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GEOMETRICAL ASPECTS OF THE SYSTEM $|\nabla v|^2=\alpha(v)$, $\nabla^2 v=\beta(v)$ AND APPLICATIONS TO THE NONLINEAR WAVE EQUATION

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1. Formulation of the problem

Let E be n-dimensional $(n \geqq 2)$ real vector space with a nondegenerate symmetric scalar product $(.|.)\colon E\times E \to R^1$ with an arbitrary signature $(p, n-p)$. Let us consider a second order partial differential equation (P.D.E.) of the form:

$$\nabla^2 v = \phi(v, |\nabla v|^2), \tag{1.1}$$

where ϕ is a given function of two variables, v is an unknown function (defined on an open subset $0 \subset E$), $|\nabla v|^2 := (\nabla v|\nabla v)$ is the square of the gradient ∇v of the function v and ∇^2, denotes the Laplace–Beltrami operator.

For simplicity we assume that all considered functions, maps and manifolds are of the class C^∞. All our considerations are of a local character. In a more rigorous formulation germs of functions (mappings, manifolds) should be used instead of functions (mappings, manifolds).

Let us consider a class of solutions of (1.1) with the property that $|\nabla v|^2$ is constant on each level of the function v, that is

$$\exists \alpha(\cdot)\colon |\nabla v|^2 = \alpha(v). \tag{1.2}$$

Obviously, in this case $\nabla^2 v$ may also be expressed by v; i.e.

$$\nabla^2 v = \beta(v),\ \beta(v) := \phi(\text{v}, \alpha(v)).$$

On the other side, if a given function u (defined on a certain domain of the space E) satisfies the system:

$$\begin{aligned} |\nabla u|^2 &= \alpha_0(u) \\ \nabla^2 u &= \beta_0(u) \end{aligned} \tag{1.3}$$

(that is, when $|\nabla u|^2$ and $\nabla^2 u$ can be expressed by u) then every function v obtained from

u by the invertible transformation $u \to v := v(u)$ has the same property:

$$\begin{cases} |\nabla v|^2 = v'(u)^2 \alpha_0(u) =: \alpha(v) \\ \nabla^2 v = v''(u)\alpha_0(u) + v'(u)\beta_0(u) =: \beta(v). \end{cases} \tag{1.4}$$

System (1.3), where α_0, β_0 are treated as arbitrary (not fixed) functions, characterizes the congruence of the levels of the function u.

Inserting $v = v(u)$ into (1.1) we obtain, by virtue of (1.4), the following second order O.D.E.

$$\alpha_0(u)v''(u) + \beta_0(u)v'(u) = \phi(v(u),\ \alpha_0(u)v'(u)^2). \tag{1.5}$$

Thus from every solution u of the system (1.3) we can generate (by integrating (1.5)) a two-parameter family of solutions of (1.1).

There are two well known examples of such a procedure. If we set $u(x) = (q|x)$ (plane wave solutions), where $q \in E$ is a constant vector, then $\alpha_0(u) = |q|^2$, $\beta_0(u) = 0$, so (1.5) has the form:

$$|q|^2 v''(u) = \phi(v(u),\ |q|^2 v'(u)^2). \tag{1.6}$$

For $u(x) = \sqrt{\pm(x|x)}$ (spherically symmetric solutions) we have $\alpha_0(u) = \pm 1$, $\beta_0(u) = \pm n - 1/u$, and (1.5) has the form:

$$v''(u) + \frac{n-1}{u} v'(u) = \pm \phi(v(u),\ \pm v'(u)^2). \tag{1.7}$$

(Note that the notion of the spherical symmetry is here understood as symmetry with respect to the group $0(p, n-p)$, not only $0(n)$-symmetry as it is usually treated.)

Let us introduce the following definitions: Function u, defined on the domain $0 \subset E$, will be called *isotropic at the point* $x_0 \in 0$ if $|\nabla u|^2 = 0$ in a certain neighbourhood $0_{x_0} \subset 0$ of the point x_0. If not it will be called *nonisotropic at the point* x_0.

