ANZIAM J. 44(2003), 539–559

MINIMAX CONTROL OF AN ELLIPTIC VARIATIONAL BILATERAL PROBLEM

QIHONG CHEN[1]

(Received 8 December, 1999; revised 27 July, 2000)

Abstract

This paper deals with a minimax control problem for semilinear elliptic variational inequalities associated with bilateral constraints. The control domain is not necessarily convex. The cost functional, which is to be minimised, is the sup norm of some function of the state and the control. The major novelty of such a problem lies in the simultaneous presence of the nonsmooth state equation (variational inequality) and the nonsmooth cost functional (the sup norm). In this paper, the existence conditions and the Pontryagin-type necessary conditions for optimal controls are established.

1. Introduction

In this paper, we consider an optimal control problem in which the state y is governed by a controlled semilinear elliptic bilateral variational inequality

$$\begin{cases} y \in H^2(\Omega) \cap H_0^1(\Omega), & \\ \varphi \le y \le \psi & \text{in} \quad \Omega, \\ (Ay - f(x, y, u))(y - \varphi) \le 0 & \text{in} \quad \Omega, \\ (Ay - f(x, y, u))(y - \psi) \le 0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

and the cost functional is taken to be

$$J(y, u) = \operatorname*{ess\,sup}_{x \in \Omega} L(x, y(x), u(x)), \tag{1.2}$$

where A is an elliptic differential operator and (y, u) is a pair satisfying (1.1).

One of the motivations for the above problem is given as follows. Consider the deformation of a membrane constrained by two obstacles. We would like to

[1]Department of Applied Mathematics, Shanghai University of Finance and Economics, 777 Guoding Road, Shanghai 200433, P. R. China; e-mail: chenqih@online.sh.cn.

design the shape of the membrane so that the largest deviation of the perpendicular displacement y from the desired position, say y_d, is minimised. In this case, we could take $L(x, y, \varphi, u) = |y - y_d(x)|^2$. Since the problem consists of minimising a "maximum", it is usually referred to as a *minimax control problem*.

Minimax control problems seem to arise more naturally in applications than the standard problem involving integral cost, especially when one is attempting to minimise the maximum deviation from the desired goal. However such problems have not been thoroughly studied (especially for infinite-dimensional systems). The minimax control problem for ordinary differential equations has been studied by several authors (see [2, 11]) and the Pontryagin maximum principle for finite-dimensional minimax problem was derived in [2]. The first infinite-dimensional version of the Pontryagin principle for the minimax problem was presented in [14] with the state equation being a second-order semilinear elliptic partial differential equation. Different aspects of optimal control problems for variational inequalities have been discussed by many authors (see for example [1, 6, 9]). However, to the best of our knowledge, minimax control problems for variational inequalities have never been discussed before. The *nonsmoothness* of the cost leads to more complicated necessary conditions for minimax control problems and this is one of the reasons for the lack of investigation thus far.

With respect to the control domain and the data involved, we make the following assumptions.

(H_1) The region $\Omega \subset \mathbb{R}^n$ is bounded with $C^{1,1}$ boundary $\partial\Omega$; U is a Polish space (a separable complete metric space) and $\mathscr{U} = \{u\colon \Omega \to U | u(\cdot) \text{ is measurable}\}$.

(H_2) Operator A is defined by

$$Ay(x) = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_i y(x))$$

with $a_{ij} \in C^1(\overline{\Omega})$, $a_{ij} = a_{ji}$, $1 \le i, j \le n$, and for some $\lambda > 0$,

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \ge \lambda \sum_{i=1}^{n} |\xi_i|^2, \quad \forall x \in \Omega,\ (\xi_1, \xi_2, \dots, \xi_n) \in \mathbb{R}^n.$$

(H_3) The function $f : \Omega \times \mathbb{R} \times U \to \mathbb{R}$ has the following properties: $f(\cdot, y, u)$ is measurable on Ω, and $f(x, \cdot, u)$ is in $C^1(\mathbb{R})$ with $f(x, \cdot, \cdot)$ and $f_y(x, \cdot, \cdot)$ continuous on $\mathbb{R} \times U$. Moreover, there exists a constant $K > 0$, such that $-K \le f_y \le 0$ on $\Omega \times \mathbb{R} \times U$ and $|f(x, 0, u)| \le K$ on $\Omega \times U$.

(H_4) The function $L\colon \Omega \times \mathbb{R} \times U \to \mathbb{R}$ satisfies the following: $L(\cdot, y, u)$ is measurable on Ω, $L(x, \cdot, u)$ is in $C^1(\mathbb{R})$ with $L(x, \cdot, \cdot)$ and $L_y(x, \cdot, \cdot)$ continuous on $\mathbb{R} \times U$, and for any $R > 0$, there exists a constant $K_R > 0$, such that $|L| + |L_y| \le K_R$ on $\Omega \times [-R, R] \times U$.

Under (H_2), the operator A is associated with a positive symmetric bilinear form $a(\cdot,\cdot)\colon\ H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbb{R}$

$$a(y,z)=\sum_{i,j=1}^{n}\int_\Omega a_{ij}(x)D_iy(x)D_jz(x)\,dx.$$

Given $\varphi,\psi\in W^{2,p}(\Omega)$ $(\forall p\geq 2)$ with $\varphi\leq 0\leq\psi$ on $\partial\Omega$, we set

$$\mathbf{K}=\{z\in H_0^1(\Omega)|\varphi\leq z\leq\psi\ \text{ a.e. in }\ \Omega\}.$$

If y solves (1.1), then

$$y\in\mathbf{K} \tag{1.3}$$

and, for any $z\in\mathbf{K}$, $(z-y)^+$ ($(z-y)^-$, resp.) can differ from 0 only where $y-\psi$ is <0 ($y-\varphi$ is >0) and therefore $Ay-f\geq 0$ ($Ay-f\leq 0$). Thus, by the divergence theorem,

$$\begin{aligned}a(y,z-y)&-\int_\Omega f(x,y,u)(z-y)\,dx\\&=\int_\Omega(Ay-f)(z-y)\,dx\\&=\int_\Omega(Ay-f)(z-y)^+\,dx-\int_\Omega(Ay-f)(z-y)^-\,dx\geq 0,\qquad\forall z\in\mathbf{K}.\end{aligned}\tag{1.4}$$

On the other hand, any $y\in H^2(\Omega)$ satisfying (1.3) and (1.4) must be a solution of (1.1). In fact, fixing any $D\subset\Omega$ and denoting by $\{\chi_n\}$ a sequence of functions from $C_c^\infty(\Omega)$ satisfying $0\leq\chi_n\leq 1$, $\chi_n\to\chi_D$ (characteristic function of D) a.e. in Ω, we can insert $z=y+\chi_n(\varphi-y)$ and $z=y+\chi_n(\psi-y)$ in (1.4) in turn and obtain

$$\int_\Omega(Ay-f)\chi_n(\varphi-y)\,dx\geq 0\quad\text{and}\quad\int_\Omega(Ay-f)\chi_n(\psi-y)\,dx\geq 0,$$

hence also

$$\int_D(Ay-f)(\varphi-y)\,dx\geq 0\quad\text{and}\quad\int_D(Ay-f)(\psi-y)\,dx\geq 0$$

after passing to the limit as $n\to\infty$. By the arbitrariness of D, we arrive at (1.1).

The above discussion yields a weak formulation of the variational bilateral problem (1.1).

