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AN EXISTENCE THEOREM FOR ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACES

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We prove the existence of bounded solution of the differential equation $y' = A(t)y + f(t, y)$ in a Banach space. The method used here is based on the concept of "admissibility" due to Massera and Schäffer when f satisfies the Caratheodory conditions and some regularity condition expressed in terms of the measure of noncompactness α .

We prove the existence of bounded solution of the differential equation $y' = A(t)y + f(t, y)$ in a Banach space. The method used here is based on the concept of "admissibility" due to Massera and Schäffer [5] when f satisfies the Caratheodory conditions and some regularity condition exprcssed in terms of the measure of noncompactness α . Our result is closely related to the results of Szufla [7].

Throughout this paper J denotes the half-line $t \geq 0$, E a Banach space with norm $\|\cdot\|$, and $L(E)$ the algebra of continuous linear operators from E into itself with induced norm $|\cdot|$. Further, we will use standard notation and some of the notation, definitions and results from the book of Massera and Schäffer [5].

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Let us denote:

by $L(J, E)$ – the vector space of strongly measurable functions from J into E, Bochner integrable in every finite subinterval I of J, with the topology of the convergence in the mean, on every such I;

by $B(J, \mathbb{R})$ – a Banach function space, provided with the norm $\|\cdot\|_{B(\mathbb{R})}$, of real-valued measurable functions on J such that

(1) $B(J, \mathbb{R})$ is stronger than $L(J, \mathbb{R})$,

(2) $B(J, \mathbb{R})$ contains all essentially bounded functions with compact support,

(3) if $u \in B(J, \mathbb{R})$ and v is a real-valued measurable function on J with $|v| \leq |u|$, then $v \in B(J, \mathbb{R})$ and $\|v\|_{B(\mathbb{R})} \leq \|u\|_{B(\mathbb{R})}$, and

(4) if v_n $(n = 1, 2, \ldots)$ are real-valued measurable functions on J such that $\lim_{n\to\infty} v_n(t) = 0$ for almost all $t \in J$ and $|v_n| \leq u$ with $u \in B(J, \mathbb{R})$, then $\lim_{n\to\infty} \|v_n\|_{B(\mathbb{R})} = 0$;

by $B^*(J, \mathbb{R})$ – the associate space to $B(J, \mathbb{R})$, that is, the Banach space of measurable functions $u : J \to \mathbb{R}$ such that

$$\|u\|_{B^*(\mathbb{R})} = \sup\left\{\int_J |v(s)u(s)|ds : v \in B(J, \mathbb{R}), \|v\|_{B(\mathbb{R})} \leq 1\right\} < \infty ;$$

by $B(J, E)$ $\left(\text{respectively } B^*(J, E)\right)$ – the Banach space of all strongly measurable functions $u : J \to E$ such that $\|u\| \in B(J, \mathbb{R})$ $\left(\text{respectively } \|u\| \in B^*(J, \mathbb{R})\right)$ provided with the norm $\|u\|_{B(E)} = \big\|\|u\|\big\|_{B(\mathbb{R})}$ $\left(\text{respectively } \|u\|_{B^*(E)} = \big\|\|u\|\big\|_{B^*(\mathbb{R})}\right)$;

by $C(J, E)$ – the vector space of all continuous functions from J to E endowed with the topology of uniform convergence on compact subsets of J.

Assume that $A \in L\big(J, L(E)\big)$. Let E_0 denote the set of all points of E which are values for $t = 0$ of bounded solutions of the

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differential equation $y' = A(t)y$. Suppose that E_0 is closed and has a closed complement, that is, there exists a closed subspace E_1 of E such that E is the direct sum of E_0 and E_1 .

Let P be the projection of E onto E_0 , and let $U : J \to L(E)$ be the solution of the equation $U' = A(t)U$ with the initial condition $U(0) = I$ (the identity mapping). For any $t \in J$ we define a function $G(t, \cdot) \in L\big(J, L(E)\big)$ by

$$G(t, s) = \begin{cases} U(t)PU^{-1}(s) & \text{for } 0 \le s \le t , \\ -U(t)(I-P)U^{-1}(s) & \text{for } s > t . \end{cases}$$

Assume in addition that there exists a constant $C > 0$ such that, for any $t \in J$, $G(t, \cdot) \in B^*\big(J, L(E)\big)$ and $\|G(t, \cdot)\|_{B^*(L(E))} \le C$.