In this paper, the general form of isotropic and nonisotropic solutions of (1.1) with the property (1.2) will be given. Examples (solutions of nonlinear field equations) illustrating the described procedures can be found in [1–5].

2. Nonisotropic solutions

We have the following:

Theorem 1. *Suppose $v: E \supset 0 \to R^1$ is a solution of* (1.1) *satisfying the condition* (1.2). *Let us assume that v is nonisotropic at the point $x_0 \in 0$. Then in a certain neighbourhood $0_{x_0} \subset 0$ of the point x_0 the function v may be described as a composite function, $v = v(u)$, where $u: E \to R^1$ is of one of the following forms:*

$$u(x) = \tfrac{1}{2}(Px|x) + (q|x) \tag{2.1(i)}$$

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where $P \in L(E, E), q \in E$ *and* $P^* = P$(*i.e.* P *is symmetric with respect to* $(.|.)$), $P^2 = 0$, $|q|^2 =: \varepsilon = \pm 1$ *and* $Pq = 0$.

$$u(x) = \sqrt{\varepsilon(P(x-a)|x-a)}, \tag{2.1)(ii}$$

where $P \in L(E, E)$ *is a non-zero orthogonal projector* ($P^* = P^2 = P$), $a \in E$ *and* $\varepsilon = \pm 1$.

The function v *is a solution of the O.D.E.* (1.5), *which has the form:*

$$v''(u) = \varepsilon\phi(v(u), \varepsilon v'(u)^2), \tag{2.2)(i}$$

if u *is of the form* (2.1)(i) *or*

$$v''(u) + \frac{r-1}{u} v'(u) = \varepsilon\phi(v(u), \varepsilon v'(u)^2), \tag{2.2)(ii}$$

where $r := \operatorname{rank} P \in \{1, 2, \ldots, n\}$, *if* u *is of the form* (2.1)(ii).

Remark 2.1. Functions u defined by (2.1)(i) or (2.1)(ii) satisfy the equations (1.3) with $\alpha_0(u) = \varepsilon = \pm 1$ and $\beta_0(u) = 0$ or $\beta_0(u) = \varepsilon(r-1)u$ respectively. Thus equations (2.2) are a special case of (1.5).

Remark 2.2. The class of solutions described by the Theorem 1 contains, in particular, plane waves and spherically symmetric solutions. They are of the type (i) with $P = 0$ and of the type (ii) with $P = 1$ respectively. The remaining (ii)-type solutions are cylindrically symmetric.

Remark 2.3. Sometimes in applications another equivalent form of equations (2.1) can be useful. To obtain it, let us consider a subspace $I := \operatorname{im} P \in E$, then $I^\perp = (\operatorname{im} P^*)^\perp = \ker P$ since P is symmetric.

For the case (i), condition $P^2 = 0$ implies $I \subset I^\perp$, i.e. I is an *isotropic subspace*. Then for any basis $(q_1, \ldots, q_r)$ of I we have $(q_i|q_j) = 0$. Let us put $Px = \sum_{i=1}^r (p^i|x)q_i$, where $p^i \in E$. Since $\ker P = I^\perp$ we have $p^i \in (I^\perp)^\perp = I$ and thus $p^i = \sum_{j=1}^r p^{ij} q_j$, where $\|p^{ij}\|$ is a non-singular and (by $P = P^*$) symmetric matrix. Change of the basis (q_i) transforms the coefficients p^{ij} to coefficients of a bilinear form. Thus (q_i) may be chosen in such a way that $p^{ij} = \varepsilon_i \delta^{ij}, \varepsilon_i = \pm 1$. Moreover $Pq = 0$ is equivalent to $q \in I^\perp$, i.e. $(q|q_i) = 0$. Hence we have

$$u(x) = \tfrac{1}{2} \sum_{1 \le i \le r} \varepsilon_i (q_i|x)^2 + (q|x),$$

$$(q_i|q_j) = 0,\ (q_i|q) = 0,\ \varepsilon_i = \pm 1 \text{ for } i, j = 1, 2, \ldots, r \tag{2.3)(i}$$

$$(q|q) = \varepsilon = \pm 1.$$

In turn, for the case (ii) $P^2 = P$ implies $I \oplus I^\perp = E$, i.e. I is a *non-singular subspace*. It