DEFINITION 1.1. Suppose $u\in\mathscr{U}$. A function $y\in H_0^1(\Omega)$ is called a weak solution of the variational bilateral problem (1.1) if

$$\begin{cases}y\in\mathbf{K},\\ a(y,z-y)\geq\displaystyle\int_\Omega f(x,y(x),u(x))(z-y)\,dx,\quad\forall z\in\mathbf{K}.\end{cases}\tag{1.5}$$

Any element $u \in \mathscr{U}$ is referred to as a control. Any pair $(y, u) \in H_0^1(\Omega) \times \mathscr{U}$ satisfying (1.5) is called a feasible pair and the corresponding y and u will be referred to as a feasible state and control, respectively. The set of all feasible pairs is denoted by $\mathscr{A}$. Clearly, under (H_1)–(H_4), $\mathscr{U}$ coincides with the set of all feasible controls and for each $u \in \mathscr{U}$, there is a corresponding unique feasible state y (see [13]) and the cost functional (1.2) is well-defined. Hereafter, we always assume (H_1)–(H_4). Thus we can write $J(y, u)$ as $J(u)$ without any ambiguity.

Our minimax control problem can now be stated as follows.

PROBLEM (M). Find a feasible control $\bar{u} \in \mathscr{U}$, such that

$$J(\bar{u}) = \inf_{u \in \mathscr{U}} J(u) \equiv \bar{J}. \tag{1.6}$$

If such a $\bar{u}$ exists, we call it an optimal control. Accordingly, the corresponding state $\bar{y}$ and the feasible pair $(\bar{y}, \bar{u}) \in \mathscr{A}$ will be called an optimal state and pair, respectively.

2. State equation

2.1. A $W^{2,p}$-estimate of state Let us start with a basic $W^{2,p}$-estimate of state.

PROPOSITION 2.1. *Let* (H_1)–(H_3) *hold and* $(y, u) \in \mathscr{A}$. *Then for any* $p \geq 2$,

$$\|y\|_{W^{2,p}(\Omega)} \leq C_p, \tag{2.1}$$

where C_p is a constant independent of the control variable u.

To prove (2.1), we define

$$\beta(t) = \begin{cases} 0, & 0 \leq t < +\infty, \\ -t^2, & -1/2 \leq t < 0, \\ t + 1/4, & -\infty < t < -1/2, \end{cases} \tag{2.2}$$

$$\gamma(t) = \begin{cases} 0, & -\infty < t < 0, \\ t^2, & 0 \leq t < 1/2, \\ t - 1/4, & 1/2 \leq t < +\infty, \end{cases} \tag{2.3}$$

and introduce a family of approximations to the state equation (1.5):

$$\begin{cases} Ay_r + r[\beta(y_r - \varphi) + \gamma(y_r - \psi)] = f(x, y_r, u) \quad \text{in } \Omega, \\ y_r|_{\partial\Omega} = 0. \end{cases} \tag*{(2.4)$_r$}$$

https://doi.org/10.1017/S1446181100012918 Published online by Cambridge University Press

It can be shown that, for any given $u \in \mathscr{U}$ and $r > 0$, $(2.4)_r$ is uniquely solvable in $W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ (see [7]). The set of all pairs $(y_r, u) \in H_0^1(\Omega) \times \mathscr{U}$ satisfying $(2.4)_r$ will be denoted by $\mathscr{A}_r$.

The estimate (2.1) results from the following two lemmas.

LEMMA 2.2. *Let* (H_1)–(H_3) *hold and* $(y_r, u) \in \mathscr{A}_r$. *Then, for any* $p \geq 2$,

$$\|r\beta(y_r - \varphi)\|_{L^p(\Omega)} \leq C_p, \tag{2.5}$$

$$\|r\gamma(y_r - \psi)\|_{L^p(\Omega)} \leq C_p \tag{2.6}$$

and consequently

$$\|y_r\|_{W^{2,p}(\Omega)} \leq C_p, \tag{2.7}$$

where C_p *is a constant independent of* $r > 0$ *and* $u \in \mathscr{U}$.

PROOF. Define, for $t \in \mathbb{R}$, $B(t) = |\beta(t)|^{p-2}\beta(t)$ and $\Gamma(t) = |\gamma(t)|^{p-2}\gamma(t)$. Then we have

$$B(t) \leq 0 \quad \text{and} \quad \Gamma(t) \geq 0 \quad \forall t \in \mathbb{R}, \tag{2.8}$$

$$B(t) = 0 \quad \forall t \geq 0 \quad \text{and} \quad \Gamma(t) = 0 \quad \forall t \leq 0, \tag{2.9}$$

$$B'(t) = (p-1)|\beta(t)|^{p-2}\beta'(t) \geq 0 \quad \text{and} \quad \Gamma'(t) = (p-1)|\gamma(t)|^{p-2}\gamma'(t) \geq 0 \tag{2.10}$$

and, as $p \geq 2$ and $\beta(0) = \gamma(0) = 0$,

$$B(y_r - \varphi),\ \Gamma(y_r - \psi) \in W_0^{1,p}(\Omega) \hookrightarrow W_0^{1,p'}(\Omega),$$

where $p' = p/(p-1) \leq p$ is the conjugate number of p.

Multiplying $(2.4)_r$ by $B(y_r - \varphi)$ and integrating by parts, noting also that (2.9) implies $\beta(y_r - \varphi)\gamma(y_r - \psi) = 0$ a.e. in Ω, we obtain

$$a(y_r, B(y_r - \varphi)) + r\int_\Omega |\beta(y_r - \varphi)|^p\,dx = \int_\Omega f(x, y_r, u)B(y_r - \varphi)\,dx. \tag{2.11}$$

From (2.8), (2.10) and the monotony of $f(x, \cdot, u)$, we see that

$$\int_\Omega f(x, y_r, u)B(y_r - \varphi)\,dx \leq \int_\Omega f(x, \varphi, u)B(y_r - \varphi)\,dx \tag{2.12}$$

and

$$a(y_r - \varphi, B(y_r - \varphi)) \geq 0. \tag{2.13}$$

https://doi.org/10.1017/S1446181100012918 Published online by Cambridge University Press

Then, from (2.11)–(2.13) and Hölder's inequality, we have

$$\begin{aligned} r\|\beta(y_r-\varphi)\|^p_{L^p(\Omega)} &\le \int_\Omega f(x,\varphi,u)B(y_r-\varphi)\,dx - a(\varphi, B(y_r-\varphi)) \\ &= \int_\Omega [f(x,\varphi,u)-A\varphi]B(y_r-\varphi)\,dx \\ &\le \|f(\cdot,\varphi(\cdot),u(\cdot))-A\varphi(\cdot)\|_{L^p(\Omega)}\|\beta(y_r-\varphi)\|^{p-1}_{L^p(\Omega)}. \end{aligned}$$

Thus, using (H_3), we get the desired estimate (2.5). The estimate (2.6) can be obtained similarly, and (2.7) follows immediately from (2.5), (2.6) and the standard elliptic L^P-estimate (see [7]).

LEMMA 2.3. *Let* (H_1)–(H_3) *hold,* $(y_r, u) \in \mathscr{A}_r$ *and* $(y,u)\in\mathscr{A}$. *Then, as* $r\to\infty$, $y_r \to y$ *weakly in* $W^{2,p}(\Omega)$ *and strongly in* $W_0^{1,p}(\Omega)$.

PROOF. By (2.7), we may assume that, as $r\to\infty$, $y_r\to y^*$ weakly in $W^{2,p}(\Omega)$ and strongly in $W_0^1(\Omega)$ for some y^*. It suffices to verify that

$$(y^*, u)\in\mathscr{A} \tag{2.14}$$

since the uniqueness will ensure that $y^*=y$.

