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THE FREE PRODUCT OF TWO GROUPS WITH A MALNORMAL AMALGAMATED SUBGROUP

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1. Introduction. In [**1**], B. Baumslag defined a subgroup U of a group G to be *malnormal* in G if $gug^{-1} \in U$, $1 \neq u \in U$, implies that $g \in U$. Baumslag considered the class of amalgamated products $(A * B; U)$ in which U is malnormal in both A and B. These amalgamated products play an important role in the investigations of B. B. Newman [**13**] of groups with one defining relation having torsion. In this paper, we shall be concerned primarily with a generalization of this class.

Let U be a subgroup of a group G and let $u \in U$. Then the *extended normalizer $E_G(u, U)$ of u relative to U in G* is defined by

$$E_G(u, U) = \{g \in G | gug^{-1} \in U\},$$

if $u \neq 1$, and by $E_G(u, U) = U$, if $u = 1$. The *extended normalizer $E_G(U)$ of U in G* is the union of all $E_G(u, U)$, u in U. We abbreviate $E_G(u, U)$ by $E_G(u)$ if the context makes clear which subgroup U is involved. The extended normalizer need not be a subgroup.

With this notation, U is malnormal in G if and only if $E_G(U) = U$.

Let $G = (A * B; U)$. Then U is *r-step malnormal in G* if the maximum syllable length $|E_G(U)|$ of an element of $E_G(U)$ does not exceed r; U is *elementwise malnormal* in G if, for each $u \in U$, $|E_G(u)| < \infty$. (Note that the syllable length of an element of U is zero.) If U is r-step malnormal in G we shall call G an *r-step malnormal product*, or simply a *malnormal product*.

If $g = g_1 \ldots g_r$ is a reduced form of an element in $E_G(u) - U$, then it is easy to see that any terminal segment $g_i \ldots g_r$ is in $E_G(u)$ and that g^{-1} is in $E_G(gug^{-1})$. Hence, *U is 0-step malnormal in G if and only if U is malnormal in both A and B.*

Moreover, *U is 1-step malnormal in G if and only if, for each $u \in U$, $E_A(u) = U$ or $E_B(u) = U$* (for, if a, b are in $E_G(u)$, then $ab^{-1} \in E_G(bub^{-1})$). This class of amalgamated products was used by T. Lewin in [**8**].

The 0-step malnormal products given by the groups $(A * B; a = b)$ where A, B are free $a \neq 1$, $b \neq 1$, and a, b are not proper powers (in A and B, respectively) were investigated by G. Baumslag in [**2**]; these include the fundamental groups of orientable two dimensional manifolds of genus $k > 1$.

B. B. Newman [**13**] has shown that if G is an infinite group with one defining

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relator having torsion, say,

$$G = \langle a, b, c, \ldots ; R^n \rangle, n > 1, \tag{1}$$

and $N \lhd G$ with G/N infinite cyclic, then N is a 0-step malnormal product, and, in fact, N can be built up from groups with one defining relator by repeatedly forming 0-step malnormal products.

Theorem 10 implies that *any group G in* (1) *can be embedded in an r-step malnormal product* $(A * B; U)$ *in which U is free, $A = X * K$, $B = Y * K$, where X, Y are infinite cyclic and K is a group on one defining relator R_0^n of shorter length than that of the original relator* (r depends upon the relator R).

Moreover (Theorem 11), *if in* (1), *R has zero exponent sum on some generator, then G is a finite extension of a* 1*-step malnormal product* $(T * S; U)$, *where T is a free group and S is a* 0*-step malnormal tree product* (i.e., any two neighbouring vertices of S together with their amalgamated subgroup form a 0-step malnormal product),

$$S = K_0 \underset{L_1}{*} K_1 \underset{L_2}{*} \ldots \underset{L_{2r-1}}{*} K_{2r-1},$$

where the K_i are isomorphic one-relator groups with relators R_i^n of shorter length than R^n, and the L_i are free groups of the same rank. (For a definition of tree product see [**6**].)

The subgroup structure in r-step malnormal products is simpler than that of the general amalgamated product $(A * B; U)$; indeed, malnormal products have a number of properties in common with free products (which clearly are 0-step malnormal products).

Let $G = (A * B; U)$ be an r-step malnormal product. The centralizer of an element of G is infinite cyclic or contained in a conjugate of A or B; the normalizer of an infinite cyclic subgroup is infinite cyclic, infinite dihedral, or contained in a conjugate of A or B; if H is a subgroup of G satisfying a non-trivial law, then H is infinite cyclic, infinite dihedral, or contained in a conjugate of A or B. Moreover, if $r = 0$, then any indecomposable (*with respect to amalgamated product*) *subgroup of G is infinite cyclic or contained in a conjugate of a factor; any two-generator subgroup of G is the free product of two cyclic groups or is contained in a conjugate of A or B.*

In the case of 0-step malnormal products, we can give a more detailed description of the general structure of a subgroup H than that given in [**6**] for arbitrary $(A * B; U)$; see Theorems 4 and 5.

Several types of examples of 0- and 1-step malnormal products are given in § 5, and other examples of r-step malnormal products are described in § 6.

An extension of some of these ideas to HNN groups is also discussed in § 6.

2. Centralizers and normalizers.

LEMMA 1. *Let $G = (A * B; U)$ be any amalgamated product. Suppose that $H < G$ and $|H| < \infty$. Then H is contained in a conjugate of A or B.*

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Proof. The proof is by induction on $|H|$. If $|H| \leqq 1$, then since H is a subgroup, $H < A$ or $H < B$. Suppose that $|H| = m > 1$, and let $h = g_1 \ldots g_s$ be a reduced form for an element of H of syllable length $s > 1$. Since $h^n \in H$, we have $g_s g_1 \in U$, and so $h = g_1 h_1 g_1^{-1}$, where $h_1 \in g_1^{-1} H g_1$ and $|h_1| = |h| - 2$. Let $k = p_1 h_2 p_1^{-1}$ (with $h_2 \in p_1^{-1} H p_1$) be any other element of H of syllable length greater than 1, where $|h_2| = |k| - 2$. Since $(hk)^n \in H$, we have that $g_1^{-1} p_1 \in U$ and, therefore, $k = g_1 h_3 g_1^{-1}$, where $|h_3| = |h_2|$. Moreover, if $g \in H$ and $|g| \leqq 1$, then $(hg)^n \in H$ implies that $g_1^{-1} g g_1 \in U$. Thus, $|g_1^{-1} H g_1| < |H|$, and so by the inductive hypothesis H is contained in a conjugate of a actor.

COROLLARY. *Under the same hypothesis as above, if $|H| = r$, then $H < U$ or $H < cAc^{-1}$ or $H < cBc^{-1}$, where $|c| = (r-1)/2$.*

Proof. If $H \nless U$, choose c of smallest syllable length ($c = 1$ if $|c| = 0$), so that H is in cAc^{-1} or cBc^{-1}, say cAc^{-1}. Then c cannot end in an A-syllable, and $cac^{-1} \in H$, for some $a \in A - U$; hence, $2|c| + 1 = r$.

THEOREM 1. *Let $G = (A * B; U)$ be an elementwise malnormal product, and let $1 \neq g \in G$. If g is not in a conjugate of a factor, then the centralizer $C(g)$ of g is infinite cyclic; otherwise, $C(g)$ is in a conjugate of a factor.*

Proof. If g is properly contained in a factor (i.e., g is in a factor but not in a conjugate of U), then $C(g)$ is contained in that factor (by [**9**, Theorem 4.5]).

If g is in U, then $C(g)$ is a subgroup of $E_G(g, U)$, so $|C(g)| < \infty$ and, by Lemma 1, $C(g)$ is in a conjugate of a factor.

Suppose now that g is not in a conjugate of a factor; then $C(g)$ has trivial intersection with any conjugate of a factor (for, if $v \neq 1$ is in $C(g)$, then g is in $C(v)$). Hence, by a theorem of H. Neumann (see [**12**] or [**9**, Corollary 4.9.2]), $C(g)$ is a free group; but $C(g)$ has a non-trivial centre, so $C(g)$ must be infinite cyclic.

T. Lewin [**8**] gave a different proof of Theorem 1 when G is 1-step malnormal.

COROLLARY. *Let $G = (A * B; U)$ be an elementwise malnormal product. Then any element not in a conjugate of a factor has at most one nth root. In particular, the class of groups in which each element has at most one nth root (n ranging over a set of positive integers) is closed under taking (elementwise) malnormal products.*

Proof. Suppose that g, h are in G and $g^n = h^n$. If g^n is not in a conjugate of a factor, then $C(g^n)$ is infinite cyclic and contains g and h; but any element in an infinite cyclic group has at most one nth root, and so $g = h$.

Suppose, moreover, that each element in a factor has at most one nth root. If g^n is in a conjugate of a factor, then $C(g^n)$ is in a conjugate of a factor, and, therefore, both g and h are in that conjugate, so $g = h$.

THEOREM 2. *Let $G = (A * B; U)$ be an elementwise malnormal product, and let $H = gp(h)$, where $1 \neq h \in G$. If h is in a conjugate of a factor, then the*

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normalizer $N(H)$ of H is in a conjugate of a factor; if h is not in a conjugate of a factor and h is not the product of two elements of order two, then $N(H) = C(h)$ is infinite cyclic; finally, if h is not in a conjugate of a factor and h is the product of two elements of order two, then $N(H)$ is infinite dihedral and $C(h)$ has index two in $N(H)$.

Proof. Suppose that h is properly contained in a factor. We show that $N(H)$ is in that same factor. Let $xhx^{-1} = h^s$, with $x \notin U$, and let $x = g_1 \ldots g_n$ be a reduced form of x. Then by examining

$$g_1 \ldots g_n h g_n^{-1} \ldots g_1^{-1},$$

it is easy to see that $n = 1$, and g_1 is in the same factor as h. Hence, $N(H)$ is in the same factor as h.

On the other hand, if $h \in U$, then $N(H)$ is a subgroup of $E_G(h, U)$, and, therefore, $N(H)$ is contained in a conjugate of a factor.

Next, suppose that h is not in a conjugate of a factor; we may assume that h is cyclically reduced. If $xhx^{-1} = h^s$, then, since h is a cyclically reduced form of the right hand side, $s = \pm 1$. This implies that $C(h)$ is of index 1 or 2 in $N(H)$.

If $s = -1$, then h is the product of two elements of order two. For, $xhx^{-1} = h^{-1}$ implies that $x^2hx^{-2} = h$. If x^2 were not in a conjugate of a factor, then h, x would both be in $C(x^2)$, which is cyclic; hence, $xhx^{-1} = h$, and so h would have order two, which is impossible. Hence, x^2 is in a conjugate of a factor. If $x^2 \neq 1$, then h is in $C(x^2)$ which is contained in a conjugate of a factor; but h is not in a conjugate of a factor. Consequently, $x^2 = 1$, $(hx)^2 = 1$, and $h = (hx)x$ is the product of two elements of order two. Thus, if h is not in a conjugate of a factor and not the product of two elements of order two, then $N(H) = C(h)$.