$(q_1,\dots,q_r)$ is an orthonormal basis of the subspace I, then we have

$$u(x)=\sqrt{\sum_{1\le i\le r}\varepsilon_i(q_i|x-a)^2}$$
$$(q_i|q_j)=\varepsilon_i\delta_{ij},\ \varepsilon_i=\pm1 \text{ for } i,j=1,\dots,r,\ \varepsilon=\pm1. \tag{2.3)(ii}$$

Remark 2.4. It follows from the Witt Lemma, that the maximal dimension of isotropic subspaces $I\subset E$ is $\min(p,n-p)$, where $(p,n-p)$ is the signature of $(.|.)$. Thus if E is *Euclidean space* (i.e. $p=n$), then solutions of the type (i) occur only with $P=0$, i.e. they are plane waves.

Let us study in more detail the case of *Minkowski space*, when $p=1$. Assume, that E is of the form $E=E_1\ominus E_m, m:=n-1$, where $E_1\cong R^1$, E_m are Euclidean spaces of dimension 1, m respectively. The symbol $\ominus$ denotes that E is the Cartesian product $E_1\times E_m$, in which the scalar product of two vectors $x=(x^0,\bar x)$, $y=(y^0,\bar y)$ is defined by $(x|y):=x^0\cdot y^0-\bar x\cdot\bar y$.

Under this assumption, formulae (2.3) can be written in the more explicit form:

$$u(x)=\frac{x^0-\beta\bar e\cdot\bar x}{|1-\beta^2|^{1/2}},\ (\text{rank } P=0), \tag{2.4)(i)$_0$}$$

$$u(x)=\frac{a^2}{2}(x^0-a^0-\bar e_1\cdot\bar x)-\bar e\cdot\bar x,\ (\text{rank } P=1), \tag{2.4)(i)$_1$}$$

$$u(x)=\left|\frac{(x^0-a^0-\beta\bar e_1\cdot(\bar x-\bar a))^2}{1-\beta^2}-\sum_{1<i\le r}(\bar e_i\cdot(\bar x-\bar a))^2\right|^{1/2} \tag{2.4)(ii}$$

Hence $\bar e,\bar e_1,\dots,\bar e_r$ denote versors in space E_m. Moreover those among them which appear explicitly in each formulae (2.4)(i)$_0$, (2.4)(i)$_1$ and (2.4)(ii) are assumed to be mutually orthogonal. Parameter $\beta(\beta^2\neq1)$ can also take the value $\pm\infty$. For example, in this case formula (2.4)(i)$_0$ reduces to $u(x)=\mp\bar e\bar x$.

The number $\varepsilon=\pm1$ (the same which occurs in O.D.E. (2.2)) is determined in the following way: $\varepsilon=\operatorname{sgn}(1-\beta^2)$ in the case (2.4)(i)$_0$, $\varepsilon=-1$ in the case (2.4)(i)$_1$; in the case (2.4)(ii) ε has the sign of the expression under the symbol $|\cdot|^{1/2}$.