First, it follows from $(2.4)_r$ that, for any $z\in\mathbf{K}$,

$$\begin{aligned} a(y_r, z-y_r) &- \int_\Omega f(x,y_r,u)(z-y_r)\,dx \\ &= -r\int_\Omega [\beta(y_r-\varphi)+\gamma(y_r-\psi)](z-y_r)\,dx \ge 0 \end{aligned}$$

(note that $\beta(y_r-\varphi)$ can differ from 0 only when $y_r<\varphi\le z$ and $\gamma(y_r-\psi)$ can differ from 0 only when $y_r>\psi\ge z$). Then the lower semicontinuity yields

$$a(y^*,y^*)\le \varliminf_{r\to 0} a(y_r,y_r)\le a(y^*,z)-\int_\Omega f(x,y^*,u)(z-y^*)\,dx \quad \forall z\in\mathbf{K}. \tag{2.15}$$

Next, for any $\eta\in H_0^1(\Omega)$ with $\eta\ge 0$ a.e. in Ω, we have from $(2.4)_r$ that

$$\int_\Omega \big[\beta(y_r-\varphi)+\gamma(y_r-\psi)\big]\eta\,dx = \frac{1}{r}\left\{\int_\Omega f(x,y_r,u)\eta\,dx - a(y_r,\eta)\right\}\to 0$$

because the terms in { } are bounded. Then, with the help of the dominated convergence theorem,

$$\int_\Omega \big[\beta(y^*-\varphi)+\gamma(y^*-\psi)\big]\eta\,dx = 0,$$

hence

$$\beta(y^* - \varphi) + \gamma(y^* - \psi) = 0 \quad \text{a.e. in } \Omega \tag{2.16}$$

due to the arbitrariness of η. By the definition of $\beta(\cdot)$ and $\gamma(\cdot)$, (2.16) implies that $y^* \in \mathbf{K}$. This, together with (2.15), proves the feasibility of (2.14).

2.2. Continuous dependence of the state on the control In the control set $\mathscr{U}$, we define the distance, called Ekeland's distance, as

$$d(u, v) = m(\{x \in \Omega | u(x) \neq v(x)\}) \quad \forall u, v \in \mathscr{U},$$

where m denotes the Lebesgue measure. We can show that $(\mathscr{U}, d)$ is a complete metric space (see [9]).

The following result is concerned with the continuity of the state y with respect to the control u under the above metric.

PROPOSITION 2.4. *Let* (H_1)–(H_3) *hold and* $(y, u), (y_k, u_k) \in \mathscr{A}$ $(k = 1, 2, \dots)$. *If* $d(u_k, u) \to 0$, *then for any* $p \geq 2$, $\|y_k - y\|_{W^{1,p}(\Omega)} \to 0$.

PROOF. From Proposition 2.1, we know that, for some subsequence $y_k \to y^*$ weakly in $W^{2,p}(\Omega)$, strongly in $W_0^{1,p}(\Omega)$. Clearly

$$\varphi(x) \leq y^*(x) \leq \psi(x) \quad \text{a.e. } x \in \Omega. \tag{2.17}$$

Note that

$$\begin{aligned}
&\|f(\cdot, y_k(\cdot), u_k(\cdot)) - f(\cdot, y^*(\cdot), u(\cdot))\|_{L^2(\Omega)} \\
&\quad \leq \|f(\cdot, y_k(\cdot), u_k(\cdot)) - f(\cdot, y_k(\cdot), u(\cdot))\|_{L^2(\Omega)} + \|f(\cdot, y_k(\cdot), u(\cdot)) \\
&\qquad - f(\cdot, y^*(\cdot), u(\cdot))\|_{L^2(\Omega)} \\
&\quad \leq C\left\{d(u_k, u)^{1/2} + \|y_k - y^*\|_{L^2(\Omega)}\right\} \to 0.
\end{aligned}$$

Passing to the limit in (1.5), in which u and y are replaced by u_k and y_k respectively, we obtain

$$a(y^*, z - y^*) \geq \int_\Omega f(x, y^*(x), u(x))(z - y^*)\, dx \quad \forall z \in \mathbf{K}.$$

This, combined with (2.17), means that y^* is a solution of (1.5). By the uniqueness, we must have that $y^* = y$ and the whole sequence $\{y_k\}$ converges to y strongly in $W_0^{1,p}(\Omega)$.

2.3. Some reductions For the sake of convenience, let us make some reductions (just as in [14]).

First of all, by scaling, we may assume that

$$m(\Omega) = 1. \tag{2.18}$$

Next, from the $W^{2,p}$-estimate of state and Sobolev's embedding, it follows that y is uniformly bounded and independent of $u \in \mathscr{U}$. Thus, by (H_4), we may assume without loss of generality that

$$|L(x, y, u)| \leq M \quad \forall (x, y, u) \in \Omega \times \mathbb{R} \times U. \tag{2.19}$$

Set

$$\tilde{L}(x, y, u) = \frac{L(x, y, u) + M + 1}{2M + 2} \quad (x, y, u) \in \Omega \times \mathbb{R} \times U.$$

By (2.19), we know that

$$0 < \frac{1}{2M + 2} \leq \tilde{L}(x, y, u) \leq \frac{2M + 1}{2M + 2} < 1 \quad \forall (x, y, u) \in \Omega \times \mathbb{R} \times U.$$

Since minimising $J(u)$ is equivalent to minimising

$$\tilde{J}(u) = \operatorname*{esssup}_{x \in \Omega} \tilde{L}(x, y(x), u(x)),$$

we may, again without loss of generality, assume at the beginning that

$$0 < a \leq L(x, y, u) \leq b < 1 \quad \forall (x, y, u) \in \Omega \times \mathbb{R} \times U \tag{2.20}$$

for some constants a and b. We will retain assumptions (2.18) and (2.20) for the rest of this paper.

3. Existence of optimal controls

This section is devoted to the existence of optimal controls. Let us first recall the following.

DEFINITION 3.1 (see [3, 9]). Let Y be a Banach space and Z be a metric space. Let $\Lambda: Z \to 2^Y$ be a multifunction. We say Λ possesses the Cesari property at $z \in Z$, if $\bigcap_{\delta > 0} \overline{\text{co}}\, \Lambda(O_\delta(z)) = \Lambda(z)$, where $\overline{\text{co}}\, E$ stands for the closed convex hull of the set E and $O_\delta(z)$ is the δ-neighbourhood of the point z. If Λ has the Cesari property at every point $z \in Z$, we simply say that Λ has the Cesari property on Z.

DEFINITION 3.2. Let $\Omega \subset \mathbb{R}^n$ be some Lebesgue measurable set and U be a Polish space. Let $\Lambda: \Omega \to 2^U$ be a multifunction. The function $u: \Omega \to U$ is called a selection of $\Lambda(\cdot)$ if $u(x) \in \Lambda(x)$ a.e. $x \in \Omega$. If such a u is measurable, then u is called a measurable selection of $\Lambda(\cdot)$.

The following gives the existence of measurable selections.

LEMMA 3.3 (see [8]). *Let $\Lambda: \Omega \to 2^U$ be measurable taking closed set values. Then $\Lambda(\cdot)$ admits a measurable selection.*

We refer the readers to [9, pp. 100–101] for the proof of Lemma 3.3.

To establish the existence of an optimal control for Problem (M), we first introduce the following set: $\Lambda(x, y) = \{(\xi, \eta) \in \mathbb{R}^2 \mid \xi \geq L(x, y, u), \eta = f(x, y, u), u \in U\}$ and make the following assumption.

(H_5) For almost all $x \in \Omega$, the mapping $y \mapsto \Lambda(x, y)$ has the Cesari property on $\mathbb{R}$.

THEOREM 3.4. *Let* (H_1)–(H_5) *hold. Then Problem* (M) *admits at least one optimal control $\bar{u} \in \mathscr{U}$.*

PROOF. The proof is essentially similar to that given in [4]. Here, we only give an outline.

