\theta(\varepsilon\delta^{(1)}) = \deg_\theta \delta^{(1)} > \deg_\theta(\delta(t-\theta^q)^m + F(t,\theta)),$$

which is a contradiction. ◻

4.2. Linear relations: statement of the main result

Theorem 4.4. *Let $w \in \mathbb{N}$. We recall that the set $\mathcal{J}'_w$ consists of positive tuples $\mathfrak{s} = (s_1, \dots, s_n)$ of weight w such that $q \nmid s_i$ for all i. Suppose that we have a nontrivial relation*

$$a + \sum_{\mathfrak{s}_i \in \mathcal{J}'_w} a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0, \quad \textit{for } a, a_i \in K.$$

Then $q - 1 \mid w$ and $a \neq 0$.

Further, if $q - 1 \mid w$, then there is a unique relation

$$1 + \sum_{\mathfrak{s}_i \in \mathcal{J}'_w} a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0, \quad \textit{for } a_i \in K.$$

Also, for indices $(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)$ with nontrivial coefficient a_i, we have $\boldsymbol{\epsilon}_i = (1, \dots, 1)$.

In particular, the ACMPL's in $\mathcal{AS}_w$ are linearly independent over K.

Remark 4.5. We emphasize that although Theorem 4.4 is a purely transcendental result, it is crucial that we need the full strength of algebraic theory for ACMPL's (i.e., Theorem 2.11) to conclude (see the last step of the proof).

As a direct consequence of Theorem 4.4, we obtain:

Theorem 4.6. *Let $w \in \mathbb{N}$. Then the ACMPL's in $\mathcal{AS}_w$ form a basis for $\mathcal{AL}_w$. In particular,*

$$\dim_K \mathcal{AL}_w = s(w).$$

Proof. By Theorem 4.4, the ACMPL's in $\mathcal{AS}_w$ are all linearly independent over K. Then by Theorem 2.11, we deduce that the ACMPL's in $\mathcal{AS}_w$ form a basis for $\mathcal{AL}_w$. Hence, $\dim_K \mathcal{AL}_w = |\mathcal{AS}_w| = s(w)$ as required. □

4.3. Proof of Theorem 4.4

We outline the ideas of the proof. Starting from such a nontrivial relation, we apply the Anderson–Brownawell–Papanikolas criterion in [2] and reduce to the solution of a system of σ-linear equations. In contrast to [31, §4 and §5], this system has a unique solution when $q - 1$ divides w. We first show that for such a weight w up to a scalar in $K^\times$ there is at most one linear relation between ACMPL's in $\mathcal{AS}_w$ and $\widetilde{\pi}^w$. Second, we show a linear relation between ACMPL's in $\mathcal{AS}_w$ and $\widetilde{\pi}^w$ where the coefficient of $\widetilde{\pi}^w$ is nonzero. For this, we use Brown's theorem for AMCPLs, that is, Theorem 2.11.

We are back to the proof of Theorem 4.4. We claim that if $q - 1 \nmid w$, then any linear relation

$$a + \sum_{\mathfrak{s}_i \in \mathcal{J}'_w} a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0$$

with $a, a_i \in K$ implies that $a = 0$. In fact, if we recall that $\overline{\mathbb{F}}_q$ denotes the algebraic closure of $\mathbb{F}_q$ in $\overline{K}$, then the claim follows from Equation (3.7) and that $\widetilde{\pi}^w \notin \overline{\mathbb{F}}_q\left(\left(\frac{1}{\theta}\right)\right)$ since $q - 1 \nmid w$.

The proof is by induction on the weight $w \in \mathbb{N}$. For $w = 1$, we distinguish two cases:

- If $q > 2$, then by the previous remark it suffices to show that if

$$a + \sum_i a_i \mathfrak{Li}(1; \epsilon_i)(\theta) = 0,$$

then $a_i = 0$ for all i. In fact, it follows immediately from Lemma 4.2.

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- If $q = 2$, then $w = q - 1 = 1$. Then the theorem holds from the facts that there is only one index $(\mathfrak{s}_1; \boldsymbol{\epsilon}_1) = (1, 1)$ and that $\mathrm{Li}(1) = \zeta_A(1) = -D_1^{-1}\widetilde{\pi}$.

Suppose that Theorem 4.4 holds for all $w' < w$. We now prove that it holds for w. Suppose that we have a linear relation

$$a + \sum_i a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0. \tag{4.1}$$

By Lemma 4.2 and its proof, we can suppose further that $\boldsymbol{\epsilon}_i$ has the same character, that is, there exists $\epsilon \in \mathbb{F}_q^\times$ such that for all i,

$$\chi(\boldsymbol{\epsilon}_i) = \epsilon_{i1} \dots \epsilon_{i\ell_i} = \epsilon. \tag{4.2}$$

We now apply Theorem 3.4 to our setting of ACMPL's. We recall that by Equation (3.6),

$$\mathfrak{Li}(\mathfrak{s}; \boldsymbol{\epsilon}) = \mathfrak{L}(\mathfrak{s}; \mathfrak{Q}'_{\mathfrak{s},\boldsymbol{\epsilon}}),$$

and also

$$\begin{aligned} I(\mathfrak{s}_i; \boldsymbol{\epsilon}_i) &= \{\emptyset, (s_{i1}; \epsilon_{i1}), \dots, (s_{i1}, \dots, s_{i(\ell_i-1)}; \epsilon_{i1}, \dots, \epsilon_{i(\ell_i-1)})\}, \\ I &= \cup_i I(\mathfrak{s}_i; \boldsymbol{\epsilon}_i). \end{aligned} \tag{4.3}$$

We know that the hypothesis are verified:

(LW) By the induction hypothesis, for any weight $w' < w$, the values $\mathfrak{Li}(\mathfrak{t}; \boldsymbol{\epsilon})(\theta)$ with $(\mathfrak{t}; \boldsymbol{\epsilon}) \in I$ and $w(\mathfrak{t}) = w'$ are all K-linearly independent.

(LD) By Equation (4.1), there exist $a \in A$ and $a_i \in A$ for $1 \le i \le n$ which are not all zero such that

$$a + \sum_{i=1}^n a_i \mathfrak{Li}(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)(\theta) = 0.$$

Thus, Theorem 3.4 implies that for all $(\mathfrak{t}; \boldsymbol{\epsilon}) \in I$, $f_{\mathfrak{t};\boldsymbol{\epsilon}}(\theta)$ belongs to K where $f_{\mathfrak{t};\boldsymbol{\epsilon}}$ is given by

$$f_{\mathfrak{t};\boldsymbol{\epsilon}} := \sum_i a_i(t) \mathfrak{Li}(s_{i(k+1)}, \dots, s_{i\ell_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{i\ell_i}).$$

Here, the sum runs through the set of indices i such that $(\mathfrak{t}; \boldsymbol{\epsilon}) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$ for some $0 \le k \le \ell_i - 1$.

We derive a direct consequence of the previous rationality result. Let $(\mathfrak{t}; \boldsymbol{\epsilon}) \in I$ and $\mathfrak{t} \neq \emptyset$. Then $(\mathfrak{t}; \boldsymbol{\epsilon}) = (s_{i1}, \dots, s_{ik}; \epsilon_{i1}, \dots, \epsilon_{ik})$ for some i and $1 \le k \le \ell_i - 1$. We denote by $J(\mathfrak{t}; \boldsymbol{\epsilon})$ the set of all such i. We know that there exists $a_{\mathfrak{t};\boldsymbol{\epsilon}} \in K$ such that

$$a_{\mathfrak{t};\boldsymbol{\epsilon}} + f_{\mathfrak{t};\boldsymbol{\epsilon}}(\theta) = 0,$$

or equivalently,

$$a_{\mathfrak{t};\boldsymbol{\epsilon}} + \sum_{i \in J(\mathfrak{t};\boldsymbol{\epsilon})} a_i \mathfrak{Li}(s_{i(k+1)}, \dots, s_{i\ell_i}; \epsilon_{i(k+1)}, \dots, \epsilon_{i\ell_i})(\theta) = 0.$$

The ACMPL's appearing in the above equality belong to $\mathcal{AS}_{w-w(\mathfrak{t})}$. By the induction hypothesis, we can suppose that $\epsilon_{i(k+1)} = \dots = \epsilon_{i\ell_i} = 1$. Further, if $q - 1 \nmid w - w(\mathfrak{t})$, then $a_i(t) = 0$ for all $i \in J(\mathfrak{t}; \boldsymbol{\epsilon})$.

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Therefore, letting $(\mathfrak{s}_i; \boldsymbol{\epsilon}_i) = (s_{i1}, \dots, s_{i\ell_i}; \epsilon_{i1}, \dots, \epsilon_{i\ell_i})$ we can suppose that $s_{i2}, \dots, s_{i\ell_i}$ are all divisible by $q-1$ and $\epsilon_{i2} = \dots = \epsilon_{i\ell_i} = 1$. In particular, for all i, $\epsilon_{i1} = \chi(\boldsymbol{\epsilon}_i) = \epsilon$.

Now, we want to solve Equation (3.12). Further, in this system we can assume that the corresponding element $b \in \mathbb{F}_q[t] \setminus \{0\}$ equals 1. We define

$$J := I \cup \{(\mathfrak{s}_i; \boldsymbol{\epsilon}_i)\},$$

where I is given as in Equation (4.3). For $(\mathfrak{t}; \boldsymbol{\epsilon}) \in J$, we denote by $J_0(\mathfrak{t}; \boldsymbol{\epsilon})$ consisting of $(\mathfrak{t}'; \boldsymbol{\epsilon}') \in I$ such that there exist i and $0 \leq j < \ell_i$ so that $(\mathfrak{t}; \boldsymbol{\epsilon}) = (s_{i1}, s_{i2}, \dots, s_{ij}; \epsilon, 1, \dots, 1)$ and $(\mathfrak{t}'; \boldsymbol{\epsilon}') = (s_{i1}, s_{i2}, \dots, s_{i(j+1)}; \epsilon, 1, \dots, 1)$. In particular, for $(\mathfrak{t}; \boldsymbol{\epsilon}) = (\mathfrak{s}_i; \boldsymbol{\epsilon}_i)$, $J_0(\mathfrak{t}; \boldsymbol{\epsilon})$ is the empty set. For $(\mathfrak{t}; \boldsymbol{\epsilon}) \in J \setminus \{\emptyset\}$, we also put

$$m_{\mathfrak{t}} := \frac{w - w(\mathfrak{t})}{q-1} \in \mathbb{Z}^{\geq 0}.$$

Then it is clear that Equation (3.12) is equivalent finding $(\delta_{\mathfrak{t};\boldsymbol{\epsilon}})_{(\mathfrak{t};\boldsymbol{\epsilon}) \in J} \in \mathrm{Mat}_{1 \times |J|}(\overline{K}[t])$ such that

$$\delta_{\mathfrak{t};\boldsymbol{\epsilon}} = \delta_{\mathfrak{t};\boldsymbol{\epsilon}}^{(-1)} (t-\theta)^{w-w(\mathfrak{t})} + \sum_{(\mathfrak{t}';\boldsymbol{\epsilon}') \in J_0(\mathfrak{t};\boldsymbol{\epsilon})} \delta_{\mathfrak{t}';\boldsymbol{\epsilon}'}^{(-1)} (t-\theta)^{w-w(\mathfrak{t})}, \quad \text{for all } (\mathfrak{t}; \boldsymbol{\epsilon}) \in J \setminus \{\emptyset\}, \tag{4.4}$$

and

$$\delta_{\mathfrak{t};\boldsymbol{\epsilon}} = \delta_{\mathfrak{t};\boldsymbol{\epsilon}}^{(-1)} (t-\theta)^{w-w(\mathfrak{t})} + \sum_{(\mathfrak{t}';\boldsymbol{\epsilon}') \in J_0(\mathfrak{t};\boldsymbol{\epsilon})} \delta_{\mathfrak{t}';\boldsymbol{\epsilon}'}^{(-1)} \gamma^{(-1)} (t-\theta)^{w-w(\mathfrak{t})}, \quad \text{for } (\mathfrak{t}; \boldsymbol{\epsilon}) = \emptyset. \tag{4.5}$$