Formulae (2.4) can be obtained from (2.3) by an appropriate choice of the vectors q,q_i. Namely, taking q in the form $q=(1,\beta\bar e)/|1-\beta^2|^{1/2}$ we get (2.4)(i)$_0$; the cases $q=(1,0)$ and $q=(0,\bar e)$ correspond to the choices $\beta=0$, and $\beta=\infty$ respectively. To obtain (2.4)(i)$_1$ we take $q_1=a(1,\bar e_1)$ with $a\neq0$ and $q=(0,\bar e)+b(1,\bar e_1)$; then $a^0:=-b/a^2$. For (2.4)(ii), one needs to insert into (2.3)(ii), $q_1=(1,\beta\bar e_1)/|1-\beta^2|^{1/2}$ (or $q_1=(0,\bar e_1)$ for $\beta=\infty$) and $q_1=(0,e_1)$ for $1<i\leqq r$.

Remark 2.5. If ϕ depends on v only, i.e. $\phi=g(v)$, then (2.2) has the form $v''(u)+r-1/uv'(u)=g(v)$. Such O.D.E.s (called Emden-type equations) were investigated in [6] for some classes of the function g.

Outline of the proof of Theorem 1. Assume that v satisfies (1.1) and (1.2). If we put $v=v(u)$, where function $v(\cdot)$ is a solution of the O.D.E.

$$v'(u)^2=|\alpha(v)|, \tag{2.5}$$

then from the transformation rule (1.4) we obtain:

$$\begin{cases}|\nabla u|^2=\varepsilon,\\ \nabla^2 u=\beta_0(u),\end{cases} \tag{2.6}$$

where $\varepsilon=\operatorname{sgn}\alpha(v)=\pm 1$ and $\beta_0(\cdot)$ is an arbitrary function of one variable. We define $f:=-\nabla^2 u$ and $X:=\nabla u$ and $D:=-\nabla X$. X is a vector field and D is a tensor field of the type $\binom{1}{1}$. It is well known [7] from the theory of symmetric polynomials, that for coefficients τ_k of the characteristic polynomial of the tensor D we have the relation $\det(D-\lambda 1)=:\sum_{0\le k\le n}\tau_k(-\lambda)^{n-k}$ and therefore τ_k may be expressed by quantities $\sigma_k:=\operatorname{tr}(D^k)$:

$$\tau_k=\frac{1}{k!}\begin{vmatrix}\sigma_1 & 1 & 0 & \dots & 0\\ \sigma_2 & \sigma_1 & 2 & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ \sigma_{k-1} & \sigma_{k-2} & \sigma_{k-3} & \dots & k-1\\ \sigma_k & \sigma_{k-1} & \sigma_{k-2} & \dots & \sigma_1\end{vmatrix} \quad \text{for } k=1,2,\dots,n. \tag{2.7}$$

The condition $(2.6)_1$ means that $(X|X)=\varepsilon$. From the definition we conclude $(X|Y)=\nabla_Y u$ and $DY=-\nabla_Y X$ for an arbitrary vector field Y. First we show that

$$\nabla_X X=0$$

This condition means that integral curves of the field X are geodesics, parametrized by an affine parameter [8]. Indeed, because of the symmetry of the Levi–Civita connection we have

$$\nabla_X(X|Y)-\nabla_Y(X|X)=(X|[X,Y])=(X|\nabla_X Y-\nabla_Y X)$$

Thus, using $\nabla_Y(X|X)=2(X|\nabla_Y X)=0$ and the Leibniz rule we obtain that $(\nabla_X X|Y)=0$ for arbitrary Y, hence

$$\nabla_X X=-D\cdot X=0; \tag{2.8}$$

so $0=\nabla(\nabla_X X)=\nabla_X(\nabla X)+\nabla X\cdot\nabla X$, that is $\nabla_X D=D^2$. Thus by simple induction with respect to $k\geqq 0$ we obtain:

$$(\nabla_X)^k f=k!\cdot\operatorname{tr}(D^{k+1})=k!\cdot\sigma_{k+1}.$$

But $\nabla_X f = -\nabla_X(\beta_0(u))$ by virtue of $(2.6)_2$, and $\nabla_X u = |\nabla u|^2 = \varepsilon$, so $\nabla_X f = -\varepsilon\beta_0'(u)$. Hence by induction for $k \geqq 0: (\nabla_X)^k f = -\varepsilon^k \beta_0^{(k)}(u)$. Therefore $\sigma_l = -\varepsilon^{l-1}/(l-1)!\beta_0^{(l-1)}$ for $l \geqq 1$, so it follows from (2.7) that τ_k can be expressed as a function of u