Let $\{u_k\} \subset \mathscr{U}$ be a minimising sequence satisfying

$$J(u_k) \leq \bar{J} + 1/k. \tag{3.1}$$

By the Mazur theorem, (H_5), and the measurable selection theorem (Lemma 3.3), we can find a feasible pair $(\bar{y}, \bar{u}) \in \mathscr{A}$, such that

$$L(x, \bar{y}(x), \bar{u}(x)) \leq \overline{L}(x) \quad \text{a.e. } x \in \Omega, \tag{3.2}$$

where

$$\overline{L}(x) = \underline{\lim}_{j \to \infty} \xi_j(x) \quad \text{a.e. } x \in \Omega \tag{3.3}$$

and

$$\xi_j(\cdot) = \sum_{i \geq 1} \alpha_{ij} L(\cdot, y_{i+j}(\cdot), u_{i+j}(\cdot)) \tag{3.4}$$

with

$$\alpha_{ij} \geq 0; \quad \sum_{i \geq 1} \alpha_{ij} = 1 \quad \forall j \tag{3.5}$$

and $(y_{i+j}, u_{i+j}) \in \mathscr{A}$ for every i, j.

https://doi.org/10.1017/S1446181100012918 Published online by Cambridge University Press

Now, from (2.18), (3.1), (3.4) and (3.5), it follows that, for any j and $p > 1$,

$$\|\xi_j\|_{L^p(\Omega)} \le \|\xi_j\|_{L^\infty(\Omega)} \le \sum_{i\ge 1} \alpha_{ij} J(u_{i+j}) \le \sum_{i\ge 1} \alpha_{ij} \left(\bar{J} + \frac{1}{i+j}\right) \le \bar{J} + \frac{1}{j}.$$

This yields

$$\varliminf_{j\to\infty} \|\xi_j\|_{L^p(\Omega)} \le \bar{J} \quad \forall p > 1. \tag{3.6}$$

Consequently, by (3.2), (3.3), (3.6) and Fatou's lemma, we obtain

$$J(\bar{u}) \le \|\overline{L}\|_{L^\infty(\Omega)} = \lim_{p\to\infty} \|\overline{L}\|_{L^p(\Omega)} \le \lim_{p\to\infty} \varliminf_{j\to\infty} \|\xi\|_{L^p(\Omega)} \le \bar{J}.$$

This means that $\bar{u}$ is an optimal control of Problem (M).

4. Regularisation

Note that, in discussing Problem (M), our difficulty is twofold: both the state equation and the cost functional are nonsmooth. Thus it is natural that both of them should be regularised.

4.1. Approximation of the state In Section 2, we introduced a family of approximate equations $(2.4)_r$ and denoted by $\mathscr{A}_r$ the set of all pairs $(y_r, u) \in H_0^1(\Omega) \times \mathscr{U}$ satisfying $(2.4)_r$. Here we will prove a useful convergence result for the approximate states.

PROPOSITION 4.1. *Let* (H_1)–(H_3) *hold and let* $\{u_r\} \subset \mathscr{U}$ *be any sequence,* $(y_r, u_r) \in \mathscr{A}_r$ *and* $(y^r, u_r) \in \mathscr{A}$. *Then, for any* $p \ge 2$,

$$\lim_{r\to\infty} \|y_r - y^r\|_{W_0^{1,p}(\Omega)} = 0. \tag{4.1}$$

PROOF. By Proposition 2.1 and Lemma 2.2, we have that, for any $p \ge 2$,

$$\|y_r\|_{W^{2,p}(\Omega)} + \|y^r\|_{W^{2,p}(\Omega)} \le C_p \tag{4.2}$$

with C_p independent of $r > 0$. Thus, we may assume that, for some subsequence, $y_r \to y$ strongly in $H_0^1(\Omega)$ and

$$\|y^r\|_{H^1(\Omega)\cap L^\infty(\Omega)} \le C \quad \text{with } C \text{ independent of } r > 0. \tag{4.3}$$

The same argument as that used in the proof of Lemma 2.3 shows that $\varphi \le y \le \psi$ a.e. in Ω.

Now, letting $z_r = y_r \vee \varphi \wedge \psi$, we have $z_r \to y$ strongly in $H_0^1(\Omega)$, and consequently,

$$\|z_r - y_r\|_{H_0^1(\Omega)} \to 0. \tag{4.4}$$

Recalling that y_r and y^r solve $(2.4)_r$ and (1.5) respectively, we have

$$\begin{aligned} a(y_r, y_r - y^r) = -r \int_\Omega [\beta(y_r - \varphi) + \gamma(y_r - \psi)](y_r - y^r)\, dx \\ + \int_\Omega f(x, y_r, u_r)(y_r - y^r)\, dx \end{aligned} \tag{4.5}$$

and

$$a(y^r, z_r - y^r) \geq \int_\Omega f(x, y^r, u_r)(z_r - y^r)\, dx. \tag{4.6}$$

By the monotonicity of $f(x, \cdot, u)$, $\beta(\cdot)$ and $\gamma(\cdot)$ we see that

$$\begin{gathered} \int_\Omega [f(x, y_r, u_r) - f(x, y^r, u_r)](y_r - y^r)\, dx \leq 0, \\ \int_\Omega \beta(y_r - \varphi)(y_r - y^r)\, dx \geq 0 \quad \text{and} \quad \int_\Omega \gamma(y_r - \psi)(y_r - y^r)\, dx \geq 0. \end{gathered} \tag{4.7}$$

Here, we have used the fact that $y^r \geq \varphi > y_r$ when $y_r < \varphi$, and $y^r \leq \psi < y_r$ when $y_r > \psi$. From (4.3)–(4.7), we may deduce that

$$\begin{aligned} a(y_r - y^r, y_r - y^r) &\leq a(y^r, z_r - y_r) - \int_\Omega f(x, y^r, u_r)(z_r - y_r)\, dx \\ &\leq C\|z_r - y_r\|_{H^1(\Omega)} \to 0 \end{aligned}$$

which implies

$$\lim_{r\to\infty} \|y_r - y^r\|_{H_0^1(\Omega)} = 0. \tag{4.8}$$

Moreover, by Lions' interpolation theorem (see [10]), we have the following lemma (see Lemma 4.2 below), which, together with (4.2) and (4.8), results in (4.1).

LEMMA 4.2. *Suppose $p > 2$, $\{w_r\}$ is bounded in $W^{2,p}(\Omega)$ and $\lim_{r\to\infty} \|w_r\|_{H^1(\Omega)} = 0$. Then $\lim_{r\to\infty} \|w_r\|_{W^{1,p}(\Omega)} = 0$.*

4.2. Approximation of the cost functional We shall now introduce a regularisation of the cost functional. We first recall a well-known real analysis result.

LEMMA 4.3. *Let Ω be bounded and $w \in L^\infty(\Omega)$. Then*

$$\lim_{r\to\infty} \|w\|_{L^r(\Omega)} = \|w\|_{L^\infty(\Omega)}.$$

The above lemma suggests that we can regularise our cost functional (1.2) by using

$$J_r(u) = \|L(\cdot, y_r(\cdot), u(\cdot))\|_{L^r(\Omega)} \quad \forall u \in \mathscr{U}, \tag{4.9}$$

where $r > 1$ and $(y_r, u) \in \mathscr{A}_r$.

We will see that the functional $J_r(\cdot)$ is continuous on $(\mathscr{U}, d)$ and is a reasonable regularisation of our nonsmooth cost functional $J(\cdot)$.

PROPOSITION 4.4. *Let* (H_1)–(H_4) *hold. Then we have the following*:

(i) *For any fixed* $r > 1$, $J_r(u)$ *is continuous on* $(\mathscr{U}, d)$;
(ii) *For any given* $u \in \mathscr{U}$, $\lim_{r\to\infty} J_r(u) = J(u)$.

Before proving the above proposition, we state a lemma, which can be easily obtained from Proposition 4.1. This lemma will play an interesting role below.

LEMMA 4.5. *Let* (H_1)–(H_4) *hold. Then, for any sequence* $\{u_r\} \subset \mathscr{U}$, *we have*

(i) $\lim_{r\to\infty} \left[J_r(u_r) - \|L(\cdot, y^r(\cdot), u_r(\cdot))\|_{L^r(\Omega)}\right] = 0$;
(ii) $\lim_{r\to\infty} \left[J(u_r) - \|L(\cdot, y_r(\cdot), u_r(\cdot))\|_{L^\infty(\Omega)}\right] = 0$,

where $(y^r, u_r) \in \mathscr{A}$ *and* $(y_r, u_r) \in \mathscr{A}_r$.