Here, $\gamma^{q-1} = \epsilon$. In fact, for $(\mathfrak{t}; \boldsymbol{\epsilon}) = (\mathfrak{s}_i; \boldsymbol{\epsilon}_i)$, the corresponding equation becomes $\delta_{\mathfrak{s}_i;\boldsymbol{\epsilon}_i} = \delta_{\mathfrak{s}_i;\boldsymbol{\epsilon}_i}^{(-1)}$. Thus, $\delta_{\mathfrak{s}_i;\boldsymbol{\epsilon}_i} = a_i(t) \in \mathbb{F}_q[t]$.

Letting y be a variable, we denote by v_y the valuation associated to the place y of the field $\mathbb{F}_q(y)$. We put

$$T := t - t^q, \quad X := t^q - \theta^q.$$

We claim that

1) For all $(\mathfrak{t}; \boldsymbol{\epsilon}) \in J \setminus \{\emptyset\}$, the polynomial $\delta_{\mathfrak{t};\boldsymbol{\epsilon}}$ is of the form

$$\delta_{\mathfrak{t};\boldsymbol{\epsilon}} = f_{\mathfrak{t}} \left(X^{m_{\mathfrak{t}}} + \sum_{i=0}^{m_{\mathfrak{t}}-1} P_{\mathfrak{t},i}(T) X^i \right),$$

   where
   - $f_{\mathfrak{t}} \in \mathbb{F}_q[t]$,
   - for all $0 \leq i \leq m_{\mathfrak{t}} - 1$, $P_{\mathfrak{t},i}(y)$ belongs to $\mathbb{F}_q(y)$ with $v_y(P_{\mathfrak{t},i}) \geq 1$.

2) For all $\mathfrak{t} \in J \setminus \{\emptyset\}$ and all $\mathfrak{t}' \in J_0(\mathfrak{t})$, there exists $P_{\mathfrak{t},\mathfrak{t}'} \in \mathbb{F}_q(y)$ such that

$$f_{\mathfrak{t}'} = f_{\mathfrak{t}} P_{\mathfrak{t},\mathfrak{t}'}(T).$$

   In particular, if $f_{\mathfrak{t}} = 0$, then $f_{\mathfrak{t}'} = 0$.

The proof is by induction on $m_{\mathfrak{t}}$. We start with $m_{\mathfrak{t}} = 0$. Then $\mathfrak{t} = \mathfrak{s}_i$ and $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_i$ for some i. We have observed that $\delta_{\mathfrak{s}_i;\boldsymbol{\epsilon}_i} = a_i(t) \in \mathbb{F}_q[t]$ and the assertion follows.

Suppose that the claim holds for all $(\mathfrak{t};\boldsymbol{\epsilon}) \in J \setminus \{\emptyset\}$ with $m_{\mathfrak{t}} < m$. We now prove the claim for all $(\mathfrak{t};\boldsymbol{\epsilon}) \in J \setminus \{\emptyset\}$ with $m_{\mathfrak{t}} = m$. In fact, we fix such $\mathfrak{t}$ and want to find $\delta_{\mathfrak{t};\boldsymbol{\epsilon}} \in \overline{K}[t]$ such that

$$\delta_{\mathfrak{t};\boldsymbol{\epsilon}} = \delta_{\mathfrak{t};\boldsymbol{\epsilon}}^{(-1)}(t-\theta)^{(q-1)m} + \sum_{(\mathfrak{t}';\boldsymbol{\epsilon}')\in J_0(\mathfrak{t};\boldsymbol{\epsilon})} \delta_{\mathfrak{t}';\boldsymbol{\epsilon}'}^{(-1)}(t-\theta)^{(q-1)m}. \tag{4.6}$$

By the induction hypothesis, for all $(\mathfrak{t}';\boldsymbol{\epsilon}') \in J_0(\mathfrak{t};\boldsymbol{\epsilon})$, we know that

$$\delta_{\mathfrak{t}';\boldsymbol{\epsilon}'} = f_{\mathfrak{t}'}\left(X^{m_{\mathfrak{t}'}} + \sum_{i=0}^{m_{\mathfrak{t}'}-1} P_{\mathfrak{t}',i}(T)X^i\right),$$

where

- $f_{\mathfrak{t}'} \in \mathbb{F}_q[t]$,
- for all $0 \le i \le m_{\mathfrak{t}'}-1$, $P_{\mathfrak{t}',i}(y) \in \mathbb{F}_q(y)$ with $v_y(P_{\mathfrak{t},i}) \ge 1$.

For $(\mathfrak{t}';\boldsymbol{\epsilon}') \in J_0(\mathfrak{t};\boldsymbol{\epsilon})$, we write $\mathfrak{t}' = (\mathfrak{t},(m-k)(q-1))$ with $0 \le k < m$ and $k \not\equiv m (\mathrm{mod}\ q)$, in particular $m_{\mathfrak{t}'} = k$. We put $f_k = f_{\mathfrak{t}'}$ and $P_{\mathfrak{t}',i} = P_{k,i}$ so that

$$\delta_{\mathfrak{t}';\boldsymbol{\epsilon}'} = f_k\left(X^k + \sum_{i=0}^{k-1} P_{k,i}(T)X^i\right) \in \mathbb{F}_q[t,\theta^q]. \tag{4.7}$$

By Lemma 4.3, $\delta_{\mathfrak{t};\boldsymbol{\epsilon}}$ belongs to $K[t]$, and $\deg_\theta \delta_{\mathfrak{t};\boldsymbol{\epsilon}} \le mq$. Further, since $\delta_{\mathfrak{t};\boldsymbol{\epsilon}}$ is divisible by $(t-\theta)^{(q-1)m}$, we write $\delta_{\mathfrak{t};\boldsymbol{\epsilon}} = F(t-\theta)^{(q-1)m}$ with $F \in K[t]$ and $\deg_\theta F \le m$. Dividing Equation (4.6) by $(t-\theta)^{(q-1)m}$ and twisting once yields

$$F^{(1)} = F(t-\theta)^{(q-1)m} + \sum_{(\mathfrak{t}';\boldsymbol{\epsilon}')\in J_0(\mathfrak{t};\boldsymbol{\epsilon})} \delta_{\mathfrak{t}';\boldsymbol{\epsilon}'}. \tag{4.8}$$

As $\delta_{\mathfrak{t}';\boldsymbol{\epsilon}'} \in \mathbb{F}_q[t,\theta^q]$ for all $(\mathfrak{t}';\boldsymbol{\epsilon}') \in J_0(\mathfrak{t};\boldsymbol{\epsilon})$, it follows that $F(t-\theta)^{(q-1)m} \in \mathbb{F}_q[t,\theta^q]$. As $\deg_\theta F \le m$, we get

$$F = \sum_{0\le i\le m/q} f_{m-iq}(t-\theta)^{m-iq}, \quad \text{for } f_{m-iq} \in \mathbb{F}_q[t].$$

Thus,

$$F(t-\theta)^{(q-1)m} = \sum_{0\le i\le m/q} f_{m-iq}(t-\theta)^{mq-iq} = \sum_{0\le i\le m/q} f_{m-iq}X^{m-i},$$
$$F^{(1)} = \sum_{0\le i\le m/q} f_{m-iq}(t-\theta^q)^{m-iq} = \sum_{0\le i\le m/q} f_{m-iq}(T+X)^{m-iq}.$$

Putting these and Equation (4.7) into Equation (4.8) gets

$$\sum_{0\le i\le m/q} f_{m-iq}(T+X)^{m-iq} = \sum_{0\le i\le m/q} f_{m-iq}X^{m-i} + \sum_{\substack{0\le k<m \\ k\not\equiv m (\mathrm{mod}\ q)}} f_k\left(X^k + \sum_{i=0}^{k-1} P_{k,i}(T)X^i\right).$$

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Comparing the coefficients of powers of X yields the following linear system in the variables $f_0, \ldots, f_{m-1}$:

$$B\Big|_{y=T}\begin{pmatrix} f_{m-1} \\ \vdots \\ f_0 \end{pmatrix} = f_m \begin{pmatrix} Q_{m-1} \\ \vdots \\ Q_0 \end{pmatrix}\Bigg|_{y=T}.$$

Here, for $0 \leq i \leq m-1$, $Q_i = \binom{m}{i} y^{m-i} \in y\mathbb{F}_q[y]$ and $B = (B_{ij})_{0\leq i,j\leq m-1} \in \mathrm{Mat}_m(\mathbb{F}_q(y))$ such that

- $v_y(B_{ij}) \geq 1$ if $i > j$,
- $v_y(B_{ij}) \geq 0$ if $i < j$,
- $v_y(B_{ii}) = 0$ as $B_{ii} = \pm 1$.

The above properties follow from the fact that $P_{k,i} \in \mathbb{F}_q(y)$ and $v_y(P_{k,i}) \geq 1$. Thus, $v_y(\det B) = 0$ so that $\det B \neq 0$. It follows that for all $0 \leq i \leq m-1$, $f_i = f_m P_i(T)$ with $P_i \in \mathbb{F}_q(y)$ and $v_y(P_i) \geq 1$, and we are done.

To conclude, we have to solve Equation (4.4) for $(\mathfrak{t};\boldsymbol{\epsilon}) = \emptyset$. We have some extra work as we have a factor $\gamma^{(-1)}$ on the right-hand side of Equation (4.5). We use $\gamma^{(-1)} = \gamma/\epsilon$ and put $\delta := \delta_{\emptyset,\emptyset}/\gamma \in \overline{K}[t]$. Then we have to solve

$$\epsilon\delta = \delta^{(-1)}(t-\theta)^w + \sum_{(\mathfrak{t}';\boldsymbol{\epsilon}')\in J_0(\emptyset)} \delta^{(-1)}_{\mathfrak{t}';\boldsymbol{\epsilon}'}(t-\theta)^w. \tag{4.9}$$

We distinguish two cases.