$$\tau_1 = \tau_1(u), \tau_2 = \tau_2(u), \ldots, \tau_n = \tau_n(u) \tag{2.9}$$

Let us notice that, what is a well known fact in differential geometry of surfaces **[8, 9]**, the coefficients $\tau_1, \ldots, \tau_{n-1}$ may be interpreted as curvatures (i.e. principal invariants of the second fundamental tensor) of the levels of the function u. Indeed let us consider the tensor $D|_{\Sigma_C}$ where Σ_C denotes C-level of function u. Since $D = -\nabla X$ and $n := X|_{\Sigma_C}$ is a normal field of unit vectors on Σ_C, then $D|_{\Sigma_C}$ acting on the vectors tangent to Σ_C coincides with the second fundamental tensor of hyperplane Σ_C. In turn (2.8) means that $D|_{\Sigma_C}$ vanishes on a vector normal to Σ_C. Hence $\tau_n = \det(D) = 0$ and values of the functions $\tau_1, \ldots, \tau_{n-1}$ are equal to principal invariants of the second fundamental tensor of the level passing through a given point. Thus (2.9) says, that curvatures $\tau_1 = \tau_1(C), \ldots, \tau_{n-1} = \tau_{n-1}(C)$ are constant on every level Σ_C.

The general form of hypersurfaces, whose curvatures are all constant is described in **[1]**. Namely we have:

Lemma. *Suppose $\Sigma \subset E$ is a hypersurface of dimension $m = n-1$ having at every point $x \in \Sigma$ a normal unit vector $n(x)$ with $|n(x)|^2 = \varepsilon = \pm 1$. Then all curvatures $\tau_1, \ldots, \tau_m$ of hypersurface Σ have constant values if Σ is (at least locally) the level of the function u of the form* (2.1)(i) or (2.1)(ii).

We have proved that the level $\Sigma_C(C := u(x_0))$ passing through the given point x_0 satisfies the assumptions of the lemma. Then Σ_C is (locally) a level of the function $\tilde{u}$, where $\tilde{u}$ is of the form (2.1)(i) or (2.1)(ii). As we noticed in Remark 2.1, function $\tilde{u}$ is also a solution of $(2.6)_1$. But it follows from the Hamilton–Jacobi theory that for two solutions $u, \tilde{u}$ of $(2.6)_1$ having a common level we must have (at least locally) $\tilde{u} = \pm u + \text{constant}$. O.D.E. (2.5) is invariant under the transformations $u \to u + \text{constant}$. (The method of characteristics applied to the problem $|\nabla u|^2 = \pm 1$, $u|_\Sigma = C$, gives the following result: $u(x) = C \pm$ (distance of point x from surface Σ) for x sufficiently close to Σ). So it may be assumed that u and $\tilde{u}$ coincide, that is u is of the form (2.1)(i) or (2.1)(ii). So it may be assumed that $u = \tilde{u}$ is of the form (2.1)(i) or (2.1)(ii). This ends the proof of Theorem 1 since equations (2.2) are a consequence of (1.5) (c.f. Remark 2.1).

3. Isotropic solutions

According to the definition from Section 1, the isotropic solution of (1.1) is a solution satisfying the condition $|\nabla v|^2 = 0$. In the case of a positive defined metric, (i.e. when E is an *Euclidean space*), the solution of the problem is evident: $|\nabla v|^2 = 0$ implies $v = \text{constant}$, so isotropic solutions are constant functions $v = v_0$ with v_0 satisfying $\phi(v_0, 0) = 0$.