PROOF OF PROPOSITION 4.4. (i) Let (y_r, u), $(y_{r,k}, u_k) \in \mathscr{A}_r$ $(k = 1, 2, \dots)$ and $d(u_k, u) \to 0$. By the continuity of the approximate state y_r with respect to the control u under Ekeland's metric $d(\cdot, \cdot)$, we know that, for any $p \geq 2$,

$$\|y_{r,k} - y_r\|_{W^{1,p}(\Omega)} \to 0.$$

Take $p > n$. By Sobolev's embedding, we have $\|y_{r,k} - y_r\|_{L^\infty(\Omega)} \to 0$. Thus we get

$$\begin{aligned} |J_r(u_k) - J_r(u)| &\leq \|L(\cdot, y_{r,k}(\cdot), u_k(\cdot)) - L(\cdot, y_r(\cdot), u(\cdot))\|_{L^r(\Omega)} \\ &\leq C[\|y_{r,k} - y_r\|_{L^r(\Omega)} + d(u_k, u)^{1/r}] \to 0 \quad (k \to \infty). \end{aligned}$$

(ii) We have that $u \in \mathscr{U}$ and $(y, u) \in \mathscr{A}$. Since

$$|J_r(u) - J(u)| \leq |J_r(u) - \|L(\cdot, y(\cdot), u(\cdot))\|_{L^r(\Omega)}| + |\|L(\cdot, y(\cdot), u(\cdot))\|_{L^r(\Omega)} - J(u)|$$

and (recalling Lemma 4.3)

$$\lim_{r\to\infty} \|L(\cdot, y(\cdot), u(\cdot))\|_{L^r(\Omega)} = J(u), \tag{4.10}$$

our conclusion is an immediate consequence of Lemma 4.5.

We point out that the convergence in (4.10) is not uniform in $u \in \mathscr{U}$. Nor is the convergence in (ii) of Proposition 4.4.

4.3. Convergence theorem Before going further, let us make an additional assumption:

(H_6) $L(x, y, u)$ is continuous on $\Omega \times \mathbb{R} \times U$ and there exists a nondecreasing continuous function ω: $[0, +\infty) \to [0, +\infty)$ with $\omega(0) = 0$, such that

$$|L(\tilde{x}, \tilde{y}, u) - L(x, y, u)| \leq \omega(|\tilde{x} - x| + |\tilde{y} - y|)$$

$\forall (x, y, u),\ (\tilde{x}, \tilde{y}, u) \in \Omega \times \mathbb{R} \times U$.

In what follows, we denote $\bar{J}_r = \inf_{u \in \mathscr{U}} J_r(u)$.

Our main result in this section is the following convergence theorem, which will be essential for deriving the optimality conditions later.

THEOREM 4.6. *Let* (H_1)–(H_4) *and* (H_6) *hold. Then*

$$\lim_{r \to \infty} \bar{J}_r = \bar{J}. \tag{4.11}$$

To prove Theorem 4.6, we need the following lemmas.

LEMMA 4.7. *Let* (H_1)–(H_4) *and* (H_6) *hold. Then, for any* $(y_r, u) \in \mathscr{A}_r$ *and* $\alpha \in \mathbb{R}$, *there exists* $(\tilde{y}_r, \tilde{u}) \in \mathscr{A}_r$ *satisfying*

$$\{x \in \Omega | \tilde{u}(x) \neq u(x)\} \subset D \tag{4.12}$$

and

$$L(x, \tilde{y}_r(x), \tilde{u}(x)) \leq \alpha + \omega(\tilde{C} m(D)^{1/n} + \|\tilde{y}_r - y_r\|_{L^\infty(\Omega)}) \quad a.e.\ x \in \Omega \tag{4.13}$$

where $D = \{x \in \Omega | L(x, y_r(x), u(x)) > \alpha\}$, $\tilde{C}$ *is a constant independent of* α, r *and* u, *and* $\omega(\cdot)$ *is the (uniform) modulus of* L *given in* (H_6).

PROOF. First, we may let $0 < m(D) \leq 1$ (recall $m(\Omega) = 1$), since (4.13) is trivially true when $m(D) = 0$. Let $\delta > 0$ be such that

$$m(D) < m(B_\delta(0)) < 2m(D), \tag{4.14}$$

where $B_\delta(x)$ denotes the open ball centred at x with radius δ. Then we can choose $x_i \in \Omega$, such that $\bigcup_{i \geq 1} B_\delta(x_i) \supset \Omega$. By (4.14), we know that for each $i \geq 1$, there exists an $\tilde{x}_i \in B_\delta(x_i) \setminus D$. For such $\tilde{x}_i$, we have $L(\tilde{x}_i, y_r(\tilde{x}_i), u(\tilde{x}_i)) \leq \alpha$.

Now we define

$$\tilde{u}(x) = \begin{cases} u(x) & x \in \Omega \setminus D, \\ u(\tilde{x}_i) & x \in D \cap \left[B_\delta(x_i) \setminus \bigcup_{j=1}^{i-1} B_\delta(x_j)\right] \end{cases}$$

https://doi.org/10.1017/S1446181100012918 Published online by Cambridge University Press

and let $(\tilde{y}_r, \tilde{u}) \in \mathscr{A}_r$.

Clearly, (4.12) holds. For any $x \in \Omega \setminus D$, we have

$$\begin{aligned} L(x, \tilde{y}_r(x), \tilde{u}(x)) &= L(x, \tilde{y}_r(x), u(x)) \\ &\leq \alpha + \omega(|\tilde{y}_r(x) - y_r(x)|) \leq \alpha + \omega(\|\tilde{y}_r - y_r\|_{L^\infty(\Omega)}). \end{aligned}$$

For $x \in D \cap [B_\delta(x_i) \setminus \bigcup_{j=1}^{i-1} B_\delta(x_j)]$, we have (note (4.14))

$$\begin{aligned} L(x, \tilde{y}_r(x), \tilde{u}(x)) &= L(x, \tilde{y}_r(x), u(\tilde{x}_i)) \\ &\leq \alpha + \omega(|x - \tilde{x}_i| + |\tilde{y}_r(x) - y_r(\tilde{x}_i)|) \\ &\leq \alpha + \omega(C\delta + \|\tilde{y}_r - y_r\|_{L^\infty(\Omega)}) \\ &\leq \alpha + \omega(\tilde{C}m(D)^{1/n} + \|\tilde{y}_r - y_r\|_{L^\infty(\Omega)}). \end{aligned}$$

Hence (4.13) follows.

In the above, we have used the fact that, for $p > n$, by (4.2) and Sobolev's embedding, $\|y_r\|_{C^1(\bar{\Omega})} \leq \|y_r\|_{W^{2,p}(\Omega)} \leq C_p$ with C_p being independent of r and u.