4.3.1. Case 1: $q-1 \nmid w$, says $w = m(q-1)+r$ with $0 < r < q-1$

We know that for all $(\mathfrak{t}';\boldsymbol{\epsilon}') \in J_0(\emptyset)$, says $\mathfrak{t}' = ((m-k)(q-1)+r)$ with $0 \leq k \leq m$ and $k \not\equiv m-r \pmod{q}$,

$$\delta_{\mathfrak{t}';\boldsymbol{\epsilon}'} = f_k\left(X^k + \sum_{i=0}^{k-1} P_{k,i}(T)X^i\right) \in \mathbb{F}_q[t,\theta^q], \tag{4.10}$$

where

- $f_k \in \mathbb{F}_q[t]$,
- for all $0 \leq i \leq k-1$, $P_{k,i}(y)$ belongs to $\mathbb{F}_q(y)$ with $v_y(P_{k,i}) \geq 1$.

By Lemma 4.3, δ belongs to $K[t]$. We claim that $\deg_\theta \delta \leq mq$. Otherwise, we have $\deg_\theta \delta_\emptyset > mq$. Twisting Equation (4.9) once gets

$$\epsilon\delta^{(1)} = \delta(t-\theta^q)^w + \sum_{(\mathfrak{t}';\boldsymbol{\epsilon}')\in J_0(\emptyset)} \delta_{\mathfrak{t}';\boldsymbol{\epsilon}'}(t-\theta^q)^w.$$

As $\deg_\theta \delta > mq$, we compare the degrees of θ on both sides and obtain

$$q\deg_\theta \delta = \deg_\theta \delta + wq.$$

Thus, $q-1 \mid w$, which is a contradiction. We conclude that $\deg_\theta \delta \leq mq$.

From Equation (4.9), we see that δ is divisible by $(t-\theta)^w$. Thus, we write $\delta = F(t-\theta)^w$ with $F \in K[t]$ and $\deg_\theta F \leq mq - w = m - r$. Dividing Equation (4.9) by $(t-\theta)^w$ and twisting once yields

$$\epsilon F^{(1)} = F(t-\theta)^w + \sum_{(\mathfrak{t}';\boldsymbol{\epsilon}')\in J_0(\emptyset)} \delta_{\mathfrak{t}'}. \tag{4.11}$$

Since $\delta_{\mathfrak{t}';\boldsymbol{\epsilon}'} \in \mathbb{F}_q[t, \theta^q]$ for all $(\mathfrak{t}'; \boldsymbol{\epsilon}') \in J_0(\emptyset)$, it follows that $F(t-\theta)^w \in \mathbb{F}_q[t, \theta^q]$. As $\deg_\theta F \leq m-r$, we write

$$F = \sum_{0 \leq i \leq (m-r)/q} f_{m-r-iq}(t-\theta)^{m-r-iq}, \quad \text{for } f_{m-r-iq} \in \mathbb{F}_q[t].$$

It follows that

$$\begin{aligned} F(t-\theta)^w &= \sum_{0 \leq i \leq (m-r)/q} f_{m-r-iq}(t-\theta)^{mq-iq} = \sum_{0 \leq i \leq (m-r)/q} f_{m-r-iq} X^{m-i}, \\ F^{(1)} &= \sum_{0 \leq i \leq (m-r)/q} f_{m-r-iq}(t-\theta^q)^{m-r-iq} = \sum_{0 \leq i \leq (m-r)/q} f_{m-r-iq}(T+X)^{m-r-iq}. \end{aligned}$$

Putting these and Equation (4.10) into Equation (4.11) yields

$$\begin{aligned} &\epsilon \sum_{0 \leq i \leq (m-r)/q} f_{m-r-iq}(T+X)^{m-r-iq} \\ &\quad = \sum_{0 \leq i \leq (m-r)/q} f_{m-r-iq} X^{m-i} + \sum_{\substack{0 \leq k \leq m \\ k \not\equiv m-r \pmod q}} f_k \left(X^k + \sum_{i=0}^{k-1} P_{k,i}(T) X^i \right). \end{aligned}$$

Comparing the coefficients of powers of X yields the following linear system in the variables $f_0, \ldots, f_m$:

$$B\Big|_{y=T} \begin{pmatrix} f_m \\ \vdots \\ f_0 \end{pmatrix} = 0.$$

Here, $B = (B_{ij})_{0 \leq i,j \leq m} \in \mathrm{Mat}_{m+1}(\mathbb{F}_q(y))$ such that

- $v_y(B_{ij}) \geq 1$ if $i > j$,
- $v_y(B_{ij}) \geq 0$ if $i < j$,
- $v_y(B_{ii}) = 0$ as $B_{ii} \in \mathbb{F}_q^\times$.

The above properties follow from the fact that $P_{k,i} \in \mathbb{F}_q(y)$ and $v_y(P_{k,i}) \geq 1$. Thus, $v_y(\det B) = 0$. Hence, $f_0 = \cdots = f_m = 0$. It follows that $\delta_\emptyset = 0$ as $\delta = 0$ and $\delta_{\mathfrak{t}';\boldsymbol{\epsilon}'} = 0$ for all $(\mathfrak{t}'; \boldsymbol{\epsilon}') \in J_0(\emptyset)$. We conclude that $\delta_{\mathfrak{t};\boldsymbol{\epsilon}} = 0$ for all $(\mathfrak{t}; \boldsymbol{\epsilon}) \in J$. In particular, for all i, $a_i(t) = \delta_{\mathfrak{s}_i;\boldsymbol{\epsilon}_i} = 0$, which is a contradiction. Thus, this case can never happen.

4.3.2. Case 2: $q-1 \mid w$, says $w = m(q-1)$

By similar arguments as above, we show that $\delta = F(t-\theta)^{(q-1)m}$ with $F \in K[t]$ of the form

$$F = \sum_{0 \leq i \leq m/q} f_{m-iq}(t-\theta)^{m-iq}, \quad \text{for } f_{m-iq} \in \mathbb{F}_q[t].$$

Thus,

$$\begin{aligned} F(t-\theta)^{(q-1)m} &= \sum_{0 \leq i \leq m/q} f_{m-iq}(t-\theta)^{mq-iq} = \sum_{0 \leq i \leq m/q} f_{m-iq} X^{m-i}, \\ F^{(1)} &= \sum_{0 \leq i \leq m/q} f_{m-iq}(t-\theta^q)^{m-iq} = \sum_{0 \leq i \leq m/q} f_{m-iq}(T+X)^{m-iq}. \end{aligned}$$

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Putting these and Equation (4.7) into Equation (4.9) gets

$$\epsilon \sum_{0\le i\le m/q} f_{m-iq}(T+X)^{m-iq} = \sum_{0\le i\le m/q} f_{m-iq}X^{m-i} + \sum_{\substack{0\le k<m \\ k\not\equiv m (\mathrm{mod}\ q)}} f_k \left(X^k + \sum_{i=0}^{k-1} P_{k,i}(T)X^i \right).$$

Comparing the coefficients of powers of X yields

$$\epsilon f_m = f_m$$

and the following linear system in the variables $f_0,\dots,f_{m-1}$:

$$B_{\big|y=T} \begin{pmatrix} f_{m-1} \\ \vdots \\ f_0 \end{pmatrix} = f_m \begin{pmatrix} Q_{m-1} \\ \vdots \\ Q_0 \end{pmatrix}_{\big|y=T}.$$

Here, for $0\le i\le m-1$, $Q_i = \binom{m}{i} y^{m-i} \in y\mathbb{F}_q[y]$ and $B = (B_{ij})_{0\le i,j\le m-1} \in \mathrm{Mat}_m(\mathbb{F}_q(y))$ such that

- $v_y(B_{ij}) \ge 1$ if $i>j$,
- $v_y(B_{ij}) \ge 0$ if $i<j$,
- $v_y(B_{ii}) = 0$ as $B_{ii} \in \mathbb{F}_q^\times$.

The above properties follow from the fact that $P_{k,i} \in \mathbb{F}_q(y)$ and $v_y(P_{k,i}) \ge 1$. Thus, $v_y(\det B)=0$ so that $\det B \ne 0$.

We distinguish two subcases.

Subcase 1: $\epsilon \ne 1$.

It follows that $f_m = 0$. Then $f_0 = \cdots = f_{m-1} = 0$. Thus, $\delta_{\mathfrak{t};\epsilon} = 0$ for all $(\mathfrak{t};\boldsymbol{\epsilon}) \in J$. In particular, for all i, $a_i(t) = \delta_{\mathfrak{s}_i;\epsilon_i} = 0$. This is a contradiction, and we conclude that this case can never happen.

Subcase 2: $\epsilon = 1$.

It follows that $\gamma \in \mathbb{F}_q^\times$ and thus

1) The polynomial $\delta_\emptyset = \delta\gamma$ is of the form

$$\delta_\emptyset = f_\emptyset \left(X^m + \sum_{i=0}^{m-1} P_{\emptyset,i}(T)X^i \right)$$

with
- $f_\emptyset \in \mathbb{F}_q[t]$,
- for all $0\le i\le m-1$, $P_{\emptyset,i}(y) \in \mathbb{F}_q(y)$ with $v_y(P_{\emptyset,i}) \ge 1$.

2) For all $(\mathfrak{t}';\boldsymbol{\epsilon}') \in J_0(\emptyset)$, there exists $P_{\emptyset,\mathfrak{t}'} \in \mathbb{F}_q(y)$ such that

$$f_{\mathfrak{t}'} = f_\emptyset P_{\emptyset,\mathfrak{t}'}(T).$$

Hence, there exists a unique solution $(\delta_{\mathfrak{t};\epsilon})_{(\mathfrak{t};\epsilon)\in J} \in \mathrm{Mat}_{1\times|J|}(K[t])$ of Equation (4.4) up to a factor in $\mathbb{F}_q(t)$. Recall that for all i, $a_i(t) = \delta_{\mathfrak{s}_i;\epsilon_i}$. Therefore, up to a scalar in $K^\times$, there exists at most one nontrivial relation

$$a\widetilde{\pi}^w + \sum_i a_i \operatorname{Li}\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} = 0$$

with $a_i \in K$ and $\operatorname{Li}\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AS}_w$. Further, we must have $\boldsymbol{\varepsilon}_i = (1,\dots,1)$ for all i.