Except for this trivial case we have found all the isotropic solutions of (1.1) only for spaces with the metric of signature $(1, n-1)$, i.e. when E is a *Minkowski space*. It turns out that (1.1) possesses nontrivial (i.e. non-constant) isotropic solutions only for the very special case $\phi(v, 0) \equiv 0$. More precisely we have:

Theorem 2. *For the function v defined in some neighbourhood of the point x_0 of Minkowski space E the following conditions are equivalent:*

(a) $|\nabla v|^2 = 0,\ \nabla^2 v = 0;$ (3.1)

(b) $|\nabla v|^2 = 0,\ \nabla^2 v = \beta(v)$ (3.2)

for a certain function β;

(c) *v may be defined in the implicit form by the equation:*

$$v = F((q(v)|x - x_0)), \tag{3.3}$$

where F is a function of one variable and $v \to q(v) \in E$ is a one-parameter family of non-zero isotropic vectors (i.e. such that $|q(v)|^2 = 0$).

Remark 3.1. The rather surprising fact that (b) implies (a) simply means that $\beta = 0$ is the compatibility condition for the system (3.2).

Remark 3.2. Let us consider the more general case of arbitrary signature $(p, n-p)$ of the metric $(.|.)$. Suppose that for $v \in R^1$, $(q^1(v), \ldots, q^r(v))$ is a basis of an isotropic subspace $I(v) \subset E$ of dimension $r \leqq \min(p, n-p)$ and $F(., \ldots, .)$ is a real function of r variables. Then the equation

$$v = F((q^1(v)|x - x_0), \ldots, (q^r(v)|x - x_0)) \tag{3.4}$$

(which is a generalization of (3.3)) defines in a neighbourhood of x_0, in implicit form, the function $x \to v(x)$. Differentiating (3.4) we obtain

$$\nabla v = \frac{\sum_{i=1}^{r} F_{,i} \cdot q^i}{1 - \sum_{i=1}^{r} F_{,i} \cdot (\dot{q}^i | x - x_0)},$$

and thus $|\nabla v|^2 = 0$. By successive differentiation we obtain

$$\nabla^2 v = \frac{2 \sum_{i=1}^{r} F_{,i}(\dot{q}^i | \nabla v)}{1 - \sum_{i=1}^{r} F_{,i}(\dot{q}^i | x - x_0)} = \frac{2 \sum_{i,j=1}^{r} F_{,i} F_{,j} (\dot{q}^i | q^j)}{\left(1 - \sum_{i=1}^{r} F_{,i}(\dot{q}^i | x - x_0)\right)^2}$$

since $(q^i|q^j) = 0$ implies $(\dot{q}^i|q^j) = -(\dot{q}^j|q^i)$. Then the function v is a solution of P.D.E. system (3.1).

This observation could suggest that Theorem 2 can be generalized for the case of an arbitrary signature $(p, n-p)$, by replacing (3.3) with (3.4). However, this generalization is not true as can be easily seen from the Example 2.

Remark 3.3. The characteristic peculiarity of function v, which may be described in the form (3.4), is that each level Σ_C of the function v is the cylinder:

$$F((q^1(C)|x-x_0),\dots,(q^r(C)|x-x_0))=C,$$

profiled by the level of F and with $\langle q^1(C),\dots,q^r(C)\rangle^{\perp}$ as the generating subspace. For the simplest case, $r=1$, (3.4) reduces to (3.3) and the cylinders become the hyperplanes.

In the theory of quasilinear P.D.E., solutions of the form (3.4) are known as nonplanar simple waves [10]. In particular (3.3) defines the so-called Riemann waves.