LEMMA 4.8. *Let* (H_1)–(H_4) *and* (H_6) *hold. Then, for any sequence* $\{u_r\} \subset \mathscr{U}$,

$$\varliminf_{r\to\infty} J_r(u_r) \geq \bar{J}. \tag{4.15}$$

PROOF. Suppose that (4.15) does not hold. Then, for some $\varepsilon > 0$ and some subsequence (still denoted by itself) $\{u_r\} \subset \mathscr{U}$, we have $J_r(u_r) \leq \bar{J} - 2\varepsilon$, $\forall r \geq r_0$. Let $(y_r, u_r) \in \mathscr{A}_r$ and $D_r = \{x \in \Omega | L(x, y_r(x), u_r(x)) > \bar{J} - \varepsilon\}$. Then

$$\bar{J} - 2\varepsilon \geq J_r(u_r) \geq \|L(\cdot, y_r(\cdot), u_r(\cdot))\|_{L^r(D_r)} \geq (\bar{J} - \varepsilon) m(D_r)^{1/r}.$$

Thus

$$m(D_r) \leq \left(\frac{\bar{J} - 2\varepsilon}{\bar{J} - \varepsilon}\right)^r \to 0 \quad (r \to \infty). \tag{4.16}$$

According to Lemma 4.7, there exists $(\tilde{y}_r, \tilde{u}_r) \in \mathscr{A}_r$ such that

$$d(\tilde{u}_r, u_r) = m\{x \in \Omega | \tilde{u}_r(x) \neq u(x)\} \leq m(D_r) \tag{4.17}$$

and $L(x, \tilde{y}_r(x), \tilde{u}_r(x)) \leq \bar{J} - \varepsilon + \omega(\tilde{C}m(D_r)^{1/n} + \|\tilde{y}_r - y_r\|_{L^\infty(\Omega)})$ a.e. $x \in \Omega$. This implies

$$\|L(\cdot, \tilde{y}_r(\cdot), \tilde{u}_r(\cdot))\|_{L^\infty(\Omega)} \leq \bar{J} - \varepsilon + \omega(\tilde{C}m(D_r)^{1/n} + \|\tilde{y}_r - y_r\|_{L^\infty(\Omega)}). \tag{4.18}$$

From the standard L^2-estimate, we can deduce that $\|\tilde{y}_r - y_r\|_{H_0^1(\Omega)} \leq Cd(\tilde{u}_r, u_r)^{1/2}$. Note that, by (4.16)–(4.17),

$$d(\tilde{u}_r, u_r)^{1/2} \leq \left(\frac{\bar{J} - 2\varepsilon}{\bar{J} - \varepsilon}\right)^{r/2} \to 0.$$

Using Lemma 4.2, we further obtain (for $p > n$)

$$\|\tilde{y}_r - y_r\|_{L^\infty(\Omega)} \leq \|\tilde{y}_r - y_r\|_{W_0^{1,p}(\Omega)} \to 0. \tag{4.19}$$

Combining (4.16), (4.18)–(4.19), we get

$$\overline{\lim_{r\to\infty}} \|L(\cdot, \tilde{y}_r(\cdot), \tilde{u}_r(\cdot))\|_{L^\infty(\Omega)} \leq \bar{J} - \varepsilon. \tag{4.20}$$

On the other hand, by Lemma 4.5 (ii), we have

$$\underline{\lim}_{r\to\infty} \|L(\cdot, \tilde{y}_r(\cdot), \tilde{u}_r(\cdot))\|_{L^\infty(\Omega)} \geq \underline{\lim}_{r\to\infty} [\|L(\cdot, \tilde{y}_r(\cdot), \tilde{u}_r(\cdot))\|_{L^\infty(\Omega)} - J(\tilde{u}_r) + \bar{J}] = \bar{J}.$$

This contradicts (4.20). Hence (4.15) holds.

PROOF OF THEOREM 4.6. Let $u_r \in \mathscr{U}$ be such that $J(u_r) < \bar{J} + 1/r$ and let $(y^r, u_r) \in \mathscr{A}$. By Hölder's inequality, we have (recall $m(\Omega) = 1$)

$$\|L(\cdot, y^r(\cdot), u_r(\cdot))_{L^r(\Omega)} \leq J(u_r) < \bar{J} + 1/r.$$

It then follows that

$$\bar{J}_r \leq J_r(u_r) \leq J_r(u_r) - \|L(\cdot, y^r(\cdot), u_r(\cdot))\|_{L^r(\Omega)} + \bar{J} + 1/r.$$

Thus, by Lemma 4.5 (i), we get

$$\overline{\lim_{r\to\infty}} \bar{J}_r \leq \bar{J}. \tag{4.21}$$

On the other hand, for any $r > 1$, one can find $u_r \in \mathscr{U}$ such that $J_r(u_r) < \bar{J}_r + 1/r$. Hence, by Lemma 4.8, we have

$$\underline{\lim}_{r\to\infty} \bar{J}_r \geq \underline{\lim}_{r\to\infty} J_r(u_r) \geq \bar{J}. \tag{4.22}$$

Finally, (4.11) follows from (4.21) and (4.22).

5. Necessary conditions

Now we are in a position to prove the following Pontryagin principle for Problem (M).

THEOREM 5.1. *Let* (H_1)–(H_4) *and* (H_6) *hold and* $(\bar{y}, \bar{u}) \in \mathscr{A}$ *be an optimal pair for Problem* (M). *Then there exist* $\bar{z} \in W^{1,p'}(\Omega)$ *with* $p' = p/(p-1) \in (1, n/(n-1))$ *and* $\bar{\lambda}, \bar{\mu} \in L^\infty(\Omega)^*$, *such that*

$$\begin{cases} A\bar{z} - f_y(x, \bar{y}, \bar{u})\bar{z} = \bar{\lambda} L_y(x, \bar{y}, \bar{u}) + \bar{\mu} & \text{in } \Omega, \\ \bar{z}|_{\partial\Omega} = 0 \end{cases} \tag{5.1}$$

https://doi.org/10.1017/S1446181100012918 Published online by Cambridge University Press

and

$$\bar{z}(x)f(x, \bar{y}(x), \bar{u}(x)) = \min_{u \in U_0(x)} \bar{z}(x)f(x, \bar{y}(x), u) \quad a.e.\ x \in \Omega_0 \tag{5.2}$$

where

$$\begin{cases} \Omega_0 = \{x \in \Omega | L(x, \bar{y}(x), \bar{u}(x)) < \bar{J}\}, \\ U_0(x) = \{u \in U | L(x, \bar{y}(x), u) < \bar{J}\} \quad x \in \Omega, \end{cases}$$

$$\bar{\lambda}(\Omega) \equiv \langle \bar{\lambda}, \chi_\Omega \rangle \geq a > 0. \tag{5.3}$$

Moreover, in the case $m(\Omega_0) > 0$, for any $0 < \sigma < m(\Omega_0)$, there exists a measurable set $S_\sigma \subset \Omega_0$ with $m(S_\sigma) \geq \sigma$, such that

$$\bar{\lambda}(S_\sigma) = 0. \tag{5.4}$$

We note that in general, the above $\bar{\lambda}$ is only a finitely additive measure and is not necessarily in $\mathscr{M}(\overline{\Omega})$. If $\bar{\lambda}$ happens to be in $\mathscr{M}(\overline{\Omega})$, then there exists a measurable set $S \subset \Omega_0$ with $m(\Omega_0 \setminus S) = 0$, such that $\bar{\lambda}(S) = 0$. This means that the support of $\bar{\lambda}$ is disjoint with Ω_0.

PROOF. Given $r > 1$ and $\alpha_r = (J_r(\bar{u}) - \bar{J}_r + 1/r)^{1/2} > 0$, from Proposition 4.4 (ii) and Theorem 4.6, we see that

$$\alpha_r \to 0. \tag{5.5}$$

Since $J_r(u)$ is continuous on $(\mathscr{U}, d)$ (recall Proposition 4.4 (i)) and

$$J_r(\bar{u}) \leq \bar{J}_r + \alpha_r^2 = \inf_{u \in \mathscr{U}} J_r(u) + \alpha_r^2,$$

by Ekeland's variational principle (see [5]), there exists a $u_r \in \mathscr{U}$, such that

$$d(u_r, \bar{u}) \leq \alpha_r, \tag{5.6}$$

$$-\alpha_r d(u, u_r) \leq J_r(u) - J_r(u_r) \quad \forall u \in \mathscr{U}. \tag{5.7}$$

Assume $\Omega_0 \neq \emptyset$ (otherwise, there is nothing to prove, see Remark 2 below) and let $s > 0$ be such that $\Omega_s \equiv \{x \in \Omega \mid L(x, \bar{y}(x), \bar{u}(x)) \leq \bar{J} - s\} \neq \emptyset$. Then denote

$$U_s(x) = \{u \in U | L(x, \bar{y}(x), u) \leq \bar{J} - s\}$$

and define

$$\Gamma_s(x) = \begin{cases} U_s(x) & x \in \Omega_s, \\ \{\bar{u}(x)\} & x \in \Omega \setminus \Omega_s. \end{cases}$$

It is clear that $\Gamma_s: \Omega \to 2^U$ is measurable and takes closed set values. Thus, by Lemma 3.3, there exists a measurable selection $v \in \mathscr{U}$. In what follows, we denote by $\mathscr{V}_s$ the set of all such selections, that is, $\mathscr{V}_s = \{v \in \mathscr{U} \mid v(x) \in \Gamma_s(x) \text{ a.e. } x \in \Omega\}$.