Finally, suppose that h is not in a conjugate of a factor and $h = h_1h_2$, where h_1, h_2 are each of order two. Now $C(h)$ is an infinite cyclic group; let $C(h) = gp(w)$, $h = w^k$. Therefore,

$$(h_1wh_1^{-1})^k = h_1hh_1^{-1} = h_2h_1 = h^{-1} = (w^{-1})^k.$$

Hence, $h_1wh_1^{-1} = w^{-1}$ (by the corollary to Theorem 1), and so $(h_1w)^2 = 1$. Let $h_3 = h_1w$; then $h = (h_1h_3)^k$ and $C(h) = gp(h_1h_3)$. Moreover, $C(h)$ has index two in $N(H)$, and $N(H)$ is generated by h_1 and h_1h_3. Since h_1h_3 generates an infinite cyclic group and $h_1(h_1h_3)h_1^{-1} = (h_1h_3)^{-1}$, it follows that h_1 and h_1h_3 generate an infinite dihedral group, and so $N(H)$ is the free product $gp(h_1) * gp(h_3)$.

3. Subgroups satisfying a law.

LEMMA 2. *Let $G = (A * B; U)$ be an r-step malnormal product. Suppose that H is a subgroup which is a union of subgroups of conjugates of A or B. Then H is contained in a conjugate of A or B.*

Proof. We first show that, if c, d are in $A \cup B$ and $c \neq 1 \neq d$ and $|p| \geqq 2r + 4$, then $g = cpdp^{-1}$ is not in a conjugate of A or B. We do this by showing that a cyclically reduced form of g has syllable length $\geqq 2$. Let p have a reduced form $p = p_1 \ldots p_s$; let i, j be the largest positive integers (if any) such that

$$c_i = p_i^{-1} \ldots p_1^{-1} c p_1 \ldots p_i, \; d_j = p_{s-j+1} \ldots p_s d p_s^{-1} \ldots p_{s-j+1}^{-1}$$

are in U. Then $i - 1, j - 1 \leqq r$, and so $s - i - j \geqq 2$. Therefore,

$$(p_{i+1}^{-1} \ldots p_1^{-1} c p_1 \ldots p_{i+1}) p_{i+2} \ldots (p_{s-j} \ldots p_s d p_s^{-1} \ldots p_{s-j}^{-1}) \ldots p_{i+2}^{-1}$$

is a cyclically reduced form of g with syllable length $\geqq 2$.

Suppose now that H is the union of groups $q_k C_k q_k^{-1}$, where C_k is a non-trivial subgroup of A or B, and that the set of non-negative integers $\{|q_k|\}$ is unbounded. Let qCq^{-1} be a fixed subgroup in $\{q_k C_k q_k^{-1}\}$, and choose q_k so that $|q_k| \geqq |q| + 2r + 4$. Then $|q^{-1} q_k| \geqq 2r + 4$, and the above argument shows that $Cq^{-1} q_k C_k q_k^{-1} q$ is not in $q^{-1} H q$, which is a contradiction.

Hence, the set $\{|q_k|\}$ is bounded, and so $|H| < \infty$; consequently, Lemma 1 applies and H is in a conjugate of a factor.

LEMMA 3. *Suppose that $G = (A * B; U)$ is an r-step malnormal product, and that H is any subgroup of G. Let $V_1, V_2, \ldots, V_n$ ($n > 1$) be the vertices in a simple path joining V_1 to V_n in the graph of the tree product base of H when H is expressed as an HNN group (according to* [**6**, *Theorem* 5]), *and let $U_1, U_2, \ldots, U_{n-1}$ be the subgroups corresponding to the edges of this path. If $V_1 \neq U_1 = U_{n-1} \neq V_n$ and $V = gp(V_1, V_n)$, then*

$$V = (V_1 * V_n; U_1 = U_{n-1}), \tag{2}$$

and this amalgamated product is an r-step malnormal product.

Proof. That V has the presentation (2) follows easily from properties of a tree product.

If $n = 2$, then $V_1 = DAD^{-1} \cap H$, $V_2 = DBD^{-1} \cap H$ (or vice versa), and $U_1 = DUD^{-1} \cap H$. If $|E_V(U_1)| > r$, it follows upon conjugation by D^{-1} that $|E_G(U)| > r$, contrary to hypothesis.

Next, let $V_1 = D_1 C_1 D_1^{-1} \cap H$ and $V_n = D_n C_n D_n^{-1} \cap H$, where

$$C_1, C_n \in \{A, B\},$$

and D_1, D_n are the appropriate double coset representives.

Suppose that neither D_1 nor D_n is an initial segment of the other. Then $D_1^{-1} D_n$ begins (ends) in a syllable in the same factor as the last syllable of $D_1(D_n)$. Moreover, $U_1 = D_1 U D_1^{-1} \cap H$, $U_{n-1} = D_n U D_n^{-1} \cap H$. Hence, if a product

$$(D_{i_1} g_1 D_{i_1}^{-1})(D_{i_2} g_2 D_{i_2}^{-1}) \ldots (D_{i_s} g_s D_{i_s}^{-1}),$$

in which the factors alternate from $V_1 - U_1$ and $V_n - U_{n-1}$, is in $E_V(U_1)$ with $s > r$, then $|E_G(U)| > r$, contrary to hypothesis.

Finally, suppose that D_1 is an initial segment of D_n and $D_n = D_1c_1Y$, where $c_1 \in C_1$ (c_1 possibly 1). Since $Y \neq 1$, we may assume that Y begins in a syllable not in C_1. In this case,

$$\begin{aligned} V_1 &= D_1c_1C_1c_1^{-1}D_1^{-1} \cap H, \\ V_n &= D_1c_1YC_nY^{-1}c_1^{-1}D_1^{-1} \cap H, \\ U_1 &= D_1c_1Uc_1^{-1}D_1^{-1} \cap H \\ &= D_1c_1YUY^{-1}c_1^{-1}D_1^{-1} \cap H, \end{aligned}$$

and again it follows easily that $|E_V(U_1)| \leqq r$.

THEOREM 3. *Let $G = (A * B;\ U)$ be an r-step malnormal product, and suppose that H is a subgroup of G satisfying a non-trivial law. Then H is infinite cyclic, infinite dihedral, or contained in a conjugate of A or B.*

Proof. According to [**6**, Theorem 7], if H is not in a conjugate of a factor, then one of the following three possibilities holds:

(3) H is an ascending union of conjugates of U, and so by Lemma 2, H is in a conjugate of a factor.

(4) H is an HNN group of the form

$$\langle t,\ U_H{}^{\delta};\ \text{rel } U_H{}^{\delta},\ tU_H{}^{\delta}t^{-1} = U_H{}^{\delta'}\rangle,$$

where $U_H{}^{\delta'} < U_H{}^{\delta}$. Therefore, $U_H{}^{\delta} < U^{t^{-k}\delta}$, for each positive integer k. Hence,

$$(\delta^{-1}t^k\delta)U(\delta^{-1}t^{-k}\delta) \cap H < U;$$

but since t is not in a conjugate of a factor of G, $\delta^{-1}t^k\delta$ has syllable length $\geqq 2k$. By choosing $k > r/2$, we see that $U_H{}^{\delta} = 1$, and so H is infinite cyclic.

(5) H is an amalgamated product $(C_1 * C_2;\ U_H{}^D)$, where $U_H{}^D$ is of index two in each C_i, and C_i are vertices in the tree product base of H; moreover, by [**6**, Theorem 3], the amalgamated subgroups corresponding to the edges of the simple path joining C_1 to C_2 (in the tree product base) are all equal to $U_H{}^D$. Hence, by Lemma 3, H is an r-step malnormal product; therefore, since $U_H{}^D$ is normal in H, we have that $U_H{}^D = 1$, and so H is infinite dihedral.

COROLLARY 1. *If, in the statement of Theorem 3, we replace "satisfying a non-trivial law" by "containing no free subgroup of rank two", then we obtain a correct result.*

Proof. [**6**, Theorem 7] holds if we replace "H satisfies a non-trivial law" by "H contains no free subgroup of rank two".

COROLLARY 2. *Let $G = (A * B;\ U)$ be such that $E_A(U) = N_A(U)$ (the normalizer of U in A), and $E_B(U) = N_B(U)$. If H is a subgroup of G satisfying a non-trivial law, then H is contained in a conjugate of a factor, or H is an infinite cyclic extension of a subgroup of a conjugate of U, or H has the form $(C_1 * C_2;\ U_H{}^D)$, where $U_H{}^D$ is of index two in each C_i, and C_i is a subgroup of a conjugate of A or B.*

Proof. The hypothesis implies that $E_G(U) = N_G(U)$. Hence, if $1 \neq U_H^{D_1} < U_H^{D_2}$, then $U_H^{D_1} = U_H^{D_2}$. Applying [**6**, Theorem 7] as above, we have the result.

For example, *if*

$$G = \langle a_1, \ldots, a_n, b_1, \ldots, b_m; w_1^p = w_2^q \rangle,$$

where $pq \neq 0$, w_1 *is a non-trivial word in the* a_i, *and* w_2 *is a non-trivial word in the* b_i, *then the only subgroups of* G *which can satisfy a non-trivial law are infinite cyclic, free abelian of rank two, or a group with presentation* $\langle a, b; a^2 = b^2 \rangle$.

4. The structure of subgroups of a malnormal product. In this section, we give a more detailed description of a subgroup H of a 0-step malnormal product $(A * B; U)$ than that given in [**6**, Theorem 5]; we also indicate a partial generalization to r-step malnormal products.

For this purpose, we require more specific information about the way in which the associated subgroups and amalgamated subgroups are situated in the tree product base S of H as an HNN group given by [**6**, Theorem 5]. In order to bring the associated subgroups explicitly into the picture, we enlarge the graph of the tree product S so as to include the associated subgroups as extremal vertices. Specifically (using the notation of [**6**]), if $D_\alpha E_u$, $D_\beta E_v$ are u-, v-double coset representives, respectively, which are neither α- nor β-double coset representatives, then we join the new vertices $U_H^{D_\alpha E_u}$, $U_H^{D_\beta E_v}$ to the vertices $A_H^{D_\alpha}$, $B_H^{D_\beta}$, respectively, of S, and make the new vertex correspond to the edge joining it to the old vertex; moreover, we extend the level function λ to the new vertices by defining the level of $U_H^{D_\alpha E_u}$, $U_H^{D_\beta E_v}$ to be the syllable length of $D_\alpha E_u$, $D_\beta E_v$, respectively. Clearly, the new vertices we have adjoined are extremal, i.e., are incident with a unique edge.

LEMMA 4. *Let* G *be an amalgamated product* $(A * B; U)$, *and let* $H < G$. *Suppose that* $U_H^{\delta_1}, U_H^{\delta_2}, \ldots, U_H^{\delta_n}$ *are the subgroups corresponding to the edges of a simple path in the enlarged graph of the tree product base* S *of* H. *Then* $\delta_1^{-1}\delta_n$ *has syllable length* $n - 1$ *in* G.

Proof. We first recall that the syllables of a word which is an α- or β- representative define elements which are not in U (by [**6**, Lemma 6, Corollary]).

Secondly, note that if DX and DY are different u- or v- double coset representatives, where X, Y are their respective last syllables, and if X, Y are both in the same factor, then $X^{-1}Y$ is not in U. For, otherwise, DX and DY end in the same type (α- or β-) of symbol, and hence are the same type of representative; therefore, DX and DY are both u- or both v- double coset representatives for the same (H, U) double coset, and so $DX = DY$.