To conclude, it suffices to exhibit such a relation with $a \neq 0$. In fact, we recall $w = (q-1)m$ and then express $\mathrm{Li}(q-1)^m = \mathrm{Li}\begin{pmatrix} 1 \\ q-1 \end{pmatrix}^m$ as a K-linear combination of ACMPL's of weight w. By Theorem 2.11, we can write

$$\mathrm{Li}(q-1)^m = \mathrm{Li}\begin{pmatrix} 1 \\ q-1 \end{pmatrix}^m = \sum_i a_i \,\mathrm{Li}\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix}, \quad \text{where } a_i \in K, \mathrm{Li}\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} \in \mathcal{AS}_w.$$

We note that $\mathrm{Li}(q-1) = \zeta_A(q-1) = -D_1^{-1}\widetilde{\pi}^{q-1}$. Thus,

$$(-D_1)^{-m}\widetilde{\pi}^w - \sum_i a_i \,\mathrm{Li}\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} = 0,$$

which is the desired relation.

5. Applications on AMZV's and Zagier–Hoffman's conjectures in positive characteristic

In this section, we give two applications of the study of ACMPL's.

First, we use Theorem 4.6 to prove Theorem A which calculates the dimensions of the vector space $\mathcal{AZ}_w$ of alternating multiple zeta values in positive characteristic (AMZV's) of fixed weight introduced by Harada [21]. Consequently, we determine all linear relations for AMZV's. To do so, we develop an algebraic theory to obtain a weak version of Brown's theorem for AMZV's. Then we deduce that $\mathcal{AZ}_w$ and $\mathcal{AL}_w$ are equal and conclude. In contrast to the setting of MZV's, although the results are clean, we are unable to obtain either sharp upper bounds or sharp lower bounds for $\mathcal{AZ}_w$ for general w without the theory of ACMPL's.

Second, we restrict our attention to MZV's and determine all linear relations between MZV's. In particular, we obtain a proof of Zagier–Hoffman's conjectures in positive characteristic in full generality (i.e., Theorem B) and generalize the work of one of the authors [31].

5.1. Linear relations between AMZV's

5.1.1. Preliminaries

For $d \in \mathbb{Z}$ and for $\mathfrak{s} = (s_1, \dots, s_n) \in \mathbb{N}^n$, recalling $S_d(\mathfrak{s})$ and $S_{<d}(\mathfrak{s})$ given in §2.1.3, and further letting $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_n \\ s_1 & \dots & s_n \end{pmatrix}$ be an array, we recall (see §2.1.3)

$$S_d\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \sum_{\substack{a_1,\dots,a_n \in A_+ \\ d = \deg a_1 > \dots > \deg a_n \geq 0}} \frac{\varepsilon_1^{\deg a_1} \dots \varepsilon_n^{\deg a_n}}{a_1^{s_1} \dots a_n^{s_n}} \in K$$

and

$$S_{<d}\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \sum_{\substack{a_1,\dots,a_n \in A_+ \\ d > \deg a_1 > \dots > \deg a_n \geq 0}} \frac{\varepsilon_1^{\deg a_1} \dots \varepsilon_n^{\deg a_n}}{a_1^{s_1} \dots a_n^{s_n}} \in K.$$

One verifies easily the following formulas:

$$S_{<d}\begin{pmatrix} 1 & \dots & 1 \\ s_1 & \dots & s_n \end{pmatrix} = S_{<d}(s_1, \dots, s_n), \quad S_d\begin{pmatrix} 1 & \dots & 1 \\ s_1 & \dots & s_n \end{pmatrix} = S_d(s_1, \dots, s_n),$$
$$S_d\begin{pmatrix} \varepsilon \\ s \end{pmatrix} = \varepsilon^d S_d(s), \quad S_d\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = S_d\begin{pmatrix} \varepsilon_1 \\ s_1 \end{pmatrix} S_{<d}\begin{pmatrix} \boldsymbol{\varepsilon}_- \\ \mathfrak{s}_- \end{pmatrix}.$$

https://doi.org/10.1017/fmp.2024.18 Published online by Cambridge University Press

Harada [21] introduced the AMZV as follows;

$$\zeta_A\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}=\sum_{d\geq 0}S_d\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}=\sum_{\substack{a_1,\dots,a_n\in A_+\\ \deg a_1>\cdots>\deg a_n\geq 0}}\frac{\varepsilon_1^{\deg a_1}\dots\varepsilon_n^{\deg a_n}}{a_1^{s_1}\dots a_n^{s_n}}\in K_\infty.$$

Using Chen's formula (see [13]), Harada proved that for $s,t\in\mathbb{N}$ and $\varepsilon,\epsilon\in\mathbb{F}_q^\times$, we have

$$S_d\begin{pmatrix}\varepsilon\\ s\end{pmatrix}S_d\begin{pmatrix}\epsilon\\ t\end{pmatrix}=S_d\begin{pmatrix}\varepsilon\epsilon\\ s+t\end{pmatrix}+\sum_i\Delta_{s,t}^iS_d\begin{pmatrix}\varepsilon\epsilon & 1\\ s+t-i & i\end{pmatrix},\tag{5.1}$$

where

$$\Delta_{s,t}^i=\begin{cases}(-1)^{s-1}\binom{i-1}{s-1}+(-1)^{t-1}\binom{i-1}{t-1} & \text{if } q-1\mid i \text{ and } 0<i<s+t,\\ 0 & \text{otherwise.}\end{cases}\tag{5.2}$$

Remark 5.1. When $s+t\leq q$, we deduce from the above formulas that

$$S_d\begin{pmatrix}\varepsilon\\ s\end{pmatrix}S_d\begin{pmatrix}\epsilon\\ t\end{pmatrix}=S_d\begin{pmatrix}\varepsilon\epsilon\\ s+t\end{pmatrix}.$$

He then proved similar results for products of AMZV's (see [21]):

Proposition 5.2. *Let* $\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}$, $\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}$ *be two arrays. Then*

1. *There exist* $f_i\in\mathbb{F}_q$ *and arrays* $\begin{pmatrix}\boldsymbol{\mu}_i\\ \mathfrak{u}_i\end{pmatrix}$ *with* $\begin{pmatrix}\boldsymbol{\mu}_i\\ \mathfrak{u}_i\end{pmatrix}\leq\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}+\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}$ *and* $\operatorname{depth}(\mathfrak{u}_i)\leq\operatorname{depth}(\mathfrak{s})+\operatorname{depth}(\mathfrak{t})$ *for all i such that*

$$S_d\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}S_d\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}=\sum_i f_iS_d\begin{pmatrix}\boldsymbol{\mu}_i\\ \mathfrak{u}_i\end{pmatrix}\quad \textit{for all } d\in\mathbb{Z}.$$

2. *There exist* $f_i'\in\mathbb{F}_q$ *and arrays* $\begin{pmatrix}\boldsymbol{\mu}_i'\\ \mathfrak{u}_i'\end{pmatrix}$ *with* $\begin{pmatrix}\boldsymbol{\mu}_i'\\ \mathfrak{u}_i'\end{pmatrix}\leq\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}+\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}$ *and* $\operatorname{depth}(\mathfrak{u}_i')\leq\operatorname{depth}(\mathfrak{s})+\operatorname{depth}(\mathfrak{t})$ *for all i such that*

$$S_{<d}\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}S_{<d}\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}=\sum_i f_i'S_{<d}\begin{pmatrix}\boldsymbol{\mu}_i'\\ \mathfrak{u}_i'\end{pmatrix}\quad \textit{for all } d\in\mathbb{Z}.$$

3. *There exist* $f_i''\in\mathbb{F}_q$ *and arrays* $\begin{pmatrix}\boldsymbol{\mu}_i''\\ \mathfrak{u}_i''\end{pmatrix}$ *with* $\begin{pmatrix}\boldsymbol{\mu}_i''\\ \mathfrak{u}_i''\end{pmatrix}\leq\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}+\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}$ *and* $\operatorname{depth}(\mathfrak{u}_i'')\leq\operatorname{depth}(\mathfrak{s})+\operatorname{depth}(\mathfrak{t})$ *for all i such that*

$$S_d\begin{pmatrix}\boldsymbol{\varepsilon}\\ \mathfrak{s}\end{pmatrix}S_{<d}\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{t}\end{pmatrix}=\sum_i f_i''S_d\begin{pmatrix}\boldsymbol{\mu}_i''\\ \mathfrak{u}_i''\end{pmatrix}\quad \textit{for all } d\in\mathbb{Z}.$$

We denote by $\mathcal{AZ}$ the K-vector space generated by the AMZV's and $\mathcal{AZ}_w$ the K-vector space generated by the AMZV's of weight w. It follows from Proposition 5.2 that $\mathcal{AZ}$ is a K-algebra.

5.1.2. Algebraic theory for AMZV's

We can extend an algebraic theory for AMZV's which follow the same line as that in §2.

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Definition 5.3. A binary relation is a K-linear combination of the form

$$\sum_i a_i S_d \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} + \sum_i b_i S_{d+1} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} = 0 \quad \text{for all } d \in \mathbb{Z},$$

where $a_i, b_i \in K$ and $\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix}, \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix}$ are arrays of the same weight.

A binary relation is called a fixed relation if $b_i = 0$ for all i.

We denote by $\mathfrak{R}_w$ the set of all binary relations of weight w. From Lemma 2.2 and the relation R_ε defined in §2.2, we obtain the following binary relation

$$R_\varepsilon: \quad S_d \begin{pmatrix} \varepsilon \\ q \end{pmatrix} + \varepsilon^{-1} D_1 S_{d+1} \begin{pmatrix} \varepsilon & 1 \\ 1 & q-1 \end{pmatrix} = 0,$$

where $D_1 = \theta^q - \theta$.

For later definitions, let $R \in \mathfrak{R}_w$ be a binary relation of the form

$$R(d): \quad \sum_i a_i S_d \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} + \sum_i b_i S_{d+1} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} = 0, \tag{5.3}$$

where $a_i, b_i \in K$ and $\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix}, \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix}$ are arrays of the same weight. We now define some operators on K-vector spaces of binary relations.

First, we define operators $\mathcal{B}^*$. Let $\begin{pmatrix} \sigma \\ v \end{pmatrix}$ be an array. We introduce

$$\mathcal{B}^*_{\sigma,v} : \mathfrak{R}_w \longrightarrow \mathfrak{R}_{w+v}$$

as follows: For each $R \in \mathfrak{R}_w$ as given in Equation (5.3), the image $\mathcal{B}^*_{\sigma,v}(R) = S_d \begin{pmatrix} \sigma \\ v \end{pmatrix} \sum_{j<d} R(j)$ is a fixed relation of the form

$$\begin{aligned} 0 &= S_d \begin{pmatrix} \sigma \\ v \end{pmatrix} \left(\sum_i a_i S_{<d} \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} + \sum_i b_i S_{<d+1} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} \right) \\ &= \sum_i a_i S_d \begin{pmatrix} \sigma \\ v \end{pmatrix} S_{<d} \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} + \sum_i b_i S_d \begin{pmatrix} \sigma \\ v \end{pmatrix} S_{<d} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} + \sum_i b_i S_d \begin{pmatrix} \sigma \\ v \end{pmatrix} S_d \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} \\ &= \sum_i a_i S_d \begin{pmatrix} \sigma & \boldsymbol{\varepsilon}_i \\ v & \mathfrak{s}_i \end{pmatrix} + \sum_i b_i S_d \begin{pmatrix} \sigma & \boldsymbol{\epsilon}_i \\ v & \mathfrak{t}_i \end{pmatrix} + \sum_i b_i \sum_j f_{i,j} S_d \begin{pmatrix} \boldsymbol{\mu}_{i,j} \\ \mathfrak{u}_{i,j} \end{pmatrix}. \end{aligned}$$

The last equality follows from Proposition 5.2.