Now, let $(y_r, u_r) \in \mathscr{A}_r$ and $v \in \mathscr{V}_s$ be fixed. By using the technique of spike variation (see [4, Proposition 4.2]) we know that, for any $\rho \in (0, 1)$, there exists a measurable set $E^\rho \subset \Omega$ with $m(E^\rho) = \rho m(\Omega) = \rho$, such that if we define

$$u_r^\rho(x) = \begin{cases} u_r(x), & \text{if } x \in \Omega \setminus (E^\rho \cap \Omega_s), \\ v(x), & \text{if } x \in E^\rho \cap \Omega_s \end{cases}$$

and let $(y_r^\rho, u_r^\rho) \in \mathscr{A}_r$, then

$$\lim_{\rho \to 0} \int_\Omega \left(1 - \frac{1}{\rho}\chi_{E^\rho}\right) [(L(x, y_r(x), v(x)))^r - (L(x, y_r(x), u_r(x)))^r]\,\chi_{\Omega_s}\,dx = 0$$

and $y_r^\rho = y_r + \rho w_r + \theta_r^\rho$ with $\lim_{\rho \to 0}(\|\theta_r^\rho\|_{W^{1,p}(\Omega)}/\rho) = 0$, where w_r satisfies the following:

$$\begin{cases} A w_r + \left\{r[\beta'(y_r - \varphi) + \gamma'(y_r - \psi) - f_y(x, y_r, u_r)\right\} w_r \\ \qquad = [f(x, y_r, v) - f(x, y_r, u_r)]\chi_{\Omega_s} & \text{in } \Omega, \\ w_r|_{\partial\Omega} = 0. \end{cases}$$

Taking $u = u_r^\rho$ in (5.7) and letting $\rho \to 0$, we obtain

$$-\alpha_r \le \frac{1}{\rho}[J_r(u_r^\rho) - J_r(u_r)] \to \int_\Omega [\lambda_r L_y(x, y_r, u_r) w_r + h_r]\,dx \quad (\rho \to 0) \tag{5.8}$$

where

$$\lambda_r(x) = \left(\frac{L(x, y_r(x), u_r(x))}{J_r(u_r)}\right)^{r-1}, \tag{5.9}$$

$$h_r(x) = \frac{[(L(x, y_r(x), v(x)))^r - (L(x, y_r(x), u_r(x)))^r]\chi_{\Omega_s}(x)}{r(J_r(u_r))^{r-1}}. \tag{5.10}$$

Let $z_r \in H_0^1(\Omega)$ be the solution of the following system:

$$\begin{cases} A z_r + \{r[\beta'(y_r - \varphi) + \gamma'(y_r - \psi)] - f_y(x, y_r, u_r)\} z_r \\ \qquad = \lambda_r L_y(x, y_r, u_r) & \text{in } \Omega, \\ z_r|_{\partial\Omega} = 0. \end{cases} \tag{5.11}$$

Then we may deduce from (5.8) that

$$\int_{\Omega_s} [z_r(f(x, y_r, v) - f(x, y_r, u_r)) + h_r]\,dx \ge -\alpha_r. \tag{5.12}$$

In what follows, we shall obtain our final conclusions by making some estimates and taking the limits in (5.11)–(5.12).

First, by (2.18) and (2.20), the function $\lambda_r(x) \geq 0$ satisfies

$$\int_\Omega \lambda_r(x)\,dx \geq \frac{1}{J_r(u_r)^{r-1}} \int_\Omega (L(x, y_r, u_r))^r\,dx = J_r(u_r) \geq a \tag{5.13}$$

and by Hölder's inequality,

$$\|\lambda_r\|_{L^1(\Omega)} = \int_\Omega \lambda_r(x)\,dx \leq \|\lambda_r\|_{L^{r/(r-1)}(\Omega)} = 1. \tag{5.14}$$

Thus we may assume

$$\begin{cases} \lambda_r \to \bar{\lambda} & \text{weakly star in } L^\infty(\Omega)^*, \\ \lambda_r L_y(\cdot, y_r(\cdot), \varphi_r(\cdot), u_r(\cdot)) \to \mu & \text{weakly star in } L^\infty(\Omega)^*. \end{cases}$$

Clearly, by (5.13), $\bar{\lambda}$ satisfies (5.3).

Let $S_\delta(t) \in C^1(\mathbb{R})$ be a family of smooth approximations to sign t, satisfying the following: $S'_\delta(t) \geq 0$, $\forall t \in \mathbb{R}$, and

$$S_\delta(t) = \begin{cases} 1 & \text{if } t > \delta, \\ 0 & \text{if } t = 0, \\ -1 & \text{if } t < -\delta. \end{cases}$$

Multiplying (5.11) by $S_\delta(z_r)$, integrating it over Ω, and letting $\delta \to 0$, we obtain

$$\|r[\beta'(y_r - \varphi) + \gamma'(y_r - \psi)]z_r\|_{L^1(\Omega)} \leq C\|\lambda_r\|_{L^1(\Omega)} \leq C. \tag{5.15}$$

Then, applying the standard estimate for an elliptic equation in L^1 (see [12]) to the system (5.11), we get

$$\|z_r\|_{W^{1,p'}(\Omega)} \leq C \tag{5.16}$$

and, using (5.11), we further get

$$\|r[\beta'(y_r - \varphi) + \gamma'(y_r - \psi)]z_r\|_{W^{-1,p'}(\Omega)} \leq C \tag{5.17}$$

where $p' = p/(p-1) \in (1, n/(n-1))$.

In all the above estimates (5.15)–(5.17), the constant C is independent of $r > 1$. Hence we may let, extracting some subsequence if necessary,

$$\begin{cases} z_r \to \bar{z} & \text{weakly in } W^{1,p'}(\Omega), \text{ strongly in } L^{p'}(\Omega), \\ r[\beta'(y_r - \varphi) + \gamma(y_r - \psi)]z_r \to \overline{\theta} & \text{weakly star in } W^{-1,p'}(\Omega) \cap L^\infty(\Omega)^*. \end{cases}$$

Let $(y^r, u_r) \in \mathscr{A}$. By (5.5)–(5.6) and the continuity of the state on the control, we have, for any $p \geq 2$,

$$\|y^r - \bar{y}\|_{W^{1,p}(\Omega)} \to 0 \tag{5.18}$$

and

$$\|y_r - \bar{y}\|_{W^{1,p}(\Omega)} \to 0 \tag{5.19}$$

where the convergence $\|y_r - y^r\|_{W^{1,p}(\Omega)} \to 0$ (see (4.1)) has been used.