Now suppose that

$$X_1^{p_1}, X_2^{p_2}, \ldots, X_{n+1}^{p_{n+1}}, n \geqq 2,$$

are the vertices of a simple path in the enlarged graph of S, where the p_i are appropriate double coset representatives. We may assume (by reversing the path if necessary)

$$\lambda(X_1^{p_1}) > \lambda(X_2^{p_2}) > \ldots > \lambda(X_k^{p_k}) \leqq \lambda(X_{k+1}^{p_{k+1}}) < \ldots < \lambda(X_{n+1}^{p_{n+1}}),$$

where $2 \leqq k \leqq n + 1$. If $U_H^{\delta_1}, \ldots, U_H^{\delta_n}$ are the subgroups associated with the edges of the path, then

$$\delta_1 = p_1, \delta_n = p_{n+1},$$

if $2 \leqq k \leqq n$, and

$$\delta_1 = p_1, \delta_n = p_n,$$

if $k = n + 1$. Moreover, because neighbouring vertices of different levels are associated with representatives which differ by a single syllable, it is clear that

$$p_2^{-1}p_1, \ldots, p_{k-1}^{-1}p_{k-2}, p_{k+1}^{-1}p_{k-1}, p_{k+1}^{-1}p_{k+2}, \ldots, p_n^{-1}p_{n+1}$$

are single syllables not in U, and alternate out of A and B. Therefore, if $k \leqq n$, then $\delta_1^{-1}\delta_n = p_1^{-1}p_{n+1}$, which has syllable length $n - 1$; and if $k = n + 1$, $\delta_1^{-1}\delta_n = p_1^{-1}p_n$, which again has syllable length $n - 1$. This completes the proof of Lemma 4.

LEMMA 5. *Suppose that $G = (A * B; U)$ is an r-step malnormal product, and $H < G$. If $U_H^{\delta_1}$, $U_H^{\delta_n}$ correspond to the first and last edge of a simple path of length n in the enlarged graph of the tree product base S of H, and $n \geqq r + 2$, then $U_H^{\delta_1} \cap U_H^{\delta_n} = 1$. Moreover, if $X_0^{p_0}$ and $X_m^{p_m}$ are vertices in the enlarged graph of S and the simple path joining $X_0^{p_0}$ to $X_m^{p_m}$ has length $m \geqq 2r + 3$, then* $\mathrm{gp}(X_0^{p_0}, X_m^{p_m})$ *is the free product $X_0^{p_0} * X_m^{p_m}$.*

Proof. By Lemma 4, $\delta_1^{-1}\delta_n$ has syllable length $n - 1 \geqq r + 1$. Therefore, since $|E_G(U)| \leqq r$, $\delta_1^{-1}\delta_n U \delta_n^{-1}\delta_1 \cap U = 1$; hence, $U_H^{\delta_1} \cap U_H^{\delta_n} = 1$.

Moreover, if $X_0^{p_0}, X_1^{p_1}, \ldots, X_m^{p_m}$ are the vertices in the simple path joining $X_0^{p_0}$ to $X_m^{p_m}$, then these $n + 1$ vertices generate their tree product. Let $U_H^{\delta_1}, \ldots, U_H^{\delta_m}$ be the subgroups corresponding to the edges of this simple path. Then

$$U_H^{\delta_1} \cap U_H^{\delta_{r+2}} = 1 = U_H^{\delta_m} \cap U_H^{\delta_{r+2}},$$

since $m \geqq 2r + 3$. Therefore,

$$X_0^{p_0} \cap U_H^{\delta_{r+2}} = 1 = X_m^{p_m} \cap U_H^{\delta_{r+2}},$$

and so $\mathrm{gp}(X_0^{p_0}, X_m^{p_m}) = X_0^{p_0} * X_m^{p_m}$.

This completes the proof of Lemma 5.

A collection of subgroups $\{L_i\}$ of a group is called *malnormal in K* if $kL_ik^{-1} \cap L_j = 1$, unless $i = j$ and $k \in L_i$.

For example, suppose that $K = A * B$, and $\{A_i\}$, $\{B_i\}$ are malnormal collections in A, B respectively; and suppose that $\{g_i\}$ is a collection of cyclically reduced elements of K such that each g_i has syllable length $\geqq 2$, each g_i has no proper root in K, and g_i is a cyclic permutation of g_j implies that $i = j$. Then the collection which is the union of $\{A_i\}$, $\{B_i\}$, $\{\mathrm{gp}(g_i)\}$ is a malnormal collection in K. Thus, if $K = \langle a, b\rangle$, then the collection of subgroups $\{\mathrm{gp}(a), \mathrm{gp}(b), \ldots, \mathrm{gp}(a^i b^j), \ldots\}$, with $ij \neq 0$, is a malnormal collection in K.

As another example, let $K = A * B$, where now A is a free group freely generated by $\{a_i\}$, $i \in I$. Let $\{b_i\}$, $i \in I$, be an indexed set of distinct elements ($\neq 1$) of B. Then A and $A' = \mathrm{gp}(a_1 b_1, a_2 b_2, \ldots)$ form a malnormal collection of (free) subgroups of $A * B$. More generally, let $\{b_{ij}\}$, $i \in I$, $j \in J$, be an indexed set of elements of B such that $b_{ij} \neq b_{ik}$, for $j \neq k$, and $b_{ij}^{-1} b_{ik} \neq b_{pj}^{-1} b_{pk}$, for $i \neq p$ and $j \neq k$; and let $A_j = \mathrm{gp}(a_1 b_{1j}, a_2 b_{2j}, \ldots)$. Then the collection of subgroups $\{A_j\}$, $j \in J$, is a malnormal collection of (free) subgroups of $A * B$. Thus, if $A = \langle x, y\rangle$ and $B = \langle b\rangle$, then $\{\mathrm{gp}(xb^j, yb^{2j})\}$, j ranging over the integers, is a malnormal collection in $\langle x, y, b\rangle$.

A tree product S is called a 0-*step malnormal tree product* if any two neighbouring vertices of S together with their amalgamated subgroup form a 0-step malnormal product.

LEMMA 6. *Let S be a* 0-*step malnormal tree product of vertices A_i.*

(a) *Then any subtree product of S is malnormal in S.*

(b) *Moreover, if L is a subgroup of a vertex A_1 of S whose conjugates in A_1 have trivial intersection with the edges of S incident with A_1, then each conjugate of L in S has trivial intersection with any subtree product of S not containing A_1 as a vertex.*

(c) *Finally, if for each i, $\{L_{ij}\}$, $j \in J_1$, is a malnormal collection of subgroups of the vertex A_i which includes all the edges of S incident with A_i, then the totality of all subgroups L_{ij} is a malnormal collection in S.*

Proof. First, we observe that A is malnormal in $G = (A * B; U)$ if and only if U is malnormal in B. For, clearly, if U is not malnormal in B, then $bub^{-1} \in U$, for some $1 \neq u \in U$, $b \in B - U$, so $b \in E_G(A) \neq A$. Conversely, suppose that U is malnormal in B, and that $gag^{-1} \in A$, $1 \neq a \in A$, $g \in G - A$, where g has shortest possible syllable length. Then g has a reduced form $g = g_1 \ldots g_s$, with $g_s \in B - U$. Since gag^{-1} has syllable length $\leqq 1$, a must be in U, and so $s = 1$; but then $gag^{-1} \in B \cap A = U$, contrary to the malnormality of U in B.

To establish (a), we first show that each vertex of S is malnormal in S. It clearly suffices to show this when S is a tree product of finitely many vertices; in this case, S has an extremal vertex, say A_0. Hence, we may write S as an amalgamated product $(A_0 * S_0; U_0)$, where S_0 is a 0-step malnormal tree product with fewer vertices than S, and U_0 is malnormal in A_0 as well as in a vertex of S_0. Then, by the transitivity of malnormality and the preceding remark, an inductive argument shows that each vertex of S is malnormal in S.

Moreover, given a subtree product of S, we may contract it to a vertex; using the transitivity of malnormality, and that each vertex in a 0-step malnormal tree product is malnormal in it, it follows that the contracted graph is a 0-step malnormal tree product; hence, the given subtree product is malnormal in S.

To show (b), suppose that a subtree product of S does not contain A_1; then $S = (S_1 * S_2; U_1)$, where U_1 is an edge incident with A_1, and S_1, S_2 are the subtree products which result from deletion of U_1 from S, and A_1 is a vertex of S_1, and the given subtree product is a subtree product of S_2. Since A_1 is malnormal in S, if $g \in S$ and $gLg^{-1} \cap U_1 \neq 1$, then $g \in A_1$, contrary to hypothesis. Therefore by [**9**, Theorem 4.6], any conjugate of L has trivial intersection with S_2 and hence with the given subtree product. Since any edge of S is contained in a vertex different from A_1, it follows that the conjugates of L in S intersect each edge of S trivially.

To show (c), let L_{ij} and L_{pq} be distinct subgroups in the given collection of subgroups. If $p = i$, then since A_i is malnormal in S, the conjugates of L_{ij} in S have trivial intersection with L_{iq}. Suppose that $i \neq p$. If L_{ij} is not an edge incident with A_i, then by (b), the conjugates of L_{ij} have trivial intersection with A_p and hence with L_{pq}. Assume, therefore, that L_{ij} is an edge incident with A_i. Now if the edge L_{ij} is deleted from S, S decomposes into two subtree products S_1, S_2, and $S = (S_1 * S_2; L_{ij})$. If L_{pq} is in S_1, then since S_1 is malnormal in S, it suffices to show that the conjugates of L_{ij} in S_1 have trivial intersection with L_{pq}. But L_{ij} is not an edge in S_1 and hence (b) applies.

THEOREM 4. *Let $G = (A * B; U)$ be a 0-step malnormal product, and let $H < G$. Then, in the description of H as an HNN group (given by [**6**, Theorem 5])*

$$H = \langle t_1, t_2, \ldots, S; \text{rel } S, t_1L_1t_1^{-1} = M_1, t_2L_2t_2^{-1} = M_2, \ldots \rangle, \tag{6}$$

each pair of associated subgroups L_i, M_i generate their free product $L_i * M_i$, and $\text{gp}(t_i, L_i)$ is the free product $\langle t_i \rangle * L_i$. In particular, H is the tree product of the groups $\langle t_i \rangle * L_i$ and S with the subgroups $L_i * M_i$ amalgamated from the single factor S. Moreover, S itself is a 0-step malnormal tree product. Finally, let V be any vertex of S; then the collection $\{U_H^{\delta_1}, U_H^{\delta_2}, \ldots\}$ of amalgamated and associated subgroups corresponding to the edges incident with V in the enlarged graph of S form a malnormal collection of subgroups in V; more generally, the collection of all amalgamated and associated subgroups corresponding to the edges in the enlarged graph of S form a malnormal collection of subgroups of S.

Proof. According to [**6**, Theorem 5], a pair of associated subgroups L_i, M_i have the form U_H^{δ}, $U_H^{\delta'}$ respectively, where the corresponding $t_i = \delta' P \delta^{-1}$ with $P \in U$. Moreover, t_i is not in any conjugate of A or B. Hence, $\delta^{-1}\delta'$ has syllable length $\geqq 2$; and so the vertices U_H^{δ}, $U_H^{\delta'}$ of the enlarged graph of S have a simple path of length $\geqq 3$ joining them. Hence, by Lemma 5,

$$\text{gp}(U_H^{\delta}, U_H^{\delta'}) = U_H^{\delta} * U_H^{\delta'}.$$

Therefore,

$$\begin{aligned} \text{gp}(t_i, U_H^{\delta}) &= \langle t_i, U_H^{\delta}, U_H^{\delta'}, \text{rel } U_H^{\delta}, \text{rel } U_H^{\delta'}, t_i U_H^{\delta} t_i^{-1} = U_H^{\delta'} \rangle \\ &= \langle t_i, U_H^{\delta}; \text{rel } U_H^{\delta} \rangle \\ &= \langle t_i \rangle * U_H^{\delta}. \end{aligned}$$

In particular, H is a tree product as asserted (see last Corollary to [**6**, Theorem 5]).