Let $\begin{pmatrix} \Sigma \\ V \end{pmatrix} = \begin{pmatrix} \sigma_1 & \dots & \sigma_n \\ v_1 & \dots & v_n \end{pmatrix}$ be an array. We define an operator $\mathcal{B}^*_{\Sigma,V}(R)$ by

$$\mathcal{B}^*_{\Sigma,V}(R) := \mathcal{B}^*_{\sigma_1,v_1} \circ \cdots \circ \mathcal{B}^*_{\sigma_n,v_n}(R).$$

Lemma 5.4. *Let* $\binom{\Sigma}{V} = \begin{pmatrix}\sigma_1 & \dots & \sigma_n \\ v_1 & \dots & v_n\end{pmatrix}$ *be an array. Under the notations of Equation (5.3), suppose that for all* i*,* $v_n + t_{i1} \le q$*, where* $\mathfrak{t}_i = (t_{i1}, \mathfrak{t}_{i-})$*. Then* $\mathcal{B}^*_{\Sigma,V}(R)$ *is of the form*

$$\sum_i a_i S_d \begin{pmatrix}\Sigma & \boldsymbol{\varepsilon}_i \\ V & \mathfrak{s}_i\end{pmatrix} + \sum_i b_i S_d \begin{pmatrix}\Sigma & \boldsymbol{\epsilon}_i \\ V & \mathfrak{t}_i\end{pmatrix} + \sum_i b_i S_d \begin{pmatrix}\sigma_1 & \dots & \sigma_{n-1} & \sigma_n \epsilon_{i1} & \boldsymbol{\epsilon}_{i-} \\ v_1 & \dots & v_{n-1} & v_n + t_{i1} & \mathfrak{t}_{i-}\end{pmatrix} = 0.$$

Proof. From the definition, we have $\mathcal{B}^*_{\sigma_n,v_n}(R)$ is of the form

$$\sum_i a_i S_d \begin{pmatrix}\sigma_n & \boldsymbol{\varepsilon}_i \\ v_n & \mathfrak{s}_i\end{pmatrix} + \sum_i b_i S_d \begin{pmatrix}\sigma_n & \boldsymbol{\epsilon}_i \\ v_n & \mathfrak{t}_i\end{pmatrix} + \sum_i b_i S_d \begin{pmatrix}\sigma_n \\ v_n\end{pmatrix} S_d \begin{pmatrix}\boldsymbol{\epsilon}_i \\ \mathfrak{t}_i\end{pmatrix} = 0.$$

For all i, since $v_n + t_{i1} \le q$, it follows from Remark 5.1 that

$$\begin{aligned} S_d \begin{pmatrix}\sigma_n \\ v_n\end{pmatrix} S_d \begin{pmatrix}\boldsymbol{\epsilon}_i \\ \mathfrak{t}_i\end{pmatrix} &= S_d \begin{pmatrix}\sigma_n \\ v_n\end{pmatrix} S_d \begin{pmatrix}\epsilon_{i1} \\ t_{i1}\end{pmatrix} S_{<d} \begin{pmatrix}\boldsymbol{\epsilon}_{i-} \\ \mathfrak{t}_{i-}\end{pmatrix} = S_d \begin{pmatrix}\sigma_n \epsilon_{i1} \\ v_n + t_{i1}\end{pmatrix} S_{<d} \begin{pmatrix}\boldsymbol{\epsilon}_{i-} \\ \mathfrak{t}_{i-}\end{pmatrix} \\ &= S_d \begin{pmatrix}\sigma_n \epsilon_{i1} & \boldsymbol{\epsilon}_{i-} \\ v_n + t_{i1} & \mathfrak{t}_{i-}\end{pmatrix}, \end{aligned}$$

hence $\mathcal{B}^*_{\sigma_n,v_n}(R)$ is of the form

$$\sum_i a_i S_d \begin{pmatrix}\sigma_n & \boldsymbol{\varepsilon}_i \\ v_n & \mathfrak{s}_i\end{pmatrix} + \sum_i b_i S_d \begin{pmatrix}\sigma_n & \boldsymbol{\epsilon}_i \\ v_n & \mathfrak{t}_i\end{pmatrix} + \sum_i b_i S_d \begin{pmatrix}\sigma_n \epsilon_{i1} & \boldsymbol{\epsilon}_{i-} \\ v_n + t_{i1} & \mathfrak{t}_{i-}\end{pmatrix} = 0.$$

Apply the operator $\mathcal{B}^*_{\sigma_1,v_1} \circ \dots \circ \mathcal{B}^*_{\sigma_{n-1},v_{n-1}}$ to $\mathcal{B}^*_{\sigma_n,v_n}(R)$, the result then follows from the definition. □

Second, we define operators $\mathcal{C}$. Let $\binom{\Sigma}{V}$ be an array of weight v. We introduce

$$\mathcal{C}_{\Sigma,V}(R) \colon \mathfrak{R}_w \longrightarrow \mathfrak{R}_{w+v}$$

as follows: For each $R \in \mathfrak{R}_w$ as given in Equation (5.3), the image $\mathcal{C}_{\Sigma,V}(R) = R(d) S_{<d+1} \binom{\Sigma}{V}$ is a binary relation of the form

$$\begin{aligned} 0 &= \left(\sum_i a_i S_d \begin{pmatrix}\boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i\end{pmatrix} + \sum_i b_i S_{d+1} \begin{pmatrix}\boldsymbol{\epsilon}_i \\ \mathfrak{t}_i\end{pmatrix} \right) S_{<d+1} \begin{pmatrix}\Sigma \\ V\end{pmatrix} \\ &= \sum_i a_i S_d \begin{pmatrix}\boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i\end{pmatrix} S_d \begin{pmatrix}\Sigma \\ V\end{pmatrix} + \sum_i a_i S_d \begin{pmatrix}\boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i\end{pmatrix} S_{<d} \begin{pmatrix}\Sigma \\ V\end{pmatrix} + \sum_i b_i S_{d+1} \begin{pmatrix}\boldsymbol{\epsilon}_i \\ \mathfrak{t}_i\end{pmatrix} S_{<d+1} \begin{pmatrix}\Sigma \\ V\end{pmatrix} \\ &= \sum_i f_i S_d \begin{pmatrix}\boldsymbol{\mu}_i \\ \mathfrak{u}_i\end{pmatrix} + \sum_i f_i' S_{d+1} \begin{pmatrix}\boldsymbol{\mu}_i' \\ \mathfrak{u}_i'\end{pmatrix}. \end{aligned}$$

The last equality follows from Proposition 5.2.

In particular, the following proposition gives the form of $\mathcal{C}_{\Sigma,V}(R_\varepsilon)$.

Proposition 5.5. *Let* $\binom{\Sigma}{V}$ *be an array with* $V = (v_1, V_-)$ *and* $\Sigma = (\sigma_1, \Sigma_-)$*. Then* $\mathcal{C}_{\Sigma,V}(R_\varepsilon)$ *is of the form*

$$S_d \begin{pmatrix}\varepsilon \sigma_1 & \Sigma_- \\ q + v_1 & V_-\end{pmatrix} + \sum_i a_i S_d \begin{pmatrix}\boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i\end{pmatrix} + \sum_i b_i S_{d+1} \begin{pmatrix}\varepsilon & \boldsymbol{\epsilon}_i \\ 1 & \mathfrak{t}_i\end{pmatrix} = 0,$$

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where $a_i, b_i \in K$ *and* $\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix}, \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix}$ *are arrays satisfying*

- $\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} \le \begin{pmatrix} \varepsilon \\ q \end{pmatrix} + \begin{pmatrix} \Sigma \\ V \end{pmatrix}$ *and* $s_{i1} < q + v_1$ *for all i;*
- $\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} \le \begin{pmatrix} 1 \\ q-1 \end{pmatrix} + \begin{pmatrix} \Sigma \\ V \end{pmatrix}$ *for all i.*

Proof. From the definition, $\mathcal{C}_{\Sigma,V}(R_\varepsilon)$ is of the form

$$S_d \begin{pmatrix} \varepsilon \\ q \end{pmatrix} S_d \begin{pmatrix} \Sigma \\ V \end{pmatrix} + S_d \begin{pmatrix} \varepsilon \\ q \end{pmatrix} S_{<d} \begin{pmatrix} \Sigma \\ V \end{pmatrix} + \varepsilon^{-1} D_1 S_{d+1} \begin{pmatrix} \varepsilon & 1 \\ 1 & q-1 \end{pmatrix} S_{<d+1} \begin{pmatrix} \Sigma \\ V \end{pmatrix} = 0.$$

It follows from Equation (5.1) and Proposition 5.2 that

$$S_d \begin{pmatrix} \varepsilon \\ q \end{pmatrix} S_d \begin{pmatrix} \Sigma \\ V \end{pmatrix} + S_d \begin{pmatrix} \varepsilon \\ q \end{pmatrix} S_{<d} \begin{pmatrix} \Sigma \\ V \end{pmatrix} = S_d \begin{pmatrix} \varepsilon\sigma_1 & \Sigma_- \\ q+v_1 & V_- \end{pmatrix} + \sum_i a_i S_d \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix},$$
$$\varepsilon^{-1} D_1 S_{d+1} \begin{pmatrix} \varepsilon & 1 \\ 1 & q-1 \end{pmatrix} S_{<d+1} \begin{pmatrix} \Sigma \\ V \end{pmatrix} = \sum_i b_i S_{d+1} \begin{pmatrix} \varepsilon & \boldsymbol{\epsilon}_i \\ 1 & \mathfrak{t}_i \end{pmatrix},$$

where $a_i, b_i \in K$ and $\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix}, \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix}$ are arrays satisfying

- $\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} \le \begin{pmatrix} \varepsilon \\ q \end{pmatrix} + \begin{pmatrix} \Sigma \\ V \end{pmatrix}$ and $s_{i1} < q + v_1$ for all i;
- $\begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} \le \begin{pmatrix} 1 \\ q-1 \end{pmatrix} + \begin{pmatrix} \Sigma \\ V \end{pmatrix}$ for all i.