Passing to the limit in (5.11), we see that $\bar{z}$ solves (5.1) with

$$\bar{\mu} = \mu - \overline{\theta} - \bar{\lambda} L_y(\cdot, \bar{y}(\cdot), \bar{u}(\cdot)).$$

Let $\delta_r = |\bar{J}_r - \bar{J}| + \omega(\|y_r - \bar{y}\|_{L^\infty(\Omega)})$, where $\omega(\cdot)$ is the (uniform) modulus of continuity for L given in (H_6). By (5.19) and Theorem 4.6, it is easy to get $\delta_r \to 0$ (as $r \to \infty$). Then, by the definition of $h_r(\cdot)$ (see (5.10)), we have

$$\begin{aligned}\int_\Omega h_r(x)\,dx &\leq \frac{J_r(u_r)}{r} \int_{\Omega_s} \left(\frac{L(x, y_r(x), v(x))}{J_r(u_r)}\right)^r dx \\ &\leq \frac{b}{r} \int_{\Omega_s} \left(\frac{L(x, \bar{y}(x), v(x)) + \delta_r}{\bar{J}_r}\right)^r dx \\ &\leq \frac{b}{r} \left(\frac{\bar{J} - s + \delta_r}{\bar{J} - \delta_r}\right)^r \to 0 \quad (r \to \infty)\end{aligned} \tag{5.20}$$

because $\delta_r \to 0$ and $s > 0$ is fixed.

Hence we take limits in (5.12) to obtain

$$\int_{\Omega_s} \bar{z}[f(x, \bar{y}, v) - f(x, \bar{y}, \bar{u})]\,dx \geq 0 \quad \forall v \in \mathscr{V}_s,\ s > 0. \tag{5.21}$$

The desired conclusion (5.2) thus follows (see Lemma 5.2 below).

Finally, using an argument analogous to that in [9] (with some necessary modifications), we can prove (5.4).

LEMMA 5.2. *Equation* (5.2) *follows from* (5.21).

PROOF. Let

$$H(x, u) = \bar{z}(x) f(x, \bar{y}(x), u) \quad x \in \Omega,\ u \in U.$$

We see that, by (H_3), for any $x \in \Omega$, $H(x, \cdot)$ is continuous on U.

As U is separable, there exists a countable dense set $U_d = \{u_i, i \geq 1\} \subset U$. For each $u_i \in U_d$ and $\Omega_{1/j} \subset \Omega_0$, we denote

$$h_{ij}(x) = [H(x, u_i) - H(x, \bar{u})]\chi_{\Omega_{1/j}}(x) \quad x \in \Omega.$$

https://doi.org/10.1017/S1446181100012918 Published online by Cambridge University Press

Since $h_{ij}(\cdot) \in L^1(\Omega)$, there exists a measurable set $D_{ij} \subset \Omega$ with $m(D_{ij}) = m(\Omega)$, such that any point in D_{ij} is a Lebesgue point of h_{ij}, namely,

$$\lim_{\delta\to 0} \frac{1}{m(B_\delta(x))} \int_{B_\delta(x)} [h_{ij}(\xi) - h_{ij}(x)]\, dx = 0 \quad \forall x \in D_{ij}.$$

Let $D_0 = \bigcap_{i,j\geq 1} D_{ij}$. Then $m(D_0) = m(\Omega)$.

To obtain (5.2), it suffices to prove that, for each $x \in \Omega_0 \cap D_0$ and $v \in U_0(x)$,

$$H(x, \bar{u}(x)) \leq H(x, v). \tag{5.22}$$

Given such an $x \in \Omega_0 \cap D_0$ and $v \in U_0(x)$, by the definitions of Ω_0 and $U_0(x)$, and the density of U_d, we can choose an integer $j > 1$ and a subsequence (still denoted by itself) $\{u_i\} \subset U_d$, such that

$$\begin{cases} x \in \Omega_{1/j}, \quad v \in U_{1/j}(x), \quad u_i \in U_{1/j}(x), \\ \lim\limits_{i\to\infty} u_i = v. \end{cases} \tag{5.23}$$

For any u_i chosen above, we define

$$v_\delta(\xi) = \begin{cases} \bar{u}(\xi) & \xi \in \Omega \setminus (B_\delta(x) \cap \Omega_{1/j}), \\ u_i & \xi \in B_\delta(x) \cap \Omega_{1/j} \quad \forall \delta > 0. \end{cases}$$

We may easily check that $v_\delta \in \mathscr{V}_{1/j}$. Taking $s = 1/j$ and $v = v_\delta$ in (5.21), we have $\int_{B_\delta(x)} h_{ij}(\xi) d\xi \geq 0$. Dividing by $\delta > 0$ and sending $\delta \to 0$, we obtain $h_{ij}(x) \geq 0$. That means $H(x, \bar{u}(x)) \leq H(x, u_i)$, $\forall i$. Hence, from (5.23) and the continuity of $H(x, u)$ in $u \in U$, (5.22) follows.

The proof of the Pontryagin principle is complete.

REMARK 1. If L is independent of u, Ω_0 and $U_0(x)$ can be replaced by Ω and U, respectively. As a matter of fact, in this case, we have $h_r = 0$ and we can carry out the proof without considering Ω_s and $\mathscr{V}_s$ *etcetera*.

REMARK 2. If $\bar{z} \neq 0$, then (5.2) gives a necessary condition for the optimal control $\bar{u}$. Whereas if $\bar{z} = 0$, then (5.2) is trivial. In this case, (5.1) tells us that

$$\bar{\lambda} L_y(x, \bar{y}, \bar{u}) + \bar{\mu} = 0. \tag{5.24}$$

This gives (implicitly, if L is independent of u) a necessary condition for $\bar{u}$. Due to (5.3), (5.24) is nontrivial.

Also, if $m(\Omega_0) = 0$, (5.2) tells us nothing. But, in this case, we must have

$$L(x, \bar{y}(x), \bar{u}(x)) = \bar{J} \qquad \text{a.e.} x \in \Omega.$$

This has already given us some information about the optimal pair $(\bar{y}, \bar{u})$.

Acknowledgement

The author is grateful to Professors Xunjing Li and Jiongmin Yong for their instructive suggestions and helpful advice.

This research has been partly supported by the National Key Project of China, the Education Ministry Science Foundation of China and the Natural Science Foundation of China under Grant 10171059.

References

[1] V. Barbu, *Optimal control of variational inequalities* (Pitman, London, 1984).

[2] E. N. Barron, "The Pontryagin maximum principle for minimax problems of optimal control", *Nonlinear Anal.* **15** (1990) 1155–1165.

[3] L. Cesari, *Existence of solutions and existence of optimal solutions*, Notes in Math. 979 (Springer, Berlin, 1983).

[4] Q. Chen, "Optimal control of semilinear elliptic variational bilateral problem", *Acta Math. Sinica* **16** (2000) 123–140.

[5] I. Ekeland, "On the variational principle", *J. Math. Anal. Appl.* **47** (1974) 324–353.

[6] A. Friedman, "Optimal control for variational inequalities", *SIAM J. Control Optim.* **24** (1986) 439–451.

[7] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed. (Springer, Berlin, 1983).

[8] C. J. Himmelberg, M. Q. Jacobs and F. S. Van Vleck, "Measurable multifunctions, selectors and Filippov's implicit functions lemma", *J. Math. Anal. Appl.* **25** (1969) 276–284.

[9] X. Li and J. Yong, *Optimal control theory for infinite dimensional systems* (Birkhäuser, Boston, 1995).

[10] J. L. Lions, *Quelques méthodes de résolution des problemes aux limites non linéaires* (Dunod, Paris, 1969).

[11] L. Neustadt, *Optimization* (Princeton Univ. Press, Princeton, NJ, 1976).

[12] G. Stampacchia, "Le probleme de Dirichlet pour les equations elliptiques du second ordre a coefficients discontinus", *Ann. Inst. Fourier Grenoble* **15** (1965) 189–258.

[13] G. M. Troianiello, *Elliptic differential equations and obstacle problems* (Plenum Press, New York, 1987).

[14] J. Yong, "A minimax control problem for second order elliptic partial differential equations", *Kodai Math. J.* **16** (1993) 468–486.

https://doi.org/10.1017/S1446181100012918 Published online by Cambridge University Press