Since neighbouring vertices and their corresponding amalgamated subgroup have the form A_H^D, B_H^D, and U_H^D, they clearly form a 0-step malnormal product.

To prove the final assertion of Theorem 4, let V be, say, $A_H^{D_\alpha}$. Then in a subgroup $U_H^{\delta_i}$, δ_i has the form $D_\alpha a_i$, where $a_i \in A$ and δ_i is a u-double coset representative. If now $h = D_\alpha a D_\alpha^{-1} \in H$ (where $a \in A$), and

$$hU_H^{\delta_i}h^{-1} \cap U_H^{\delta_j} \neq 1,$$

then $aa_i = a_j u$, where $u \in U$. Hence, $HD_\alpha a_i U = HD_\alpha aa_i U = HD_\alpha a_j U$, and so $\delta_i = \delta_j$. Moreover, since $U_H^{\delta_i}$ is malnormal in $A_H^{\delta_i} = A_H^{D_\alpha}$, we have that $h \in U_H^{\delta_i}$. Lemma 6 (c) implies that the totality of all amalgamated and associated subgroups form a malnormal collection in S. This completes the proof of Theorem 4.

COROLLARY 1. *Let $G = (A * B;\, U)$ be an r-step malnormal product and let $H < G$. In the description* (6) *of H as an HNN group, if the syllable length of a cyclically reduced form of $t_i \geqq 2r + 2$, then* $\mathrm{gp}(L_i, M_i) = L_i * M_i$ *and*

$$\mathrm{gp}(t_i, L_i) = \langle t_i \rangle * L_i.$$

COROLLARY 2. *Let $G = (A * B;\, U)$ be an r-step malnormal product, and let $H < G$. In the description* (6) *of H as an HNN group, if the syllable length of a cyclically reduced form of t_i is τ and $\sigma \cdot \tau \geqq r + 1$, then* $\mathrm{gp}(t_i^{2\sigma}, L_i) = \langle t_i^{2\sigma} \rangle * L_i$.

Proof. First we establish the following: *If*

$$H = \langle t, K;\, \mathrm{rel}\, K,\, tLt^{-1} = M \rangle$$

is an HNN group such that $t^\sigma L t^{-\sigma} \cap L = 1$, then $\mathrm{gp}(t^{2\sigma}, L) = \langle t^{2\sigma} \rangle * L$.

For, the normal subgroup N_q of H generated by K and t^q is an *HNN* group with free part generated by t^q; its base S_q is the tree product of the factors $K_i = t^i K t^{-i}$, $0 \leqq i < q$, with the subgroups $t^{(i-1)} M t^{-(i-1)} = t^i L t^{-i}$ amalgamated between K_{i-1} and K_i; moreover, the pair of associated subgroups for N_q are L, $t^q L t^{-q}$. (This follows easily by using the Reidemeister-Schreier theorem on N_q.)

Now

$$S_{2\sigma} = (S_\sigma * t^\sigma S_\sigma t^{-\sigma};\, t^\sigma L t^{-\sigma}).$$

Since $L \cap t^\sigma L t^{-\sigma} = 1 = t^{2\sigma} L t^{-2\sigma} \cap t^\sigma L t^{-\sigma}$, we have that $\mathrm{gp}(L, t^{2\sigma} L t^{-2\sigma}) = L * t^{2\sigma} L t^{-2\sigma}$; hence, $\mathrm{gp}(t^{2\sigma}, L) = \langle t^{2\sigma} \rangle * L$.

To prove the corollary, let $L_i = U_H^{\delta}$, $M_i = U_H^{\delta'}$, and $t_i = \delta' P \delta^{-1}$, $P \in U$. Then $L_i \cap t_i^\sigma L_i t_i^{-\sigma} = 1$; for, $\delta^{-1} t_i^\sigma \delta$ has syllable length $\sigma \cdot \tau \geqq r + 1$, and G is r-step malnormal.

COROLLARY 3. *Any indecomposable* (*with respect to amalgamated product*) *subgroup of a* 0-*step malnormal product is either infinite cyclic or contained in a conjugate of a factor.*

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Proof. This follows easily from [**6**, Theorem 6], Theorem 4, and Lemma 2.

COROLLARY 4. *Under the same hypotheses as in Theorem* 4,

$$\mathrm{gp}(t_i, S) = ((t_i * L_i) * S; L_i * M_i),$$

which is a 1-*step malnormal product.*

Proof. That $\mathrm{gp}(t_i, S)$ is the amalgamated product indicated holds because $\langle t_i \rangle * L_i$ and S generate their subtree product in H.

To establish the 1-step malnormality, we show that the hypotheses of Lemma 8 of § 6 are satisfied. Now L_i, M_i occur in different vertices of S. Hence one can find an edge Q and S such that $S = (C * D; Q)$, where C, D are the subtree products of S which result from deletion of Q from S, and the vertex of S containing L_i is a vertex of C whereas the vertex of S containing M_i is a vertex of D. By Lemma 6, each conjugate of L_i or M_i in S intersects Q trivially. Hence, Lemma 8 applies.

THEOREM 5. *Let $G = (A * B; U)$ be a* 0-*step malnormal product in which A, B are both finite. Then any subgroup H of G is the free product of a free group (possibly trivial) and factors of the type $A_H{}^D$, $B_H{}^D$, or $(A_H{}^D * B_H{}^D; U_H{}^D)$.*

Proof. In the *HNN* description of H as given by [**6**, Theorem 5], each of the associated and amalgamated subgroups is malnormal in its corresponding vertices (by Theorem 4). Now in a finite group any two proper ($\neq 1$) malnormal subgroups are conjugate (see the remarks at the beginning of § 5). Hence, by the last part of Theorem 4, any vertex in the tree product base of H cannot contain more than one corresponding non-trivial associated or amalgamated subgroup. Hence, the tree product base S decomposes into the free product of groups of the type $A_H{}^D$, $B_H{}^D$, or $(A_H{}^D * B_H{}^D; U_H{}^D)$.

Moreover, any generator t of the free part of H which has trivial associated subgroups can be factored out of H as a free factor. Furthermore, any generator t of the free part of H with a non-trivial associated subgroup corresponds to vertices, $A_H{}^\delta$, $B_H{}^{\delta'}$ all of whose other associated and amalgamated subgroups must be trivial; hence,

$$T = \mathrm{gp}(t, A_H{}^\delta, B_H{}^{\delta'}) = \langle t, A_H{}^\delta, B_H{}^{\delta'}; \mathrm{rel}\, A_H{}^\delta, \mathrm{rel}\, B_H{}^{\delta'}, tU_H{}^\delta t^{-1} = U_H{}^{\delta'} \rangle$$

is a factor of H (as a free product). Moreover, $t = \delta' P \delta^{-1}$, where $P \in U$. Hence, $A_H{}^{\delta'} = tA_H{}^\delta t^{-1}$, and

$$T = \langle t, A_H{}^{\delta'}, B_H{}^{\delta'}; \mathrm{rel}\, A_H{}^{\delta'}, \mathrm{rel}\, B_H{}^{\delta'}, U_H{}^{\delta'} = U_H{}^{\delta'} \rangle = \langle t \rangle * (A_H{}^{\delta'} * B_H{}^{\delta'}; U_H{}^{\delta'}).$$

Consequently, H is a free product as claimed.

The two generator subgroups of a 0-step malnormal product have a particularly simple description:

THEOREM 6. *If $G = (A * B; U)$ is a* 0-*step malnormal product, then any two-generator subgroup of G is the free product of two cyclic groups or is contained*

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in a conjugate of a factor. More generally, if G is a 0*-step malnormal tree product, then the same result holds.*

Proof. The proof is essentially the same as that in B. Baumslag [**1**]. Assume the theorem false and let α be the least possible syllable length of an element of G, which together with another element of G makes the theorem false. Furthermore let y be of minimal syllable length with respect to the property that there exists an x of syllable length α and $\mathrm{gp}(x, y)$ makes the theorem false.

Case 1. $\alpha = 0$, i.e., $x \in U$.

Clearly, we may assume that $\beta > 1$ and that $y = f_1 \dots f_\beta$ is a reduced form for y. Since U is closed in A (i.e., $a^n \in U, n > 0$, implies that $a \in U$) and in B, it is easy to show that if $y^k \neq 1$, $k > 0$, then y^k has a reduced form which begins with f_1 and ends with f_β. Since $f_1^{-1}x^\gamma f_1 \notin U$ when $x^\gamma \neq 1$, if $f_\beta = f_1^{-1}$, then $\mathrm{gp}(x, y) = \mathrm{gp}(x) * \mathrm{gp}(y)$. Moreover, if $f_\beta x^\gamma f_1 \in U$, then y can be replaced (without changing $\mathrm{gp}(x, y)$) by an element whose first and last syllables are inverses. Hence, we may assume that when $x^\gamma \neq 1$, neither $f_\beta x^\gamma f_1$, nor $f_1^{-1}x^\gamma f_\beta^{-1}$, nor $f_\beta x^\gamma f_\beta^{-1}$ is in U. It then follows again that $\mathrm{gp}(x, y) = \mathrm{gp}(x) * \mathrm{gp}(y)$.

Case 2. $\alpha = 1$, $\beta = 1$.

To be specific, suppose that $x \in A - U$. If $\beta = 1$, then $y \in B - U$. Therefore, $\mathrm{gp}(x) \cap U = 1 = \mathrm{gp}(y) \cap U$. Hence, $\mathrm{gp}(x, y) = \mathrm{gp}(x) * \mathrm{gp}(y)$.

The remaining cases follow as in B. Baumslag [**1**], simply by using the fact that U is closed in A and B, and by replacing conditions such as "x^γ, $\gamma \neq 0$" by "$x^\gamma \neq 1$".

If G is a 0-step malnormal tree product then one can assume that G has finitely many vertices and use a standard inductive argument.

COROLLARY. *Let $G = (A * B; U)$ be a* 0*-step malnormal product, and suppose that x, y are two elements of finite order which are not both in a single conjugate of a factor. Then* $\mathrm{gp}(x, y) = \mathrm{gp}(x) * \mathrm{gp}(y)$. *More generally, the result holds if G is a* 0*-step malnormal tree product.*

Proof. If $H = \mathrm{gp}(x, y)$ is cyclic, then clearly it is contained in a conjugate of a factor. Hence, $H = \mathrm{gp}(p) * \mathrm{gp}(q)$; moreover, we may assume (after conjugation) that $x \in \mathrm{gp}(p)$. Since the conjugates of $\mathrm{gp}(p)$ cannot generate H, y is in a conjugate of $\mathrm{gp}(q)$. But $\mathrm{gp}(p)$ and a conjugate of $\mathrm{gp}(q)$ generate their free product; thus, $H = \mathrm{gp}(x) * \mathrm{gp}(y)$.