This proves the proposition. □

Finally, we define operators $\mathcal{BC}$. Let $\varepsilon \in \mathbb{F}_q^\times$. We introduce

$$\mathcal{BC}_{\varepsilon,q} : \mathfrak{R}_w \longrightarrow \mathfrak{R}_{w+q}$$

as follows: For each $R \in \mathfrak{R}_w$ as given in Equation (5.3), the image $\mathcal{BC}_{\varepsilon,q}(R)$ is a binary relation given by

$$\mathcal{BC}_{\varepsilon,q}(R) = \mathcal{B}^*_{\varepsilon,q}(R) - \sum_i b_i \mathcal{C}_{\boldsymbol{\epsilon}_i,\mathfrak{t}_i}(R_\varepsilon).$$

Let us clarify the definition of $\mathcal{BC}_{\varepsilon,q}$. We know that $\mathcal{B}^*_{\varepsilon,q}(R)$ is of the form

$$\sum_i a_i S_d \begin{pmatrix} \varepsilon & \boldsymbol{\varepsilon}_i \\ q & \mathfrak{s}_i \end{pmatrix} + \sum_i b_i S_d \begin{pmatrix} \varepsilon & \boldsymbol{\epsilon}_i \\ q & \mathfrak{t}_i \end{pmatrix} + \sum_i b_i S_d \begin{pmatrix} \varepsilon \\ q \end{pmatrix} S_d \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} = 0.$$

Moreover, $\mathcal{C}_{\boldsymbol{\epsilon}_i,\mathfrak{t}_i}(R_\varepsilon)$ is of the form

$$S_d \begin{pmatrix} \varepsilon & \boldsymbol{\epsilon}_i \\ q & \mathfrak{t}_i \end{pmatrix} + S_d \begin{pmatrix} \varepsilon \\ q \end{pmatrix} S_d \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} + \varepsilon^{-1} D_1 S_{d+1} \begin{pmatrix} \varepsilon \\ 1 \end{pmatrix} S_{<d+1} \begin{pmatrix} 1 \\ q-1 \end{pmatrix} S_{<d+1} \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} = 0.$$

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Combining with Proposition 5.2, we have that $\mathcal{BC}_{\varepsilon,q}(R)$ is of the form

$$\sum_i a_i S_d \begin{pmatrix} \varepsilon & \boldsymbol{\varepsilon}_i \\ q & \mathfrak{s}_i \end{pmatrix} + \sum_{i,j} b_{ij} S_{d+1} \begin{pmatrix} \varepsilon & \boldsymbol{\epsilon}_{ij} \\ 1 & \mathfrak{t}_{ij} \end{pmatrix} = 0,$$

where $b_{ij} \in K$ and $\begin{pmatrix} \boldsymbol{\epsilon}_{ij} \\ \mathfrak{t}_{ij} \end{pmatrix}$ are arrays satisfying $\begin{pmatrix} \boldsymbol{\epsilon}_{ij} \\ \mathfrak{t}_{ij} \end{pmatrix} \le \begin{pmatrix} 1 \\ q-1 \end{pmatrix} + \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix}$ for all j.

Proposition 5.6. 1) *Let* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_n \\ s_1 & \dots & s_n \end{pmatrix}$ *be an array such that* $\operatorname{Init}(\mathfrak{s}) = (s_1, \dots, s_{k-1})$ *for some* $1 \le k \le n$. *Then* $\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ *can be decomposed as follows:*

$$\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \underbrace{\sum_i a_i \zeta_A \begin{pmatrix} \boldsymbol{\varepsilon}'_i \\ \mathfrak{s}'_i \end{pmatrix}}_{\text{type 1}} + \underbrace{\sum_i b_i \zeta_A \begin{pmatrix} \boldsymbol{\epsilon}'_i \\ \mathfrak{t}'_i \end{pmatrix}}_{\text{type 2}} + \underbrace{\sum_i c_i \zeta_A \begin{pmatrix} \boldsymbol{\mu}_i \\ \mathfrak{u}_i \end{pmatrix}}_{\text{type 3}},$$

where $a_i, b_i, c_i \in K$ *such that for all i, the following properties are satisfied:*

- *For all arrays* $\begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$ *appearing on the right-hand side,*

$$\operatorname{depth}(\mathfrak{t}) \ge \operatorname{depth}(\mathfrak{s}) \quad \text{and} \quad T_k(\mathfrak{t}) \le T_k(\mathfrak{s}).$$

- *For the array* $\begin{pmatrix} \boldsymbol{\varepsilon}' \\ \mathfrak{s}' \end{pmatrix}$ *of type* 1 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, *we have* $\operatorname{Init}(\mathfrak{s}) \preceq \operatorname{Init}(\mathfrak{s}')$ *and* $s'_k < s_k$.
- *For the array* $\begin{pmatrix} \boldsymbol{\epsilon}' \\ \mathfrak{t}' \end{pmatrix}$ *of type* 2 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, *for all* $k \le \ell \le n$,

$$t'_1 + \dots + t'_\ell < s_1 + \dots + s_\ell.$$

- *For the array* $\begin{pmatrix} \boldsymbol{\mu} \\ \mathfrak{u} \end{pmatrix}$ *of type* 3 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, *we have* $\operatorname{Init}(\mathfrak{s}) \prec \operatorname{Init}(\mathfrak{u})$.

2) *Let* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_k \\ s_1 & \dots & s_k \end{pmatrix}$ *be an array such that* $\operatorname{Init}(\mathfrak{s}) = \mathfrak{s}$ *and* $s_k = q$. *Then* $\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ *can be decomposed as follows:*

$$\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \underbrace{\sum_i b_i \zeta_A \begin{pmatrix} \boldsymbol{\epsilon}'_i \\ \mathfrak{t}'_i \end{pmatrix}}_{\text{type 2}} + \underbrace{\sum_i c_i \zeta_A \begin{pmatrix} \boldsymbol{\mu}_i \\ \mathfrak{u}_i \end{pmatrix}}_{\text{type 3}},$$

where $b_i, c_i \in K$ *such that for all i, the following properties are satisfied:*

- *For all arrays* $\begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$ *appearing on the right-hand side,*

$$\operatorname{depth}(\mathfrak{t}) \ge \operatorname{depth}(\mathfrak{s}) \quad \text{and} \quad T_k(\mathfrak{t}) \le T_k(\mathfrak{s}).$$

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- *For the array* $\begin{pmatrix} \boldsymbol{\epsilon}' \\ \mathfrak{t}' \end{pmatrix}$ *of type* 2 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$,

$$t_1' + \cdots + t_k' < s_1 + \cdots + s_k.$$

- *For the array* $\begin{pmatrix} \boldsymbol{\mu} \\ \mathfrak{u} \end{pmatrix}$ *of type* 3 *with respect to* $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, *we have* $\operatorname{Init}(\mathfrak{s}) \prec \operatorname{Init}(\mathfrak{u})$.

Proof. The proof follows the same line as in [31, Proposition 2.12 and 2.13]. We outline the proof here and refer the reader to [25] for more details. For Part 1, since $\operatorname{Init}(\mathfrak{s}) = (s_1, \dots, s_{k-1})$, we get $s_k > q$. Set $\begin{pmatrix} \Sigma \\ V \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon_{k+1} & \dots & \varepsilon_n \\ s_k - q & s_{k+1} & \dots & s_n \end{pmatrix}$. By Proposition 5.5, $\mathcal{C}_{\Sigma,V}(R_{\varepsilon_k})$ is of the form

$$S_d \begin{pmatrix} \varepsilon_k & \dots & \varepsilon_n \\ s_k & \dots & s_n \end{pmatrix} + \sum_i a_i S_d \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} + \sum_i b_i S_{d+1} \begin{pmatrix} \varepsilon_k & \boldsymbol{\epsilon}_i \\ 1 & \mathfrak{t}_i \end{pmatrix} = 0, \tag{5.4}$$

where $a_i, b_i \in K$ and $\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix}, \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix}$ are arrays satisfying

$$\begin{aligned} \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} &\le \begin{pmatrix} \varepsilon_k \\ q \end{pmatrix} + \begin{pmatrix} \Sigma \\ V \end{pmatrix} = \begin{pmatrix} \varepsilon_k & \dots & \varepsilon_n \\ s_k & \dots & s_n \end{pmatrix} \quad \text{and} \quad s_{i1} < q + v_1 = s_k; \\ \begin{pmatrix} \boldsymbol{\epsilon}_i \\ \mathfrak{t}_i \end{pmatrix} &\le \begin{pmatrix} 1 \\ q-1 \end{pmatrix} + \begin{pmatrix} \Sigma \\ V \end{pmatrix} = \begin{pmatrix} 1 & \varepsilon_{k+1} & \dots & \varepsilon_n \\ s_k - 1 & s_{k+1} & \dots & s_n \end{pmatrix}. \end{aligned} \tag{5.5}$$

For $m \in \mathbb{N}$, we recall that $q^{\{m\}}$ is the sequence of length m with all terms equal to q. Setting $s_0 = 0$, we may assume that there exists a maximal index j with $0 \le j \le k-1$ such that $s_j < q$, hence $\operatorname{Init}(\mathfrak{s}) = (s_1, \dots, s_j, q^{\{k-j-1\}})$. Then the operator $\mathcal{BC}_{\varepsilon_{j+1},q} \circ \cdots \circ \mathcal{BC}_{\varepsilon_{k-1},q}$ applied to the relation (5.4) gives

$$\begin{aligned} S_d \begin{pmatrix} \varepsilon_{j+1} & \dots & \varepsilon_{k-1} & \varepsilon_k & \dots & \varepsilon_n \\ q & \dots & q & s_k & \dots & s_n \end{pmatrix} &+ \sum_i a_i S_d \begin{pmatrix} \varepsilon_{j+1} & \dots & \varepsilon_{k-1} & \boldsymbol{\varepsilon}_i \\ q & \dots & q & \mathfrak{s}_i \end{pmatrix} \\ &+ \sum_i b_{i_1 \dots i_{k-j}} S_{d+1} \begin{pmatrix} \varepsilon_{j+1} & \boldsymbol{\epsilon}_{i_1 \dots i_{k-j}} \\ 1 & \mathfrak{t}_{i_1 \dots i_{k-j}} \end{pmatrix} = 0, \end{aligned} \tag{5.6}$$

where $b_{i_1 \dots i_{k-j}} \in K$ and

$$\begin{pmatrix} \boldsymbol{\epsilon}_{i_1 \dots i_{k-j}} \\ \mathfrak{t}_{i_1 \dots i_{k-j}} \end{pmatrix} \le \begin{pmatrix} \varepsilon_{j+2} & \dots & \varepsilon_k & 1 & \varepsilon_{k+1} & \dots & \varepsilon_n \\ q & \dots & q & s_k - 1 & s_{k+1} & \dots & s_n \end{pmatrix}. \tag{5.7}$$