Let α denote the Kuratowski measure of noncompactness in E , the properties of which may be found in [2] and [3]. Suppose $f : J \times E \to E$ is a function which satisfies the following conditions:

1°. for each $x \in E$ the mapping $t \mapsto f(t, x)$ is strongly measurable, and for each $t \in J$ the mapping $x \mapsto f(t, x)$ is continuous;

2°. $\|f(t, x)\| \le m(t)$ for all $(t, x) \in J \times E$, where $m \in B(J, \mathbb{R})$;

3°. for any $\varepsilon > 0$, $t_0 > 0$ and bounded subset X of E there exists a closed subset Q of $[0, t_0]$ such that $\operatorname{mes}\big([0, t_0]\setminus Q\big) < \varepsilon$ and

$$\alpha(f[I \times X]) \le \sup\{g(t) : t \in I\} \cdot h\big(\alpha(X)\big)$$

for each closed subset I of Q , where g and h are functions of J into itself, g is measurable, h is non-decreasing and

$$\sup\left\{\int_J \|G(t, s)\| g(s) ds : t \in J\right\} \cdot h(t) < t$$

for all $t > 0$.

Under the above hypotheses our result reads as follows.

THEOREM. *For each* $x_0 \in E_0$ *with sufficiently small norm there exists a bounded solution of the differential equation*

$$y'(t) = A(t)y(t) + f(t, y(t))$$

on J *such that* $Py(0) = x_0$.

Proof. The result can be proved by the fixed point theorem given in [6] as Theorem 2.

According to Theorem 4.1 of [4] there is a constant $M > 0$ such that every bounded solution of $y' = A(t)y$ satisfies the estimate $\|y(t)\| \leq M\|y(0)\|$ for $t \in J$. Pick $r > C\|m\|_{B(\mathbf{R})}$ and assume that $x_0 \in E_0$ with $\|x_0\| \leq M^{-1}\left(r - C\|m\|_{B(\mathbf{R})}\right)$.

Denote by K the set of all $y \in C(J, E)$ such that $\|y(t)\| \leq r$ on J , and

$$\|y(t_1) - y(t_2)\| \leq r\left|\int_{t_1}^{t_2} \|A(s)\| ds\right| + \left|\int_{t_1}^{t_2} m(s) ds\right|$$

for t_1, t_2 in J . Define a mapping T as follows:

$$(Ty)(t) = U(t)x_0 + \int_J G(t, s)(Fy)(s)ds$$

for $y \in K$, where $(Fy)(t) = f(t, y(t))$.

Let $y_0 \in K$. By the Hölder inequality

$$\begin{aligned} \|(Ty_0)(t)\| &\leq M\|U(0)x_0\| + \int_J \|G(t, s)\| m(s) ds \\ &\leq M\|x_0\| + C\|m\|_{B(\mathbf{R})} \leq r \end{aligned}$$

on J . Since Ty_0 is a solution of the equation $y' = A(t)y + Fy_0$, we have

$$\|(Ty_0)(t_1) - (Ty_0)(t_2)\| \leq \left|\int_{t_1}^{t_2} \|A(s)(Ty_0)(s) + (Fy_0)(s)\| ds\right|$$

for $t_1, t_2 \in J$. Thus $Ty_0 \in K$. Evidently,

$$\|(Tu)(t)-(Tv)(t)\| \le C\|Fu-Fv\|_{B(E)} \quad \text{for} \quad u, v \in K .$$

Now, from this and from 2°, (3) and (4), we conclude that T is continuous as a map of K into itself.

Let us put $\Phi(Y) = \sup\{\alpha(Y(t)) : t \in J\}$ for any nonempty subset Y of K ; here $Y(t)$ stands for the set of all $y(t)$ with $y \in Y$. By the corresponding properties of α , $\Phi(Y_1) \le \Phi(Y_2)$ whenever $Y_1 \subset Y_2$, $\Phi(Y \cup \{y\}) = \Phi(Y)$ for $y \in K$, and $\Phi(\overline{\text{conv}}\, Y) = \Phi(Y)$. If $\Phi(Y) = 0$ then $\overline{Y(t)}$ is compact for every $t \in J$; therefore Ascoli's theorem implies that $\overline{Y}$ is compact in $C(J, E)$.

Assume that Y is a nonempty subset of K with $\Phi(Y) > 0$. We shall prove that $\Phi(T[Y]) < \Phi(Y)$.

Let $t \in J$ be fixed. Let $\varepsilon > 0$ be arbitrary. Since $\lim_{n\to\infty} \|\chi_{[t,\infty)}m\|_{B(\mathbb{R})} = 0$, so $C\|\chi_{[a,\infty)}m\|_{B(\mathbb{R})} < \varepsilon$ for some $a \ge t$. Further, let $\delta = \delta(\varepsilon) > 0$ be a number such that $\int_A \|G(t, s)\| m(s)ds < \varepsilon$ for each measurable $A \subset [0, a]$ with $\text{mes}(A) < \delta$. By the Łuzin theorem there exists a closed subset Z_1 of $[0, a]$ with $\text{mes}([0, a]\backslash Z_1) < \delta/2$ and the function g is continuous on Z_1 .