5. Examples of 0- and 1-step malnormal products. As noted in the introduction, $(A * B; U)$ is a 0-step malnormal product if and only if U is malnormal in A and B. Thus, to construct examples of 0-step malnormal products, we need to be able to construct groups having malnormal subgroups. Moreover, if U is malnormal in A or in B, then any amalgamated product $(A * B; U)$ is 1-step malnormal. In this section, we describe several types of examples of groups having malnormal subgroups.

(Note, however, that the condition "$(A * B; U)$ is 1-step malnormal" is not equivalent to the condition that "U is malnormal in A or in B". For example, let $A = \langle a, u; au^2 = u^2a, u^6\rangle$, $B = \langle b, u; bu^3 = u^3b, u^6\rangle$, and $U = \langle u; u^6\rangle$. Then in $(A * B; U)$, clearly $E_A(U) = A \neq U \neq B = E_B(U)$; on the other hand, $E_A(u^n) = U$, for $n = 1, 3, 5$, and $E_B(u^n) = U$, for $n = 1, 2, 4, 5$, so $(A * B; U)$ is 1-step malnormal. This example is a special case of a class of 1-step malnormal products (called "free products with centralized subgroups") which we describe at the end of this section.)

If $1 \neq U < A$, the condition $E_U(A) = U$ is equivalent to U being its own normalizer and having trivial intersection with each of its distinct conjugates. Thus, if A is finite and U is malnormal in A with $1 \neq U \neq A$, then A is a familiar type of group, namely, a Frobenius group with complement U. As is well known, when A is finite the elements of A outside of U and its conjugates, together with 1, form a normal subgroup of A (called the kernel of A) with complement U; moreover, any pair of Frobenius complements are conjugates (see, for example, [**14**, p. 354]). (If A is infinite, neither of these results necessarily holds.)

Obvious examples of such groups are obtained by taking A to be a transitive permutation group in which each permutation different from the identity permutation has at most one fixed point, and taking U to be the subgroup that leaves a given point fixed. (Indeed, this permutation description of a group with a malnormal subgroup is equivalent to the abstract description.) For example, let A be the group of linear functions (under resultant composition) $f(x) = ax + b$ over a field F, where b ranges over F and a ranges over a subgroup M of the multiplicative group of F. The subgroup U of functions, which have fixed point 0, viz., ax with $a \in M$, is a malnormal subgroup of A.

Since the literature on finite groups having malnormal subgroups is extensive, we shall concentrate our attention on infinite groups having malnormal subgroups.

First, we prove a theorem which allows us to determine the malnormal cyclic subgroups of an r-step malnormal product.

THEOREM 7. *Let $G = (A * B; U)$ be an elementwise malnormal product. Suppose that $h(\neq 1)$ is in G and $H = \mathrm{gp}(h)$. Then, if h is not in a conjugate of A or B, $E_G(H) = N(H)$; if, additionally, h is not the product of two elements of order two and h has no proper roots (i.e., H is a maximal cyclic subgroup of G), then $E_G(H) = H$.*

If h is in a factor A or B but H intersects each conjugate of U trivially, then $E_G(H)$ is in that same factor.

Finally, if $h^n(\neq 1)$ is in U, then $E_G(H)$ is in a conjugate of a factor, provided that either U is malnormal in A or in B, or h has prime power order, or h has infinite order and $|E_G(\mathrm{gp}(h^n))| < \infty$.

Proof. Let $x \in E_G(H)$ and $xh^px^{-1} = h^q$, where $h^p \neq 1$.

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Suppose that h is not in a conjugate of a factor. Then we may assume that h is cyclically reduced and of length $\geqq 2$. Since $xh^p x^{-1} = h^q$, it is easy to see (by using length arguments on cyclically reduced forms) that $p = \pm q$. Therefore, $(xhx^{-1})^p = h^{\epsilon p}$, $\epsilon = \pm 1$. Hence (by the corollary to Theorem 1), $xhx^{-1} = h^\epsilon$, i.e., x is in $N(H)$, and so $E_G(H) = N(H)$. Moreover, if h is not the product of two elements of order two and H is maximal cyclic, then $N(H) = C(h)$ is cyclic, and so $E_G(H) = H$.

Suppose next that h is in a factor, but H intersects each conjugate of U trivially. Then h^p, h^q are properly contained in a factor and so x must also be in that factor. Hence, $E_G(H)$ is in the same factor as h.

Suppose now that U is malnormal in A or B, say $E_A(U) = U$. Hence, by the first remark in the proof of Lemma 6, B is malnormal in G; therefore, $E_G(U)$ is in B. If $h^n (\neq 1)$ is in U for some n, then h is in $E_G(h^n)$, and so h is in B; hence, $H < B$ and $E_G(H)$ is in B.

Suppose next that h has infinite or prime power order. We first observe that $E_G(H)$ is a subgroup. For, clearly $x \in E_G(H)$ implies that $x^{-1} \in E_G(H)$. Suppose also that $y \in E_G(H)$ and $yh^s y^{-1} = h^t$, $h^s \neq 1$. If h has infinite order, then $xyh^{sp}y^{-1}x^{-1} = h^{tq}$ with $h^{sp} \neq 1$, and so $E_G(H)$ is a subgroup. If h has prime power order ${p_1}^k$ then we may assume that $p = {p_1}^\alpha$ and $s = {p_1}^\beta$. Since h^q has order ${p_1}^{k-\alpha}$ and h^t has order ${p_1}^{k-\beta}$, we have that $p|q$ and $s|t$. Letting $\gamma = \operatorname{lcm}(p, s)$, we see that $h^\gamma \neq 1$ and

$$xyh^\gamma y^{-1}x^{-1} = x(yh^s y^{-1})^{\gamma/s} x^{-1} = xh^{\gamma(t/s)}x^{-1} = h^{(\gamma q t)/(ps)},$$

and again $E_G(H)$ is a subgroup.

Let $h^n (\neq 1)$ be in U. If h has infinite order, then $xh^{pn}x^{-1} = h^{qn}$; hence, x is in $E_G(\mathrm{gp}(h^n))$. If h has prime power order ${p_1}^k$, then we may assume that $n = {p_1}^\delta$ and $p = {p_1}^\alpha$. Letting $d = \operatorname{lcm}(n, p)$ we have that

$$xh^d x^{-1} = (xh^p x^{-1})^{d/p} = h^{d/(qp)};$$

hence, x is again in $E_G(\mathrm{gp}(h^n))$. In both cases, $E_G(H)$ is a subgroup of $E_G(\mathrm{gp}(h^n))$, and therefore by Lemma 1, $E_G(H)$ is in a conjugate of a factor.

COROLLARY 1. *Let G be a 0-step malnormal tree product, $1 \neq h \in G$, and $H = \mathrm{gp}(h)$. If h is in some vertex then $E_G(H)$ is in that vertex. Suppose that h is not in a conjugate of any vertex of G, h has no proper roots (i.e., H is maximal cyclic in G), and h is not the product of two elements of order two; then H is malnormal in G.*

Proof. Clearly, it suffices to show that the assertions hold when G has only finitely many vertices. Since each vertex of G is malnormal in G, if h is in some vertex, $E_G(H)$ is also. Suppose, then, that h satisfies the last conditions in the hypotheses above. If A_1 is an extremal vertex of G, then $G = (A_1 * G_1; U_1)$, where G_1 is the subtree product arising from G by deleting A_1 and its edge U_1. If h is in a conjugate of G_1, then by inductive hypothesis, H is malnormal in that conjugate of G_1, and therefore H is malnormal in G. Otherwise, Theorem 7 applies and again H is malnormal in G.

COROLLARY 2. *If we take an amalgamated product of two free groups with the amalgamated subgroup maximal cyclic in one of the factors, then take an amalgamated product of two such groups with the amalgamated subgroup again maximal cyclic in one of the factors, and continue to repeat the process finitely often, we always arrive at* 1-*step malnormal products; if at each stage the amalgamated subgroup is maximal cyclic in both factors, then we always arrive at* 0-*step malnormal products.*

COROLLARY 3. *Let* $G = (A * B; U)$ *be an r-step malnormal product. Suppose that* $a \in A - U$, $b \in B - U$. *If either coset* Ua *or* Ub *has no elements of order two, then* $\mathrm{gp}(ab)$ *is malnormal in* G.

Proof. Clearly, $H = \mathrm{gp}(ab)$ is maximal cyclic in G. It is easy to show by using length arguments that, if $ab = cd$ where c and d have order two, then Ua and Ub have elements of order two.

As an application, *let G be presented by*

$$(7) \qquad G = \langle a_1, b_1, \ldots, a_p, b_p, s_1, \ldots, s_d, c_1, \ldots, c_r; \\ s_1^{n_1}, \ldots, s_d^{n_d}, k_1^{q_1} \ldots k_p^{q_p} s_1^{t_1} \ldots s_d^{t_d} c_1^{m_1} \ldots c_r^{m_r}\rangle$$

where $k_i = [a_i, b_i]$, $0 < t_i < n_i$, $0 < m_i$; it follows from elementary number theory that without loss of generality we may assume that $t_i | n_i$. *Then G can be expressed as a* 1-*step malnormal product* (*in which the amalgamated subgroup is malnormal in one of the factors*) *unless one of the following degenerate cases occurs:* $p + d + r \leqq 2$; $p = r = 0$, $d = 3$; $p = r = 0$, $d = 4$, *and each* $n_i/t_i = 2$; $p = 1$, $r = 0$, $d = 2$, *and each* $n_i/t_i = 2$; $p = 0$, $r = 1$, $d = 2$, *and each* $n_i/t_i = 2$. For example, if $p = r = 0$, $d = 4$, and $n_1/t_1 \neq 2$, then

$$G = (\langle s_1, s_2; s_1^{n_1}, s_2^{n_2}\rangle * \langle s_3, s_4; s_3^{n_3}, s_4^{n_4}\rangle; s_1^{t_1}s_2^{t_2} = s_3^{-t_3}s_4^{-t_4});$$

hence, by Corollary 3 above, with $A = \mathrm{gp}(s_1)$, $B = \mathrm{gp}(s_2)$, $U = 1$, $a = s_1^{t_1}$, $b = s_2^{t_2}$, we see that G is a malnormal product as stated. *Thus, except for the degenerate cases, the groups G in* (7) *have the cyclic centralizer property; moreover, the only subgroups satisfying a non-trivial law are cyclic.* In each of the excluded cases there are groups which fail to have the cyclic centralizer property.

The groups G in (7) include as a special case the finitely generated orientation-preserving discrete groups of motions of the hyperbolic plane; these hyperbolic groups all have the cyclic centralizer property (see Greenberg [**3**]).

Next, we generalize an example of T. Lewin [**8**].

THEOREM 8. *Let* $\{L_i\}$, $i \in I$, *be a malnormal collection of subgroups of a group* K, *and let* $\{x_i\}$ *be a collection of distinct elements of a group* X. *Then the subgroup* U *generated by the subgroups* $x_iL_ix_i^{-1}$,

$$U = \prod_i{}^* (x_iL_ix_i^{-1}),$$

is malnormal in $X * K$.

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Proof. The proof is similar to that of [**1**, Lemma 2]. As in that proof, let $A = X * K$, and let $\{x_\sigma\}$ denote the elements of X, where σ ranges over an index set containing I. Then the normal subgroup N of A generated by K, $N = K^A$, is the free product of the conjugates K^{x_σ}. Moreover, if $a_1 \in A - U$, then a_1 has the form

$$a_1 = x_\sigma \cdot k_1^{y_1} \ldots k_r^{y_r} . u = au,$$

where $y_i = x_{\sigma i}$, $i = 1, \ldots, r$, $x_\sigma, x_{\sigma_i} \in X$, $k_i \in K$, $\sigma_i \neq \sigma_{i+1}$, $u \in U$, and if $\sigma_r \in I$ then $k_r \notin L_{\sigma_r}$.