We let $\begin{pmatrix} \Sigma' \\ V' \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_j \\ s_1 & \dots & s_j \end{pmatrix}$, and we apply $\mathcal{B}^*_{\Sigma',V'}$ to Equation (5.6). Since $s_j < q$, that is, $s_j + 1 \le q$, we can deduce from Lemma 5.4 that

$$\begin{aligned} \zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = &- \sum_i a_i \zeta_A \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_j & \varepsilon_{j+1} & \dots & \varepsilon_{k-1} & \boldsymbol{\varepsilon}_i \\ s_1 & \dots & s_j & q & \dots & q & \mathfrak{s}_i \end{pmatrix} - \sum_i b_{i_1 \dots i_{k-j}} \zeta_A \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_j & \varepsilon_{j+1} & \boldsymbol{\epsilon}_{i_1 \dots i_{k-j}} \\ s_1 & \dots & s_j & 1 & \mathfrak{t}_{i_1 \dots i_{k-j}} \end{pmatrix} \\ &- \sum_i b_{i_1 \dots i_{k-j}} \zeta_A \begin{pmatrix} \varepsilon_1 & \dots & \varepsilon_{j-1} & \varepsilon_j \varepsilon_{j+1} & \boldsymbol{\epsilon}_{i_1 \dots i_{k-j}} \\ s_1 & \dots & s_{j-1} & s_j + 1 & \mathfrak{t}_{i_1 \dots i_{k-j}} \end{pmatrix}. \end{aligned} \tag{5.8}$$

The first term, the second term and the third term on the right-hand side of Equation (5.8) are referred to as type 1, type 2 and type 3, respectively. From Equations (5.5) and (5.7) and Remark 2.1, one verifies that the arrays of type 1, type 2 and type 3 satisfy the desired conditions. We have proved Part 1.

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The proof of Part 2 is similar to that of Proposition 2.7. We first begin with the relation R_{ε_k}. Next, we apply $\mathcal{BC}_{\varepsilon_{j+1},q} \circ \cdots \circ \mathcal{BC}_{\varepsilon_{k-1},q}$ to R_{ε_k} and then apply $\mathcal{B}^*_{\Sigma',V'}$. We can decompose $\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ in terms of type 2 and type 3 as in Equation (5.8). One verifies that these terms satisfy the desired conditions. We finish the proof. □

Proposition 5.7. *For all $k \in \mathbb{N}$ and for all arrays $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, $\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ can be expressed as a K-linear combination of $\zeta_A \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$'s of the same weight such that $\mathfrak{t}$ is k-admissible.*

Proof. The proof follows the same line as that of [31, Proposition 3.2]. We outline the proof here and refer the reader to [25] for more details. We consider the following statement:

(H_k) For all arrays $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, we can express $\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ as a K-linear combination of $\zeta_A \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$'s of the same weight such that $\mathfrak{t}$ is k-admissible.

We will show that (H_k) holds for all $k \in \mathbb{N}$ by induction on k. For $k = 1$, we prove that (H_1) holds by induction on the first component s_1 of $\mathfrak{s}$. If $s_1 \le q$, then either $\mathfrak{s}$ is 1-admissible, or $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon \\ q \end{pmatrix}$. We deduce from the relation R_ε that (H_1) holds for the case $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon \\ q \end{pmatrix}$. If $s_1 > q$, we assume that (H_1) holds for the array $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$, where $s_1 < s$. We need to shows that (H_1) holds for the array $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ where $s_1 = s$. Indeed, assume that $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \cdots & \varepsilon_n \\ s_1 & \cdots & s_n \end{pmatrix}$. Set $\begin{pmatrix} \Sigma \\ V \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \varepsilon_2 & \cdots & \varepsilon_n \\ s_1 - q & s_2 & \cdots & s_n \end{pmatrix}$. Applying $C_{\Sigma,V}$ to the relation R_1 and using Proposition 5.5, we can deduce that

$$\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = -\sum_i a_i \zeta_A \begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix} - \sum_i b_i \zeta_A \begin{pmatrix} 1 & \boldsymbol{\epsilon}_i \\ 1 & \mathfrak{t}_i \end{pmatrix},$$

where $a_i, b_i \in K$ and $\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix}$are arrays satisfying $s_{i1} < s$ for all i. From the induction hypothesis, we deduce that (H_1) holds for $\begin{pmatrix} \boldsymbol{\varepsilon}_i \\ \mathfrak{s}_i \end{pmatrix}$, and therefore for $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$.

We next assume that (H_{k-1}) holds. We need to show that (H_k) holds. The rest of the proof is similar to that of Proposition 2.9. We can restrict our attention to the array $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \cdots & \varepsilon_n \\ s_1 & \cdots & s_n \end{pmatrix}$, where $\mathfrak{s}$ is not k-admissible and $\operatorname{depth}(\mathfrak{s}) \ge k$. We show that (H_k) holds for the array $\begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ by induction on $s_1 + \cdots + s_k$. For the induction step, by using Proposition 5.6 and the induction hypothesis, we can give an algorithm to decompose $\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$as a K-linear combination of $\zeta_A \begin{pmatrix} \boldsymbol{\epsilon} \\ \mathfrak{t} \end{pmatrix}$'s of the same weight such that $\mathfrak{t}$ is k-admissible. From Proposition 5.6 and similar arguments as in [31, Proposition 3.2], we can show that this algorithm stops after a finite number of steps. This proves the result. □

Consequently, we obtain a weak version of Brown's theorem for AMZV's as follows.

Proposition 5.8. *The set of all elements $\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ such that $\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} \in \mathcal{AT}_w$ forms a set of generators for $\mathcal{AZ}_w$. Here, we recall that $\mathcal{AT}_w$ is the set of all AMZV's $\zeta_A \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix} = \operatorname{Li} \begin{pmatrix} \boldsymbol{\varepsilon} \\ \mathfrak{s} \end{pmatrix}$ of weight w such*

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that $\begin{pmatrix} \varepsilon \\ \mathfrak{s} \end{pmatrix} = \begin{pmatrix} \varepsilon_1 & \cdots & \varepsilon_n \\ s_1 & \cdots & s_n \end{pmatrix}$ *with* $s_1, \dots, s_{n-1} \le q$ *and* $s_n < q$ *introduced in the paragraph preceding Proposition 2.10.*

Proof. It follows from Proposition 5.7 in the case of $k = w$. □

5.2. *Proof of Theorem A*

As a direct consequence of Proposition 2.10 and Proposition 5.8, we get

Theorem 5.9. *The K-vector space* $\mathcal{AZ}_w$ *of AMZV's of weight w and the K-vector space* $\mathcal{AL}_w$ *of ACMPL's of weight w are the same.*

By this identification, we apply Theorem 4.6 to obtain Theorem A.

5.3. *Zagier–Hoffman's conjectures in positive characteristic*

5.3.1. Known results

We use freely the notation introduced in §1.2.1. We recall that for $w \in \mathbb{N}$, $\mathcal{Z}_w$ denotes the K-vector space spanned by the MZV's of weight w and $\mathcal{T}_w$ denotes the set of $\zeta_A(\mathfrak{s})$, where $\mathfrak{s} = (s_1, \dots, s_r) \in \mathbb{N}^r$ of weight w with $1 \le s_i \le q$ for $1 \le i \le r-1$ and $s_r < q$.

Recall that the main results of [31] state that

- For all $w \in \mathbb{N}$ we always have $\dim_K \mathcal{Z}_w \le d(w)$ (see [31, Theorem A]).
- For $w \le 2q-2$, we have $\dim_K \mathcal{Z}_w \ge d(w)$ (see [31, Theorem B]). In particular, Conjecture 1.7 holds for $w \le 2q-2$ (see [31, Theorem D]).

However, as stated in [31, Remark 6.3] it would be very difficult to extend the method of [31] for general weights.

As an application of our main results, we present a proof of Theorem B which settles both Conjectures 1.6 and 1.7.

5.3.2. Proof of Theorem B

As we have already known the sharp upper bound for $\mathcal{Z}_w$ (see [31, Theorem A]), Theorem B follows immediately from the following proposition.

Proposition 5.10. *For all* $w \in \mathbb{N}$ *we have* $\dim_K \mathcal{Z}_w \ge d(w)$.

Proof. We denote by $\mathcal{S}_w$ the set of CMPL's consisting of $\mathrm{Li}(s_1, \dots, s_r)$ of weight w with $q \nmid s_i$ for all i.

Then $\mathcal{S}_w$ can be considered as a subset of $\mathcal{AS}_w$ by assuming $\boldsymbol{\epsilon} = (1, \dots, 1)$. In fact, all algebraic relations in §2 hold for CMPL version, that is, for $\mathrm{Si}_d(s_1, \dots, s_r) = \mathrm{Si}_d \begin{pmatrix} 1 & \dots & 1 \\ s_1 & \dots & s_r \end{pmatrix}$ and $\mathrm{Li}(s_1, \dots, s_r) = \mathrm{Li} \begin{pmatrix} 1 & \dots & 1 \\ s_1 & \dots & s_r \end{pmatrix}$. It follows that $\mathcal{S}_w$ is contained in $\mathcal{Z}_w$ by Theorem 5.9. Further, by §2.4.1, $|\mathcal{S}_w| = d(w)$. By Theorem 4.4, we deduce that elements in $\mathcal{S}_w$ are all linearly independent over K. Therefore, $\dim_K \mathcal{Z}_w \ge |\mathcal{S}_w| = d(w)$. □

5.4. *Sharp bounds without ACMPL's*

To end this paper, we mention that without ACMPL's it seems very hard to obtain for arbitrary weight w

- either the sharp upper bound $\dim_K \mathcal{AZ}_w \le s(w)$,
- or the sharp lower bound $\dim_K \mathcal{AZ}_w \ge s(w)$.

We can only do this for small weights with ad hoc arguments. We collect the results below, sketch some ideas for the proofs, and refer the reader to [26] for full details.

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Proposition 5.11. *Let* $w \leq 2q-2$. *Then* $\dim_K \mathcal{AZ}_w \leq s(w)$.

Proof. We outline the proof and refer the reader to [25] for more details. We denote by $\mathcal{AT}^1_w$ the subset of AMZV's $\zeta_A\begin{pmatrix}\boldsymbol{\epsilon}\\ \mathfrak{s}\end{pmatrix}$ of $\mathcal{AT}_w$ such that $\epsilon_i = 1$ whenever $s_i = q$ and by $\langle \mathcal{AT}^1_w\rangle$ the K-vector space spanned by the AMZV's in $\mathcal{AT}^1_w$. We see that $|\mathcal{AT}^1_w| = s(w)$. Thus, it suffices to prove that $\langle \mathcal{AT}^1_w\rangle = \mathcal{AZ}_w$.

