THE SHAPE OF DRUMLINS

By Richard J. Chorley

(School of Geography, Oxford University)

Abstract. The lemniscate loop, previously employed in the approximate generic description of the shapes of erosional drainage basins, is shown to bear a close genetic relationship to streamlined forms. This is illustrated by comparison with the shape of airfoils, the snowdrift and certain eggs. Drumlins have been recognized as similar streamlined forms and the lemniscate loop is suggested as providing a quantitative genetic description of their shapes.

Zusammenfassung. Es wird gezeigt, dass die typische Lemniskat Kurve die bereits früher in der generischen Beschreibung von erosionsartigen Abzugsbeckenformen angewandt wurde, in naher genetischer Beziehung zu Stromlinienformen steht. Dies wird durch Vergleich mit der Form von „airfoils", Schneeverwehungen und gewissen Eiern erläutert. Drumlins sind als ähnliche Stromlinienformen erkannt worden, und es wird darauf hingewiesen, dass die Lemniskat Kurve eine quantitative genetische Erläuterung für die Formen der Drumlins gibt.

Of all the topographic forms associated with the action of land ice, none consistently possesses greater geometrical regularity or symmetry than the drumlin. Alden [1] stated: "We may regard as a typical drumlin a hill of glacial drift which approximates the form of a segment of an elongated ovoid, of which the widest part of the basal outline and the highest point of the crest are not more distant from the stoss end than one-third the length of the major axis, and whose major axis is oriented parallel to the direction of movement of the glacier which formed it." Within drumlin swarms the recurrence of similar forms has led to qualitative attempts to describe what is considered to be the characteristic shape. Alden employed the term "half-torpedo",[2] Flint likened them to "the inverted bowl of a spoon",[3] whereas other authors have consistently compared the shape of "characteristic" drumlins to that of eggs. This regularity of form associated with the drumlin is now so generally recognized that the term "drumlin-shaped" is used without embarrassment, sometimes to describe the shape of a feature which has no connexion with glacial action.

The basal outline of drumlins, as well as their significantly similar half outline in side elevation, has naturally been associated with the moulding of the till into a streamlined equilibrium form by the continuous medium of the ice moving around and over it. Flint [4] pointed out that: "The presence of such forms establishes the existence of an actively flowing glacier at the time of formation", and Charlesworth [5] that: "Asymmetrical longitudinal profiles are likewise consistent with ice-moulding. The iceward end is generally broader, blunter, steeper and higher, the distal end, narrower and more tapering . . . This asymmetry harmonizes with physical law. Drumlins, as experimental evidence confirms, are streamlined; they present their steeper face to the moving medium, in order to offer the minimum resistance to the flow by hindering the formation of vortices in the rear (which act as a drag on the moving body)." Charlesworth concluded that: "Drumlins are an expression of the equilibrium between the erosive action of ice and the opposing forces of the solidity and cohesion of the material."[6] With this recognition of form regularity and, particularly, with its association with the mechanics of genesis, it is strange that no serious attempt has yet been made to produce a quantitative standard, having a genetic basis, whereby drumlin shape may be expressed. Such expression would be valuable not only on the purely descriptive level but, through its association with the mechanics of formation, would throw significant light upon the local conditions of drumlin origin—particularly in respect of relative velocities of basal ice flow and of till resistance to ice moulding.

A streamlined form has been defined by Von Mises [7] as one wherein the flow past it is free (or practically free) of discontinuity surfaces separating the dead-flow region from the stream. The same author [8] has given examples of such airfoil forms in a standard wing profile without camber and the U.S. Navy No. 1 Strut (Fig. 1, a and b). Two points of especial interest immediately present themselves from a study of airfoils; firstly, the maximum wing width is located at a point 3/10ths of the length from the leading edge and, secondly, that the forms adapted for greater air speed around them are more elongate. Alden [9] pointed out

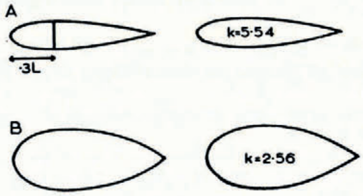

Fig. 1. (a) Standard aircraft wing profile, without camber. (b) Section of a U.S. Navy No. 1 Strut (after Von Mises). Approximating lemniscate loops, together with the values of k, appear throughout

that: "The drumlins generally increase in length toward the axis of the movement of the (ice) lobe", and Charlesworth [10] that: "The broader and shorter drumlins, often found in rising ground, were probably fashioned in more sluggish ice, the longer and more tapering ones in more active flow." Although the most obvious application of the streamlined form in glaciology is to the plan of a snowdrift built up around an obstruction (Fig. 2),[11] it is clear

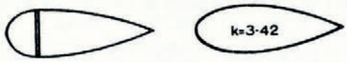

Fig. 2. Plan view of a snowdrift caused by an obstruction (after Seligman)

that airfoils and similar equilibrium forms can be genetically applied as a standard to the quantitative description of drumlins.

To be both significant and of practical value, such a family of idealized geometrical forms must be as closely as possible a description of the most regular forms found in nature, as well as providing a very close approximation to the ideal genetic form with which the natural form is being compared (in this instance, the airfoil), if not indeed an exact expression of this form. This family of descriptive shapes should be also closely related to each other and should be mathematically simple enough for ease of practical application. A family of lemniscate loops which would seem to be well suited to the above conditions governing the description of drumlins has already been applied to the description of drainage basin shapes.[12] Such an application of the lemniscate loop to drainage basins was made in a purely generic sense, in preference to the circularity ratio hitherto employed in their description, and mention of these forms of fluvial erosion is not meant to imply any relationship between the shapes of drumlins and drainage basins from the point of view of their mechanics, but is made solely as a link with the extended mathematical treatment of the properties of lemniscate loops in a previous paper. Horton has calculated the average shape of six of the great drainage basins of the world [13] (Fig. 3a) and the ideal drainage basin shape assumed by the cutting of approximately parabolic slopes by sheetwash into an inclined surface [14, 15] (Fig. 3b). Whereas the fitted lemniscate loops show only a generic approximation to both the actual and theoretical drainage basin shapes, indicating that they may be used merely as a useful quantitative measure of the basin shapes found in nature, the lemniscate loops fitted to the airfoil and snowdrift forms in Figs. 1 and 2 stress the genetic nature of their descriptive application here. It can be shown that similar genetic application of the lemniscate loop to drumlin shapes is possible.