Suppose that $E_A(U) \neq U$; let $d(\neq 1)$ be an element of U of smallest syllable length s such that $ada^{-1} \in U$, for some $a \in E_A(U) - U$. Now d has the form

$$d = c_1^{z_1} \ldots c_s^{z_s},$$

where $z_i = x_{\tau_i}$, $i = 1, \ldots, s$, $\tau_i \in I$, $c_i \in L_{\tau_i}$, and $\tau_i \neq \tau_{i+1}$; hence,

$$ada^{-1} = k_1^{x_\sigma y_1} \ldots k_r^{x_\sigma y_r} c_1^{x_\sigma z_1} \ldots c_s^{x_\sigma z_s} k_r^{-x_\sigma y_r} \ldots k_1^{-x_\sigma y_1}.$$

First, suppose that $x_\sigma = 1$. If $\sigma_r \neq \tau_1$ or $\sigma_r \neq \tau_s$, then the presence of the syllable $k_r^{y_r}$ or $k_r^{-y_r}$ prevents ada^{-1} from being in U; if $\sigma_r = \tau_1 = \tau_s$, then the presence of the syllable $(k_r c_1)^{z_1}$, $(c_s k_r^{-1})^{z_s}$, or $(k_r c_1 k_r^{-1})^{z_1}$ prevents ada^{-1} from being in U.

Hence, we may suppose that $x_\sigma \neq 1$. In this case, $c_i^{x_\sigma z_i}$ cannot be a syllable of ada^{-1}; for, if $x_\sigma x_{\tau_i} = x_j$, $j \in I$, then since $L_j \cap L_{\tau_i} = 1$, we have $c_i \notin L_j$ and so $ada^{-1} \notin U$. Hence, $r > 0$ and $\tau_1 = \sigma_r = \tau_s$. If $s > 1$, then $a' = ac_1^{z_1}$, $d' = c_2^{z_1} \ldots (c_s c_1)^{z_s}$ would have the same properties as a, d although d' has shorter syllable length than d. Finally, if $s = 1$, then $x_\sigma x_{\tau_1} = x_j$, $j \neq \tau_1$, $k_r L_{\tau_1} k_r^{-1} \cap L_j = 1$, so again we have $ada^{-1} \notin U$. This completes the proof of Theorem 8.

As an illustration, let $K = \mathrm{gp}(u) * A$, and let $X = \mathrm{gp}(v)$, where A is arbitrary and X is infinite cyclic. If $a_1, a_2, \ldots$ are distinct elements $(\neq 1)$ of A, then $\{\mathrm{gp}(u), \mathrm{gp}(ua_1), \mathrm{gp}(ua_2) \ldots\}$ is a malnormal collection of subgroups of K (see the first illustration following the definition of a malnormal collection of subgroups). Hence,

$$\mathrm{gp}(u, v^{-1}ua_1v, v^{-2}ua_2v^2, \ldots)$$

is a malnormal in $\mathrm{gp}(u) * \mathrm{gp}(v) * A$. This special case was proved by T. Lewin (see [**8**, p. 394, lemma]).

COROLLARY. *Let $\mathscr{C}$ be a class of groups including the infinite cyclic group and let $\mathscr{D}$ be a class of groups including the infinite cyclic and infinite dihedral groups. If A is a countable group having one of the following properties, then A can be embedded in a three-generator group having that property: the centralizer of any non-trivial element of A belongs to $\mathscr{C}$; the normalizer of any infinite cyclic subgroup of A belongs to $\mathscr{D}$; the subgroups of A satisfying a particular non-trivial law belong to $\mathscr{D}$; any indecomposable (with respect to amalgamated product)* *subgroup of A belongs to $\mathscr{C}$; any two-generator subgroup of A is free (or more generally is a free product of two cyclic groups). Moreover, if A is an n-relator group, then the three-generator group is an n-relator group.*

Proof. We modify an argument in T. Lewin [**8**]. Let $a_1, a_2, \ldots$ be distinct generators $(\neq 1)$ of A; let $F_1 = \langle u, v\rangle$ and $F_2 = \langle x, y\rangle$. Then $u, v^{-1}ua_1v, v^{-2}ua_2v^2, \ldots$ freely generate a malnormal subgroup S of $F_1 * A$. Moreover, it can be shown that $xyx, x^2y^2x^2, x^3y^3x^3, \ldots$ freely generate a malnormal subgroup T of F_2. Hence, the amalgamated product $(F_2 * (F_1 * A); T = S)$ is a 0-step malnormal product, which is clearly generated by x, y, v. Furthermore, the relations resulting from $T = S$, merely serve to define $u, a_1, a_2, \ldots$ in terms of x, y, v. These relations may be deleted provided we replace the a_i in the defining relators of A by their equivalents in terms of x, y, v. Hence, if A is an n-relator group, then the three-generator group is an n-relator group.

An interesting class of groups with malnormal subgroups is provided by groups with one defining relation

$$G = \langle a, b, c, \ldots; R^n\rangle, \ n > 1 \tag{8}$$

having torsion. B. B. Newman [**13**] has shown that the subgroup generated by any proper subset of the generators of G is malnormal in G. Hence, if we use the standard Magnus embedment to embed G in a one-relator group with torsion having zero exponent sum on one of the generators, then the normal subgroup N generated by all the other generators is a 0-step malnormal tree product whose vertices are "shorter one-relator" groups having torsion.

Also, in (8), *if R is not itself a proper power in the free group on $a, b, c, \ldots,$ then* $\mathrm{gp}(R)$ *is malnormal in G* (see [**5**]).

Finally, we describe two other types of examples of 0- and 1-step malnormal products: *free products with commuting subgroups and free products with centralized subgroups* ([**9**, pp. 220–221, exercises 22 and 27]). Specifically, let H_1, K_1 be malnormal subgroups of A_1, B_1, respectively.

For the first construction, form the group G with presentation obtained by writing down a presentation for $A_1 * B_1$ and then adjoining defining relations which state that each word in a given set of generators for H_1 commutes with each word in a given set of generators for K_1. It is easily shown that $G = (A * B; U)$, where $A = \mathrm{gp}(A_1, K_1) = (A_1 * (H_1 \times K_1); H_1)$,

$$B = \mathrm{gp}(H_1, B_1) = ((H_1 \times K_1) * B_1; K_1),$$

and $U = H_1 \times K_1$ (direct product). Since H_1, K_1 are malnormal in A_1, B_1, respectively, U is malnormal in A and in B (see the first remark in the proof of Lemma 6). Thus, G is a 0-step malnormal product. For example, the group

$$G = \langle a, b, h, k; a^3, h^2, (ah)^2, b^3, k^2, (bk)^2, hk = kh\rangle$$

is the free product of two symmetric groups of degree three with commuting cyclic subgroups of order two. It then follows, e.g., that the abelian subgroups of G are infinite cyclic, cyclic of order two or three, or $Z_2 \times Z_2$.

For the second construction, form the group G with presentation obtained by writing down a presentation for $A_1 * B_1$ and then adjoining defining relations which state that each A_1-generator commutes with a set of words which generate K_1 and that each B_1-generator commutes with a set of words which generate H_1. It is easily shown that $G = (A * B; U)$, where $A = A_1 \times K_1$, $B = H_1 \times B_1$, and $U = H_1 \times K_1$. Now in G, if $u = (h_1, k_1)$, then $E_A(u) = U$, when $h_1 \neq 1$, and $E_B(u) = U$, when $k_1 \neq 1$. Thus, G is a 1-step malnormal product (although ordinarily U is not malnormal in A or in B since $E_A(U) = A$ and $E_B(U) = B$).

For example, if $A_1 = \langle a, c; c^2\rangle$, $B_1 = \langle b, d; d^3\rangle$, $H_1 = \mathrm{gp}(c)$, $K_1 = \mathrm{gp}(d)$, then the group

$$G = \langle a, b, c, d; ad = da, bc = cb, cd = dc, c^2, d^3\rangle,$$

is the free product of A_1 and B_1 with the centralized subgroups H_1 and K_1; since $U = H_1 \times K_1 = \mathrm{gp}(u)$ is cyclic of order six,

$$G = \langle a, b, u; au^2 = u^2a, bu^3 = u^3b, u^6\rangle,$$

which is the group given near the beginning of this section.

6. Examples of r-step malnormal products. The examples of r-step malnormal products we give in this section all arise from HNN groups

$$G = \langle t_1, t_2, \ldots, K; \text{rel } K, t_1L_1t_1^{-1} = \varphi_1(L_1), t_2L_2t_2^{-1} = \varphi_2(L_2), \ldots\rangle, \tag{9}$$

where $\{\varphi_i\}$ is a collection of isomorphisms of $\{L_i\}$ into K; as usual we denote $\varphi_i(L_i)$ by M_i or L_{-i}. To construct these examples, we use the standard embedment (of Higman, Neumann, and Neumann [**4**]) of the HNN group given by (9) in the amalgamated product

$$E = (A * B; U) = X * G = Y * G, \tag{10}$$

where

$$A = X * K, B = Y * K, U = K * \ldots * x_iL_ix_i^{-1} * \ldots$$
$$= K * \ldots * y_1My_i^{-1} * \ldots,$$

and X, Y are free groups on x_i, y_i, respectively, and $t_i = y_i^{-1}x_i$, $i = 1, 2, \ldots$.

LEMMA 7. *Let A be the free product $X * K$ of two groups X and K, and let $x_0 = 1, x_1, \ldots, x_n$ be distinct elements of X such that $x_ix_j^{-1} \neq x_px_q^{-1}$, unless $i = j$ or $i = p$, where $0 \leqq i, j, p, q \leqq n$. Suppose that $L_0 = K, L_1, \ldots, L_n$ are subgroups of K and that $U = x_0L_0x_0^{-1} * \ldots * x_nL_nx_n^{-1}$, and let $a_1 \in A - U$. If $a_1Ua_1^{-1} \cap U \neq 1$, then $a_1Ua_1^{-1} \cap U = ax_iL_ix_i^{-1}a^{-1} \cap U$, where $a_1 = au_1$, $u_1 \in U$. Moreover, if $L_i' < L_i$, then the intersection $ax_iL_i'x_i^{-1}a^{-1} \cap U$, if non-trivial, is $ux_j(L_j \cap kL_i'k^{-1})x_j^{-1}u^{-1}$, for some $0 \leqq j \leqq n$, $u \in U$, and $k \in K$; furthermore, $a = ux_jkx_i^{-1}$, $k \neq 1$ or $k \notin L_j$, and $i \neq 0$ or $j \neq 0$.*

Proof. This is the same as that of [**7**, Lemma 2].