Let $U = (u_1, \dots, u_n)$ and $W = (w_1, \dots, w_r)$ be tuples of positive integers such that $w(U)+w(W)+q = w, u_n \leq q-1$ and $w_1, \dots, w_r \leq q$. Let $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_n) \in (\mathbb{F}_q^\times)^n$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in (\mathbb{F}_q^\times)^r$. By direct calculations, we can obtain an explicit formula for $\mathcal{B}_{\boldsymbol{\epsilon},U}\mathcal{C}_{\lambda,W}(R_\epsilon)$. We give briefly the form of this formula as follows:

$$\begin{aligned} &S_d\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon & \boldsymbol{\lambda}\\ U & q & W\end{pmatrix} + S_d\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon\lambda_1 & \boldsymbol{\lambda}_-\\ U & q+w_1 & W_-\end{pmatrix} \\ &+ \sum_i \alpha_i S_d\begin{pmatrix}\boldsymbol{\epsilon}'_i & \lambda'_{i1} & \boldsymbol{\lambda}'_{i-}\\ U'_i & q+w'_{i1} & W'_{i-}\end{pmatrix} + \sum_i \beta_i S_d\begin{pmatrix}\boldsymbol{\epsilon}''_i & \lambda_1 & \boldsymbol{\lambda}_-\\ U''_i & q+w_1-1 & W_-\end{pmatrix} + \sum_i \gamma_i S_d\begin{pmatrix}\mu_i\\ V_i\end{pmatrix} = 0. \end{aligned} \tag{5.9}$$

Here, the coefficients $\alpha_i, \beta_i, \gamma_i \in K$. For the third term, we have $W'_i = (w'_{i1}, W'_{i-})$ are tuples of positive integers such that $\operatorname{depth}(W'_i) < r$. For the last term, since $w(U)+w(W) = w-q \leq q-2$, we have V_i are tuples of positive integers such that all components are less than or equal to $q-1$.

We denote by H_r the following claim: For any tuples of positive integers U and $W = (w_1, \dots, w_r)$ of depth r, $\boldsymbol{\epsilon} \in (\mathbb{F}_q^\times)^{\operatorname{depth}(U)}$ of any depth, $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in (\mathbb{F}_q^\times)^r$, and $\epsilon \in \mathbb{F}_q^\times$ such that $w(U)+w(W)+q = w$, the AMZV's $\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon & \boldsymbol{\lambda}\\ U & q & W\end{pmatrix}$ and $\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon\lambda_1 & \boldsymbol{\lambda}_-\\ U & q+w_1 & W_-\end{pmatrix}$ belong to $\langle \mathcal{AT}^1_w\rangle$.

We will show that H_r holds for all $r \geq 0$ by induction on r. For $r = 0$, we know that $W = \emptyset$. The explicit expression for $\mathcal{B}_{\boldsymbol{\epsilon},U}(R_\epsilon)$ is given by

$$S_d\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon\\ U & q\end{pmatrix} + \epsilon^{-1}D_1 S_d\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon & 1\\ U & 1 & q-1\end{pmatrix} + \epsilon^{-1}D_1 S_d\begin{pmatrix}\epsilon_1 & \dots & \epsilon_{n-1} & \epsilon_n\epsilon & 1\\ u_1 & \dots & u_{n-1} & u_n+1 & q-1\end{pmatrix} = 0.$$

Since $u_i \leq w(U) = w-q \leq q-2$, we deduce that $\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon\\ U & q\end{pmatrix} \in \langle \mathcal{AT}^1_w\rangle$ as required.

Suppose that $H_{r'}$ holds for any $r' < r$. We now show that H_r holds. We proceed again by induction on w_1. For $w_1 = 1$, we apply the formula (5.9). As $w(U)+w(W) = w-q \leq q-2$, by induction we deduce that all the terms except the first two ones in this expression belong to $\langle \mathcal{AT}^1_w\rangle$. Thus, for any $\epsilon \in \mathbb{F}_q^\times$,

$$\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon & \boldsymbol{\lambda}\\ U & q & W\end{pmatrix} + \zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon\lambda_1 & \boldsymbol{\lambda}_-\\ U & q+1 & W_-\end{pmatrix} \in \langle \mathcal{AT}^1_w\rangle. \tag{5.10}$$

We take $\epsilon = 1$. As the first term lies in $\mathcal{AT}^1_w$ by definition, we deduce that

$$\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \lambda_1 & \boldsymbol{\lambda}_-\\ U & q+1 & W_-\end{pmatrix} \in \langle \mathcal{AT}^1_w\rangle.$$

Thus, in Equation (5.10) we now know that the second term lies in $\langle \mathcal{AT}^1_w\rangle$, which implies that

$$\zeta_A\begin{pmatrix}\boldsymbol{\epsilon} & \epsilon & \boldsymbol{\lambda}\\ U & q & W\end{pmatrix} \in \langle \mathcal{AT}^1_w\rangle.$$

We suppose that H_r holds for all $W' = (w'_1, \dots, w'_r)$ such that $w'_1 < w_1$. We have to show that H_r holds for all $W = (w_1, \dots, w_r)$. The proof is similar to that of the base step $w_1 = 1$. We first consider the formula (5.9). As $w(U)+w(W) = w-q \leq q-2$, we can deduce from the induction hypothesis that

$$\zeta_A\begin{pmatrix}\epsilon & \epsilon & \lambda \\ U & q & W\end{pmatrix} + \zeta_A\begin{pmatrix}\epsilon & \epsilon\lambda_1 & \lambda_- \\ U & q+w_1 & W_-\end{pmatrix} \in \langle \mathcal{AT}^1_w\rangle. \tag{5.11}$$

From similar arguments as in the base step $w_1 = 1$, we can deduce that the first term and the second term of Equation (5.11) belong to $\langle \mathcal{AT}^1_w\rangle$. The proof is complete. □

Remark 5.12. The condition $w \leq 2q-2$ is essential in the previous proof as it allows us to significantly simplify the expression of $\mathcal{B}_{\epsilon,U}\mathcal{C}_{\lambda,W}(R_\epsilon)$ (see Equation (5.11)). For $w = 2q-1$, the situation is already complicated but we can manage to prove Proposition 5.11. Unfortunately, we are not able to extend it to $w = 2q$.

Proposition 5.13. *Let either $w \leq 3q-3$, or $w = 3q-2, q = 2$. Then* $\dim_K \mathcal{AZ}_w \geq s(w)$.

Proof. We outline a proof of this theorem and refer the reader to [26] for more details. For $1 \leq w \leq 3q-2$, we denote by $\mathcal{I}'_w$ the set of tuples $\mathfrak{s} = (s_1,\dots,s_r) \in \mathbb{N}^r$ of weight w as follows:

- For $1 \leq w \leq 2q-2$, $\mathcal{I}'_w$ consists of tuples $\mathfrak{s} = (s_1,\dots,s_r) \in \mathbb{N}^r$ of weight w, where $s_i \neq q$ for all i.
- For $2q-1 \leq w \leq 3q-3$, $\mathcal{I}'_w$ consists of tuples $\mathfrak{s} = (s_1,\dots,s_r) \in \mathbb{N}^r$ of weight w of the form
  - either $s_i \neq q, 2q-1, 2q$ for all i,
  - or there exists a unique integer $1 \leq i < r$ such that $(s_i, s_{i+1}) = (q-1, q)$.
- For $w = 3q-2$ and $q > 2$, $\mathcal{I}'_w$ consists of tuples $\mathfrak{s} = (s_1,\dots,s_r) \in \mathbb{N}^r$ of weight w of the form
  - either $s_i \neq q, 2q-1, 2q, 3q-2$ for all i,
  - or there exists a unique integer $1 \leq i < r$ such that $(s_i, s_{i+1}) \in \{(q-1,q), (2q-2,q)\}$, but $\mathfrak{s} \neq (q-1, q-1, q)$,
  - or $\mathfrak{s} = (q-1, 2q-1)$.
- For $q = 2$ and $w = 3q-2 = 4$, $\mathcal{I}'_w$ consists of the following tuples: $(2,1,1)$, $(1,2,1)$ and $(1,3)$.

We denote by $\mathcal{AT}'_w$ the subset of AMZV's given by

$$\mathcal{AT}'_w := \left\{\zeta_A\begin{pmatrix}\epsilon \\ \mathfrak{s}\end{pmatrix} : \mathfrak{s} \in \mathcal{I}'_w, \text{ and } \epsilon_i = 1 \text{ whenever } s_i \in \{q, 2q-1\}\right\}.$$

Thus, if either $w \leq 3q-3$, or $w = 3q-2, q = 2$, then one shows that

$$|\mathcal{AT}'_w| = s(w).$$

Further, for $w \leq 3q-3$ and any $(\mathfrak{s};\epsilon) = (s_1,\dots,s_r;\epsilon_1,\dots,\epsilon_r) \in \mathbb{N}^r \times (\mathbb{F}_q^\times)^r$, if $\zeta_A\begin{pmatrix}\varepsilon \\ \mathfrak{s}\end{pmatrix} \in \mathcal{AT}'_w$, then $\zeta_A\begin{pmatrix}s_1 & \dots & s_{r-1} \\ \epsilon_1 & \dots & \epsilon_{r-1}\end{pmatrix}$ belongs to $\mathcal{AT}'_{w-s_r}$. This property allows us to apply Theorem 3.4 and show by induction on $w \leq 3q-3$ that the AMZV's in $\mathcal{AT}'_w$ are all linearly independent over K. The proof is similar to that of Theorem 4.4. We apply Theorem 3.4 and reduce to solve a system of σ-linear equations. By direct but complicated calculations, we show that there does not exist any nontrivial solutions and we are done. For $w = 3q-2$ and $q = 2$, it can be treated separately by the same method. □

Remark 5.14. 1) We note that the MZV's $\zeta_A(1, 2q-2)$ and $\zeta_A(2q-1)$ (resp. $\zeta_A(1, 3q-3)$ and $\zeta_A(3q-2)$) are linearly dependent over K by [28, Theorem 3.1]. This explains the above ad hoc construction of $\mathcal{AT}'_w$.

2) Despite extensive numerical experiments, we cannot find a suitable basis $\mathcal{AT}'_w$ for the case $w = 3q-1$.

Acknowledgements. This project is carried out within the framework of the France–Korea International Research Laboratory/Network in Mathematics (FKmath).

Funding statement. The first named author (B.-H. Im) was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MSIT) (No. 2020R1A2B5B01001835) and (NRF-2023R1A2C1002385). Two of the authors (KN. L. and T. ND.) were partially supported by the Excellence Research Chair 'L-functions in positive characteristic and applications' funded by the Normandy Region. The fourth author (T. ND.) was partially supported by the ANR Grant COLOSS ANR-19-CE40-0015-02. The fifth author (LH. P.) was supported by the grant ICRTM.02-2021.05 funded by the International Center for Research and Postgraduate Training in Mathematics (VAST, Vietnam).

Competing interests. The authors have no competing interest to declare.

Ethical standards. The research meets all ethical guidelines, including adherence to the legal requirements of the study country.

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