Remark 3.4. Using the notation of Remark 2.4 we assume $E=E_1\ominus E_m$. Then isotropic vector $q(v)$ can be (after suitable normalization) written in the form $q(v)=(1,\bar{e}(v))$, where $\bar{e}(v)\in E_m$ is a versor. So (3.3) takes the form.

$$v=F(x^0-x_0^0-\bar{e}(v)\cdot(\bar{x}-\bar{x}_0)),\ \bar{e}(v)^2=1, \tag{3.5}$$

Outline of the Proof of Theorem 2. Implication (a)⇒(b) is obvious; (c)⇒(a) was proved in Remark 3.2. Thus it remains to prove that (b)⇒(c). It follows from (3.2) that (in Cartesian coordinates) $\sum_{\alpha,\beta,\mu,\nu=1}^{n} g^{\alpha\beta}g^{\mu\nu}v_{,\alpha\mu}v_{,\beta\nu}=0$, where $\|g^{\mu\nu}\|$ denotes the inverse matrix $\|g_{\mu\nu}\|$ of the scalar product $(.|.)$ hence from $\nabla|\nabla v|^2=0$ (i.e. $\sum_{\lambda,\mu=1}^{n} g^{\lambda\mu}v_{,\lambda}v_{,\nu\mu}=0$), with the assumption $\operatorname{sgn}(.|.)=(1,n-1)$, it can be deduced by algebraic considerations, that $v_{,\mu\nu}=v_{,\mu\beta\nu}+v_{,\nu\beta\mu}$ where $\beta\mu$ denotes coefficients of some covariant vector. Therefore ∇v has constant direction on the levels of function v. So $\nabla v=\phi\cdot q(v)$ where $q(v)$ is an isotropic vector and ϕ is a function defined in a neighbourhood of x_0. But then we have

$$\nabla v=\frac{\phi}{1+\phi(\dot{q}(v)|x-x_0)}\nabla(q(v)|x-x_0),$$

which means that v remains constant on each level of the function $(q(v)|x-x_0)$. So (3.3) holds.

Example 1. Let $n=3$ and (x^0,x^1,x^2) denote Cartesian coordinates in E such that $|x|^2=(x^0)^2-(x^1)^2-(x^2)^2$. Then formula

$$v:=\phi+\arccos\frac{x^0}{r}$$

where $x^1=r\cos\phi$, $x^2=r\sin\phi$, denotes a multivalued function outside the light cone, satisfying (3.1). It can be easily checked, that in the neighbourhood of an arbitrary point x_0^0, $x^1=r_0\cos\phi_0$, $x^2=r_0\ \sin\phi_0$ (outside the light cone), the equation (3.5) with $\bar{e}(v):=(\cos v,\sin v)$ and $F(z):=\phi_0+\arccos z+x_0^0/r_0$ describes the single-valued branch of function v.

Example 2. We shall now construct a solution v for the system (3.1), which can not be written in the form (3.4), for the case when $E=E_2\ominus E_3$, which has signature $(2,3)$. Let $R^1\ni p\to q(p)\in E$ be a curve such that

$$|q(p)|^2=0,\ |\dot{q}(p)|^2=0, \tag{3.6}$$

and $q(p)$, $\dot{q}(p)$, $\ddot{q}(p)$ are linearly independent vectors for each $p \in R^1$. For example the curve $g(p) := (\bar{e}_2(p), \bar{e}_3(p))$, where $\bar{e}_2(p) := a(\cos p/a, \sin p/a) \in E_2$, $\bar{e}_3(p) := (\cos p, \sin p, \sqrt{a^2-1}) \in E_3$ (with $a^2 > 1$), satisfies the above conditions. (Note that such a curve $q(p)$ does not exist for the case of signature $(1, n-1)$.) Moreover, let $p \to \lambda(p) \in R^1$ be a function satisfying conditions $\lambda(0) = 0$, $\ddot{\lambda}(0) = 0$. Then the system of equations:

$$\begin{cases} v = (q(p)|x) - \lambda(p) \\ 0 = (q(p)|x) - \dot{\lambda}(p) \end{cases} \tag{3.7}$$

defines in the implicit form (in the neighbourhood of $x = 0$), functions $x \to v(x)$, $x \to p(x)$ such that $v(0) = -\lambda(0)$, $p(0) = 0$. Moreover, differentiating (3.6) we have

$$\nabla v = q(p),\ \nabla p = \frac{\dot{q}(p)}{\ddot{\lambda}(p) - (\ddot{q}(p)|x)},\ \nabla^2 u = \frac{|\dot{q}(p)|^2}{\ddot{\lambda}(p) - (\ddot{q}(p)|x)} \tag{3.8}$$

Thus, by virtue of (3.6), the function v satisfies (3.1).