THEOREM 9. *Let G be an HNN group*

$$G = \langle t, K; \text{rel } K, tLt^{-1} = M \rangle$$

satisfying the following conditions:

(a) *L and M are malnormal in K.*

(b) *There exists $r > 0$ such that for any $k_1, \ldots, k_r \in K$,*

$$L \cap L^{t^{-1}k_1} \cap L^{t^{-1}k_1t^{-1}k_2} \cap \ldots \cap L^{t^{-1}k_1 \ldots t^{-1}k_r} = 1. \tag{11}$$

Then the amalgamated product corresponding to G, $E = (A * B; U)$, described in (10), *is a 2r-step malnormal product.*

Proof. Now, condition (b) is equivalent to

$$M \cap M^{tk_1} \cap M^{tk_1tk_2} \cap \ldots \cap M^{tk_1 \ldots tk_r} = 1, \tag{12}$$

for any $k_1, \ldots, k_r \in K$ (simply conjugate the left hand side of (11) by $tk_r^{-1}t \ldots k_1^{-1}t$).

We first show that if

$$baxL'x^{-1}a^{-1}b^{-1} \cap U \tag{13}$$

is non-trivial, where $L' < L$, $a \in A - U$, $b \in B - U$, then (13) equals

$$u_1x(L \cap t^{-1}kL'k^{-1}t)x^{-1}u_1^{-1},$$

where $u_1 \in U$. For, by Lemma 7,

$$axL'x^{-1}a^{-1} \cap U = u_2x_j(L_j \cap k_1L'k_1^{-1})x_j^{-1}u_2^{-1}, \tag{14}$$

and $a = u_2x_jk_1x^{-1}$. Now $j = 0$, for, otherwise, $L_j = L > L'$, and so $k_1 \in L$ which implies that $a \in U$, contrary to the definition of a. Hence,

$$axL'x^{-1}a^{-1} \cap U = u_2L'u_2^{-1},$$

and so (13) equals

$$bu_2(y_0L'y_0^{-1})u_2^{-1}b^{-1} \cap U. \tag{15}$$

Hence, again applying Lemma 7, we have that $bu_2 = u_1yky_0^{-1}$ and (15) becomes

$$u_1y(M \cap kL'k^{-1})y^{-1}u_1^{-1}; \tag{16}$$

since $t^{-1} = x^{-1}y$, (16) equals

$$u_1xt^{-1}(M \cap kL'k^{-1})tx^{-1}u_1^{-1} = u_1x(L \cap t^{-1}kL'k^{-1}t)x^{-1}u_1^{-1}. \tag{17}$$

It is now easy to show by induction that

$$b_ra_r \ldots b_1a_1xLx^{-1}a_1^{-1}b_1^{-1} \ldots a_r^{-1}b_r^{-1} \cap U$$
$$= ux(L \cap L^{t^{-1}k_1} \cap L^{t^{-1}k_1t^{-1}k_2} \ldots \cap L^{t^{-1}k_1 \ldots t^{-1}k_r})x^{-1}u^{-1},$$

which by hypothesis is trivial.

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Clearly, because of the symmetric conditions on L and M, we may assume that

$$a_r b_r \ldots a_1 b_1 y M y^{-1} b_1^{-1} a_1^{-1} \ldots b_r^{-1} a_r^{-1} \cap U = 1.$$

To prove Theorem 9, suppose that $p \in E_E(U)$ and $|p| \geqq 2r + 1$. Now, using Lemma 7, it is easy to show that

$$pUp^{-1} \cap U = p_1 x_i L_i x_i^{-1} p_1^{-1} \cap U, \ i = 0 \text{ or } 1,$$

where $|p_1| = |p|$. If $i = 1$, the preceding argument shows that $pUp^{-1} \cap U = 1$; hence, $i = 0$. Because of the symmetrical hypothesis, we may assume that $p_1 = qa$, where a, the last syllable of p_1, is in $A - U$. But then by Lemma 7,

$$aKa^{-1} \cap U = uxLx^{-1}u^{-1},$$

and, therefore,

$$pUp^{-1} \cap U = (qu)xLx^{-1}(qu)^{-1} \cap U,$$

where $|qu| \geqq 2r$. Thus, again by the preceding argument, $pUp^{-1} \cap U = 1$. This completes the proof of Theorem 9.

We shall see (in Theorem 10) that the groups G in Theorem 9 include one-relator groups having torsion and zero exponent sum on some generator.

Using similar arguments we may prove Corollaries 1 and 2.

COROLLARY 1. *Let G be an HNN group*

$$G = \langle t, K; \text{rel } K, tLt^{-1} = M \rangle$$

satisfying the following conditions:

(a) *L and M are malnormal in K.*

(b) *$kLk^{-1} \cap M = 1$ unless $k \in L \cap M$.*

(c) *There exists an integer $r > 0$ such that $L \cap t^r L t^{-r} = 1$.*

Then the amalgamated product corresponding to G, $E = (A * B; U)$, described in (10), *is a $2r$-step malnormal product.*

For example, let

$$G = \langle t, a_1, a_2, \ldots, a_n; tw_1t^{-1} = w_2 \rangle,$$

where w_1, w_2 are words in the a_i which are not proper powers, and which, when cyclically reduced, are not cyclic permutations of each other. Then the conditions of Corollary 1 are clearly satisfied with $L = \text{gp}(w_1)$, $M = \text{gp}(w_2)$, $K = \langle a_1, \ldots, a_n \rangle$ and $r = 1$.

COROLLARY 2. *Let G be an HNN group*

$$G = \langle t_1, t_2, \ldots, K; \text{rel } K, t_1 L_1 t_1^{-1} = L_{-1}, t_2 L_2 t_2^{-1} = L_{-2}, \ldots \rangle$$

satisfying the following conditions:

(a) *For each $k \in K$, and each pair i, j, $kL_ik^{-1} \cap L_j = 1$, unless $k \in L_i \cap L_j$.*

(b) *There exists an integer $r > 0$ such that, if $W(t_1, \ldots, t_r)$ is a freely*

reduced word of length r in the free part of G, then $WL_iW^{-1} \cap L_j = 1$, *for each* i, j *for which* $ij > 0$.
Then the amalgamated product E in (10) *is a* $2r$-*step malnormal product.*

Any *HNN* group satisfying the conditions of Theorem 9, Corollary 1, or Corollary 2 will be called an *HNN group with r-step malnormal associated subgroups*, or simply *an r-step malnormal HNN group*.

It follows, for example, that *in an r-step malnormal HNN group, the centralizer of an element* ($\neq 1$) *is infinite cyclic or contained in a conjugate of K, the normalizer of a cyclic subgroup* ($\neq 1$) *is infinite cyclic, infinite dihedral, or contained in a conjugate of K; a subgroup of G satisfying a non-trivial law is infinite cyclic, infinite dihedral, or contained in a conjugate of K.*

COROLLARY 3. *Let G be an HNN group*

$$G = \langle t_1, t_2, \ldots, K; \text{rel } K, t_1L_1t_1^{-1} = L_{-1}, t_2L_2t_2^{-1} = L_{-2}, \ldots \rangle$$

in which the centralizer of an element ($\neq 1$) *is cyclic or contained in a conjugate of K; then the free part* $T = \text{gp}(t_1, t_2, \ldots)$ *is malnormal in G. In particular, if G is an r-step malnormal HNN group, then its free part is malnormal in G.*

Proof. If $N = K^G$, then any element g of G has the form $g = nw$, where $n \in N$, $w \in T$. Suppose that w_1 ($\neq 1$), w_2 are in T and $nww_1w^{-1}n^{-1} = w_2$. Since T is a retract of G with kernel N, $ww_1w^{-1} = w_2$, and so n is in the centralizer of w_2. If $n \neq 1$, then a non-zero power of w_2 would be in N, which is impossible.

COROLLARY 4. *Let G be an HNN group*

$$G = \langle t_1, t_2, \ldots, K; \text{rel } K, t_1L_1t_1^{-1} = L_{-1}, t_2L_2t_2^{-1} = L_{-2}, \ldots \rangle$$

in which the collection of associated subgroups $\{L_i\}$ *is a malnormal collection in the base K. Then G is a* 1-*step malnormal HNN group; moreover, each associated subgroup* L_i *is malnormal in G.*

Proof. Let $N = K^G$. Then N is the tree product of the vertices wKw^{-1}, where w ranges over all freely reduced words in the free part T of G; for each w the collection of subgroups $\{wL_iw^{-1}\}$ corresponds to the collection of edges incident with the vertex wKw^{-1}, and is malnormal in wKw^{-1}. Hence, N is a 0-step malnormal tree product; moreover, Lemma 6 (c) implies that the collection of subgroups corresponding to all the edges of N is malnormal in N. It follows, therefore, that the hypotheses of Corollary 2 above are satisfied with $r = 1$, and so G is a 1-step malnormal *HNN* group.

To show that L_i is malnormal in G, suppose that $nwL_iw^{-1}n^{-1} \cap L_i \neq 1$, where $n \in N$, $w \in T$. If $w \neq 1$, then wL_iw^{-1} and L_i are part of a malnormal collection in N; hence, $w = 1$ and $n \in L_i$, so L_i is malnormal in G. This completes the proof of Corollary 4.

It should also be noted that for particular cases of r-step malnormal *HNN* groups, the standard embedment may not be the most economical. For example, if G is an HNN group whose base is a free product $K = K_1 * K_{-1}$ such that $\{L_i\}, \{M_i\}$ are malnormal collections of subgroups of K_1, K_{-1}, respectively, then G is a 1-step malnormal HNN group; the standard embedment E yields a 2-step malnormal product. On the other hand, G can be embedded in the group

$$((X * K_1) * (Y * K_{-1}); \prod_i{}^* x_i L_i x_i^{-1} = \prod_i{}^* y_i M_i y_i^{-1}) \tag{18}$$

where X and Y are as in the standard embedment; using Theorem 8, we see that (18) is a 0-step malnormal product.

An important class of r-step malnormal HNN groups is provided by one-relator groups G having torsion. If the relator R^n, $n > 1$, is such that some generator of G occurs exactly once in R, then G is the free product of a free group and a cyclic group of order n; if some generator does not occur in R, then again G is a free product. Hence, we may restrict ourselves to relators R^n in which each generator occurs at least twice in R.

THEOREM 10. *A group with one defining relator R^n, $n > 1$, such that R has zero exponent sum on some generator involved in R is an r-step malnormal HNN group*

$$G = \langle t, K; \text{rel } K, tLt^{-1} = M\rangle,$$

where K is a group on one relator P^n with P of shorter length than R, and L, M are free.

Proof. Let

$$G = \langle a, b, c, \ldots, R^n\rangle,$$

where R has zero exponent sum on a; for convenience of notation, we assume that G has three generators a, b, c. Moreover, as Moldavanski [**10**] has observed, G is an HNN group

$$G = \langle t, K; R_0^n, tLt^{-1} = M\rangle, \tag{19}$$

where $t = a$; R_0 is the word obtained from R by rewriting R in terms of the conjugates $b_i = a^i ba^{-1}$ and $c_i = a^i ca^{-i}$; K is the group with the single defining relator R_0^n and with generators b_i, c_j, where i ranges between the minimum subscript $\lambda = \lambda(b)$ and the maximum subscript $\mu = \mu(b)$ occuring on b in R_0, and, similarly, $\lambda' = \lambda(c) \leqq j \leqq \mu' = \mu(c)$; and L is the free group on

$$b_\lambda, \ldots, b_{\mu-1}, c_{\lambda'}, \ldots, c_{\mu'-1}.$$

Moreover, R_0 has shorter length then R. Also, since each generator occurs at least twice in Q, we may assume that $\lambda < \mu$.

We shall prove that, if $r = \max(\mu - \lambda, \mu' - \lambda')$, then G (when written as in (19)), satisfies the conditions of Theorem 9. To do this, we use some results of B. B. Newman [**13**].