Let $X_0 = \bigcup\{Y(s) : 0 \le s \le a\}$. It follows from 3° that there exists a closed subset Z_2 of $[0, a]$ such that $\text{mes}([0, a]\backslash Z_2) < \delta/2$ and $\alpha(f[I \times X_0]) \le \sup\{g(s) : s \in I\} \cdot h(\alpha(x_0))$ for each closed subset I of Z_2 .

Define $A = ([0, a]\backslash Z_1) \cup ([0, a]\backslash Z_2)$ and $Z = [0, a]\backslash A$. For any given $\varepsilon' > 0$ there exists a $\eta > 0$ such that if $|s'-s''| < \eta$ with $s', s'' \in [0, t] \cap Z$ or $s', s'' \in [t, a] \cap Z$, then $\|G(t, s')-G(t, s'')\| < \varepsilon'$ and $|g(s')-g(s'')| < \varepsilon'$. Now we devide the interval $[0, a]$ into $2n$ parts:

$$t_0 = 0 < t_1 < \ldots < t_n = t < \ldots < t_{2n} = a$$

with $t_i - t_{i-1} < \eta$. Denote by I_i $(i = 1, 2, \ldots, 2n)$ the set $[t_{i-1}, t_i]\setminus A$. Moreover, let

$$c_1 = \sup\{\|G(t, s)\| : s \in Z\} , \quad c_2 = \sup\{g(s) : s \in Z\} ,$$

and let p_i, r_i be points in I_i such that

$$\|G(t, p_i)\| = \sup\{\|G(t, s)\| : s \in I_i\}$$

and

$$g(r_i) = \sup\{g(s) : s \in I_i\} .$$

It is not hard to see that if H is a continuous mapping from a compact subinterval I to $L(E)$ and W is a bounded subset of E , then $\alpha(\bigcup\{H(s)W : s \in I\}) \leq \sup\{\|H(s)\| : s \in I\} \cdot \alpha(W)$. Hence

$$\alpha\big(\bigcup\{G(t, s)f[I_i \times X_0] : s \in I_i\}\big) \leq \|G(t, p_i)\| g(r_i) \cdot h(\alpha(X_0))$$

for $i = 1, 2, \ldots, 2n$.

Applying the integral mean value theorem, we get

$$\begin{aligned}
\alpha\left(\left\{\int_Z G(t, s)(Fy)(s)ds : y \in Y\right\}\right)
&\leq \alpha\left(\sum_{i=1}^{2n} \operatorname{mes}(I_i)\overline{\operatorname{conv}}\big(\bigcup\{G(t, s)f[I_i \times X_0] : s \in I_i\}\big)\right) \\
&\leq h(\alpha(X_0)) \cdot \sum_{i=1}^{2n} \|G(t, p_i)\| g(r_i) \operatorname{mes}(I_i) \\
&\leq h(\alpha(X_0)) \cdot \sum_{i=1}^{2n} \int_{I_i} \big(\|G(t, p_i)-G(t, s)\| g(r_i) \\
&\qquad + \|G(t, s)\| |g(r_i)-g(s)| + \|G(t, s)\| g(s)\big) ds \\
&\leq h(\alpha(X_0)) \cdot \left[a(c_1+c_2)\varepsilon' + \int_Z \|G(t, s)\| g(s) ds\right] .
\end{aligned}$$

Since Y is almost equicontinuous and bounded, we can apply Lemma 2.2 of [1] to get

$$\alpha(X_0) = \sup\{\alpha(Y(s)) : 0 \leq s \leq a\} \leq \Phi(Y) .$$

Consequently

$$\begin{aligned}
&\alpha\big(T[Y](t)\big)\\
&\le 2\int_A \|G(t,s)\|m(s)ds + h\big(\alpha(S_0)\big)\int_Z \|G(t,s)\|g(s)ds + 2\int_a^\infty \|G(t,s)\|m(s)ds\\
&< 2\varepsilon + h\big(\Phi(Y)\big)\int_Z \|G(t,s)\|g(s)ds + 2C\|\chi_{[a,\infty)}m\|_{B(\mathbb{R})}\\
&< 4\varepsilon + h\big(\Phi(Y)\big)\int_Z \|G(t,s)\|g(s)ds\ .
\end{aligned}$$

This proves

$$\alpha\big(T[Y](t)\big) \le h\big(\Phi(Y)\big)\cdot\sup\left\{\int_J \|G(t,s)\|g(s)ds \ :\ t\in J\right\}$$

for each $t \in J$, and our claim is proved.

The set K is closed and convex subset of $C(J, E)$. Thus all assumptions of our fixed point theorem are satisfied; T has a fixed point in K which ends the proof.

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