Assume that v can be described by the implicit equation (3.4). In Remark 3.2 we found that $\nabla v \in I(v) := \langle q^1(p), \dots, q^r(p) \rangle$. Thus, by (3.8), $q(p) \in I(v)$. But (3.8) also implies, that ∇v and ∇p are linearly independent, so v and p are functionally independent. Hence $q(p) \in I(v)$ implies $\dot{q}(p)$ and $\ddot{q}(p) \in I(v)$. But $\dim I(v) = r \leqq \min(2, 3) = 2$ due to the isotropy of $I(v)$. This is in contradiction with the assumption that $q(p)$, $\dot{q}(p)$, $\ddot{q}(p)$ are linearly independent, so the function v cannot be written in the form (3.4).

Remark 3.5. For a long time there has been much interest in localized solutions of the nonlinear wave equation $\nabla^2 v = g(v)$ in Minkowski space, with signature $(1, 3)$. The procedure given by us reduces the problem of finding static, spherically symmetric solutions of the wave equation to the Emden equation [11]

$$v''(u) + \frac{2}{u} v'(u) = g(v). \tag{3.9}$$

In the literature [12] one can find existence theorems for the solution of (3.9) with property $v'(0) = 0$ and $\lim_{u \to \infty} v(u) = 0$ for the right hand side of the form

$$g(v) = \sin v,\ g(v) = \cos v,\ g(v) = \sinh v, g(v) = \cosh v,$$

$$g(v) = \exp v,\ g(v) = v^n,\ g(v) = \mu v + \lambda v^3$$

For some solutions their Taylor expansions were found and others are given in the tables [12].

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REFERENCES

1. G. Cieciura and A. M. Grundland, A certain class of solutions of the non-linear wave equations, *J. Math. Phys.* **25** (1984), 3460–3469.

2. C. B. Collins, All solutions to a nonlinear system of complex potential equations, *J. Math. Phys.* **21** (1980), 240–248.

3. C. B. Collins, Complex equations, special relativity and complex Minkowski space-time, *J. Math. Phys.* **21** (1980), 249–255.

4. A. M. Grundland, J. Harnad and P. Winternitz, Solutions of the multidimensional Sine–Gordon equation obtained by symmetry reduction, *Kinam. Rev. Fis.* **4** (1982), 333–344.

5. R. Hermann, *Differential Geometry and the Calculus of Variations* (Math. Sci. Press, Brookline, Massachusetts, 1977).

6. H. Berestycki and P. L. Lions, Existence of stationary states in nonlinear scalar field equation, in C. Bardos and D. Bessis (eds.), *Bifurcation Phenomena in Mathematical Physics and Related Topics* (D. Reidel Publ. Comp., New York, 1980).

7. S. Lang, *Algebra* (Addison-Wesley Publ. Comp., London, 1970).

8. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vols. I and II (J. Wiley-Interscience, New York, 1962).

9. S. Sternberg, *Lectures on Differential Geometry* (Prentice-Hall, Englewood Cliffs, NY, 1964).

10. Z. Peradzynski, Riemann invariants for the nonplanar k-waves, *Bull. Acad. Polon. Sci. Ser. Tech.* **19** (1971), 67–74.

11. R. Emden, *Gaskugeln* (Berlin and Leipzig, 1907).

12. H. Davis, *Introduction to Nonlinear Differential and Integral Equations* (Dover, New York, 1960).

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