First of all, L, M are malnormal in K; indeed, as we have already indicated, any proper subset of the generators of a one relator group having torsion generates a malnormal subgroup of the group.

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Next, we show that if $k \in K$ and

$$D = k\,\mathrm{gp}(b_\lambda, \ldots, b_{\mu-s}, c_{\lambda'}, \ldots, c_{\mu'-s})k^{-1} \cap M \neq 1,$$

then

$$D = m_1\,\mathrm{gp}(b_{\lambda+1}, \ldots, b_{\mu-s}, c_{\lambda'+1}, \ldots, c_{\mu'-s})m_1^{-1},\ m_1 \in M. \tag{20}$$

For, suppose that $klw_1l^{-1}k^{-1} = mw_2m^{-1}$, where l, w_1 are in

$$\mathrm{gp}(b_\lambda, \ldots, b_{\mu-s}, c_{\lambda'}, \ldots, c_{\mu'-s}),$$

$w_1\ (\neq 1)$ is cyclically reduced, $m, w_2 \in M$, and $w_2 (\neq 1)$ is cyclically reduced. Now, w_1, w_2 are both in $L \cap M$. For, it follows from [**13**, Lemma 2.3.1], that either w_1 or w_2 is in $L \cap M$; if $w_2 \in L \cap M$, since L is malnormal in K, $l_1 = m^{-1}kl \in L$. But $L = \mathrm{gp}(b_\lambda, c_{\lambda'}) * (L \cap M)$ (since by the Spelling Theorem in [**13**], $L \cap M = \mathrm{gp}(b_{\lambda+1}, \ldots, b_{\mu-1}, c_{\lambda'+1}, \ldots, c_{\mu'-1})$) and w_1 is a cyclically reduced element of L which is conjugate in L to an element of $L \cap M$; therefore, w_1 is also in $L \cap M$. Similarly, $w_1 \in L \cap M$ implies that $w_2 \in L \cap M$. Moreover, since $L \cap M$ is malnormal in K, $l_1 \in L \cap M$. Therefore, if $m_1 = ml_1$,

$$\begin{aligned} D &= m_1(\mathrm{gp}(b_\lambda, \ldots, b_{\mu-s}, c_{\lambda'}, \ldots, c_{\mu'-s}) \cap M)m_1^{-1} \\ &= m_1\,\mathrm{gp}(b_{\lambda+1}, \ldots, b_{\mu-s}, c_{\lambda'+1}, \ldots, c_{\mu'-s})m_1^{-1}. \end{aligned}$$

Consequently,

$$\begin{aligned} L \cap L^{t^{-1}k_1} &= (M \cap L^{k_1})^{t^{-1}} \\ &= (m_1\,\mathrm{gp}(b_{\lambda+1}, \ldots, b_{\mu-1}, c_{\lambda'+1}, \ldots, c_{\mu'-1})m_1^{-1})^{t^{-1}} \\ &= l_2\,\mathrm{gp}(b_\lambda, \ldots, b_{\mu-2}, c_{\lambda'}, \ldots, c_{\mu'-2})l_2^{-1},\ l_2 \in L. \end{aligned}$$

By induction on s, it follows that

$$\begin{aligned} &L \cap L^{t^{-1}k_1} \cap \ldots \cap L^{t^{-1}k_1 \cdots t^{-1}k_s} \\ &\qquad = l_{s+1}\,\mathrm{gp}(b_\lambda, \ldots, b_{\mu-s-1}, c_{\lambda'}, \ldots, c_{\mu'-s-1})l_{s+1}^{-1}, \end{aligned} \tag{21}$$

where $l_{s+1} \in L$. Hence, if $s = r = \max(\mu - \lambda, \mu' - \lambda')$, then (21) is trivial. This completes the proof of Theorem 10.

It follows then, for example, *that in a one-relator group G having torsion, the centralizer of an element* $(\neq 1)$ *is cyclic* (this is also proved in [**13**]); *an element of infinite order has at most one qth root for any q; the normalizer of a cyclic subgroup is cyclic, or infinite dihedral; a maximal cyclic subgroup $H = \mathrm{gp}(h)$ of G is malnormal in G unless h is the product of two elements of order two; and a subgroup of G satisfying a non-trivial law must be cyclic or infinite dihedral* (for another proof of this last result, see [**7**]).

COROLLARY. *Using the notation of Theorem* 10, *no subgroup H of G can be a union of subgroups of conjugates of K unless H is in a conjugate of K.*

LEMMA 8. *Let G be the HNN group*

$$G = \langle t, S; \mathrm{rel}\ S, tLt^{-1} = M \rangle,$$

where $S = (C * D; Q)$ *is a* 0-*step malnormal product*, L, M *are malnormal in* C, D, *respectively, and the conjugates of* L *in* C *and of* M *in* D *each intersect* Q *trivially. Then*

$$G = ((\mathrm{gp}(t) * L) * (C * D; Q); L * M), \tag{22}$$

and is a 1-*step malnormal product.*

Proof. Since $L \cap Q = 1 = M \cap Q$, it follows that $U = \mathrm{gp}(L, M) = L * M$, $T = \mathrm{gp}(t, L) = \mathrm{gp}(t) * L$, and, hence, that G can be written as in (22).

Next, we note that by [**7**, Lemma 2] (using $X = \mathrm{gp}(t)$, $K = L$, $x_1 = t$, $L_1 = L$), if $\tau \in T - U$, then $\tau U \tau^{-1} \cap U < uLu^{-1}$ or $\tau U \tau^{-1} \cap U < uMu^{-1}$, for some $u \in U$.

Moreover, if $\sigma \in S$, and $(\sigma(L \cup M)\sigma^{-1}) \cap (L * M) \neq 1$, then $\sigma \in L * M$. For, suppose, e.g., that $\sigma l \sigma^{-1} = pnp^{-1}$, where $1 \neq l \in L, p, n \in L * M$, n cyclically reduced in $L * M$. Now the syllable length of an element of $L * M$ in the free product $L * M$ is the same as its syllable length in $(C * D; Q)$. Hence, by [**6**, Theorem 4.6], n has syllable length 1; moreover, since in D the conjugates of M intersect Q trivially, $n \in L$. But L is malnormal in $(C * D; Q)$; hence, $p^{-1}\sigma \in L$ and so $\sigma \in L * M$.

To prove Lemma 8, suppose that some element of G with syllable length $\geqq 2$ is in $E_G(U)$. It follows, then, that there is an element $\sigma\tau$ of syllable length two in $E_G(U)$, where $\sigma \in S - U, \tau \in T - U$. Suppose that $\sigma\tau u_1 \tau^{-1}\sigma^{-1} \in U$, where $1 \neq u_1 \in U$. Then $\tau u_1 \tau^{-1} \in u(L \cup M)u^{-1}$, for some $u \in U$ (by the remark of the second paragraph). Hence, $\sigma u(L \cup M)u^{-1}\sigma^{-1} \cap (L * M) \neq 1$; and, therefore, by the preceding paragraph, σu and hence σ are in $L * M$, which is a contradiction.

THEOREM 11. *Using the same notation as in Theorem* 10, G *is a finite extension of a* 1-*step malnormal product whose factors are* $T = \mathrm{gp}(t^{2r}) * L$, *and the* 0-*step malnormal tree product* S *given by*

$$S = K_0 \underset{L_1}{*} K_1 \underset{L_2}{*} \dots \underset{L_{2r-1}}{*} K_{2r-1},$$

where $K_i = t^i K t^{-i}$, $L_i = t^i L t^{-i}$; *the subgroup amalgamated between* T *and* S *is* $L * L_{2r}$.

Proof. It follows, as in the proof of Corollary 2 of Theorem 4, that the normal subgroup N_{2r} of G generated by t^{2r} and K is the HNN group

$$\langle t^{2r}, S; \mathrm{rel}\, S, t^{2r} L t^{-2r} = L_{2r} \rangle.$$

We view S as $(C * D; Q)$, where

$$C = \mathrm{gp}(K_0, \dots, K_{r-1}), \quad D = \mathrm{gp}(K_r, \dots, K_{2r-1}), \quad \text{and} \quad Q = L_r,$$

and show that the hypotheses of Lemma 8 (with $L = L$, $M = L_{2r}$) are satisfied. Since S is clearly a 0-step malnormal tree product, each vertex K_i is malnormal in S. Hence, each L_i is malnormal in S. Therefore, L, Q are malnormal in C, and L_{2r}, Q are malnormal in D.

It remains to show that $cLc^{-1} \cap Q = 1$, for each $c \in C$, and that $dL_{2r}d^{-1} \cap Q = 1$, for each $d \in D$.

To this end, we first show by induction on s that, if $g \in \text{gp}(K_0, \ldots K_{s-1})$, then

$$(23) \qquad g\,\text{gp}(b_\lambda, \ldots, b_{\mu-p}, c_{\lambda'}, \ldots, c_{\mu'-p})g^{-1} \cap L_s$$

is contained in

$$(24) \qquad t^s l\,\text{gp}(b_\lambda, \ldots, b_{\mu-p-s}, c_{\lambda'}, \ldots, c_{\mu'-p-s})l^{-1}t^{-s},$$

with $p \geqq 1$ and $l \in L$. If $s = 0$, this is clear. Suppose that $s > 0$. Now

$$(25) \qquad \text{gp}(K_0, \ldots, K_{s-1}) = (K_0 * \text{gp}(K_1, \ldots, K_{s-1}); L_1).$$

If $gL'g^{-1}$ is in (23) and $g = g_2g_1$, where $g_1 \in K_0$, and g_2, when written in reduced form in (25), does not end in a K_0-syllable, then $g_1L'g_1^{-1} < L_1$. Hence, as shown in the proof of (20) of Theorem 10,

$$\begin{aligned} g_1L'g_1^{-1} &< g_1\,\text{gp}(b_\lambda, \ldots, b_{\mu-p}, c_{\lambda'}, \ldots, c_{\mu'-p})g_1^{-1} \cap M \\ &= m_1\,\text{gp}(b_{\lambda+1}, \ldots, b_{\mu-p}, c_{\lambda'+1}, \ldots, c_{\mu'-p})m_1^{-1}, \ m_1 \in M. \end{aligned}$$

Therefore, $g_2 \in \text{gp}(K_1, \ldots, K_{s-1})$, since this factor of (25) is malnormal in (25). Hence, (23) is contained in

$$g_2m_1\,\text{gp}(b_\lambda, \ldots, b_{\mu-p'}\, c_{\lambda}{'}_{+1}, \ldots, c_{\mu'-p})m_1^{-1}g_2^{-1} \cap L_s,$$

which equals

$$(26) \qquad t(g_3\,\text{gp}(b_\lambda, \ldots, b_{\mu-p-1}, c_{\lambda'}, \ldots, c_{\lambda'-p-1})g_3^{-1} \cap L_{s-1})t^{-1},$$

where $g_3 \in \text{gp}(K_0, \ldots, K_{s-2})$. By the inductive hypothesis, (26) and hence (23) are contained in (24).

Since $r = \max(\mu - \lambda, \mu' - \lambda')$, it follows that $cLc^{-1} \cap L_r = 1$. Moreover, $dL_{2r}d^{-1} \cap L_r = t^r(cL_rc^{-1} \cap L)t^{-r} = 1$. This completes the proof of Theorem 11.

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