The equation of the lemniscate loop employed is the simple polar-coordinate form:

$$\rho = l \cos k\theta \tag{1}$$

where l is the length of the long axis (*i.e.* the value of ρ when $\theta = 0$) and k is a dimensionless number expressing the elongation of the lemniscate loop, such that when k equals unity

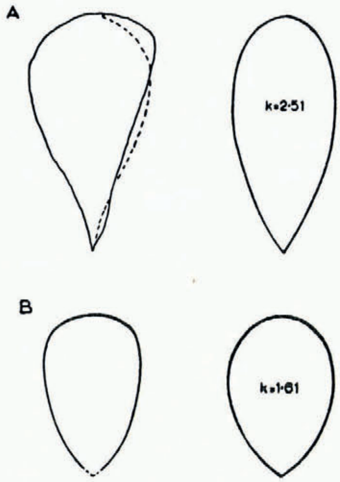

Fig. 3. (a) Average shape of six major drainage basins (Yukon, Irrawaddy, Indus, Mackenzie, Tigris and Euphrates, Ganges). The dashed line is the mirror image of the more regular half of the basin outline. (b) Ideal drainage basin outline (after Horton)

the form is circular and that k increases with the elongation of the lemniscate loop. The value of k is given by the equation

$$k = \frac{l^2\pi}{4A} \tag{2}$$

where A is the area of the loop.

To obtain the best-fit lemniscate loop for approximating an actual shape, one has only to measure the actual values of l and A, substitute them in equation (2) to obtain the value of k, and then to substitute both l and k in equation (1).

It is not claimed that this particular group of curves is the best-fit for the ideal airfoil, but it is apparent from Fig. 1 that it combines the qualities of goodness of fit and simplicity for a well-knit family of forms regulated by the value of k. Thus the shape of a streamlined form may be accurately expressed in terms of the value of k for the fitted lemniscate loop.

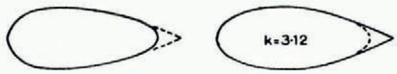

Fig. 4. U.S. "C" Class Airship (after Von Mises). The dashed portion completes the downstream tapering

The contoured form of drumlins in map view does not always show a complete tapering of the down-ice trailing edge. This is often due to problems of mapping, as Alden [16] showed by superimposing the surveyed outlines of drumlin bases on contour maps. Mechanically, however, it is of interest that streamlined forms around which flow is relatively slow in comparison with their magnitude often lack this extreme tapering. This is illustrated in Fig. 4 by the U.S. "C" Class airship,[17] and a suitable lemniscate loop can be fitted after completing the truncated tapering of the downstream end.

Another observed feature of some drumlins is that they tend to be elliptical (oval) and symmetrical about two axes at right angles, rather than streamlined and symmetrical only about the longitudinal axis. This two-fold symmetry resulting in elliptical forms seems most

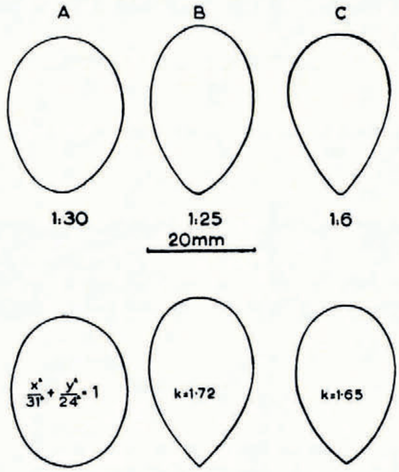

Fig. 5. The shapes of the eggs of the (a) Common bittern (Botaurus stellaris); (b) Great crested grebe (Podicipes cristatus); (c) Golden plover (Charadrius pluvialis) (after Romanoff and Romanoff). The mean ratios of the egg and bird weights are given. An ellipse and two lemniscate loops are fitted

common, although not exclusively confined, to drumlins of smaller size (Fig. 6). A mechanical analogue for this may be found perhaps in another streamlined form of somewhat similar origin—the egg. Thompson [18] has pointed out that: " . . . the case of the egg is somewhat akin to a hydrodynamical problem; for as it lies in the oviduct we may look upon it as a stationary body round which waves are flowing, with the same result as when a body moves through a fluid at rest. Thus we may treat it as a hydrodynamical problem, but a very simple one—simplified by the absence of all eddies and every form of turbulence; and we come to look on the egg as a streamlined structure, though its streamlines are of a very simple kind." The analogy with the moulding of till into drumlins does not seem too inappropriate because of the non-turbulent nature of the moulding media and the observation that, for example, hen's eggs are always laid with the blunt end foremost [19]—in other words, the blunt end simulates the stoss end facing the direction of greatest stress. The egg is thus also an equilibrium form, to which the hard shell is added at a comparatively late stage when the contents of the egg, surrounded by a membrane, have assumed a shape best adjusted to the forced passage of the egg through the oviduct. It has been determined further that the shape of an egg, as regards whether it is elliptical (oval) or streamlined (tapering or lemniscate), is most commonly determined by the relation of the size of the egg to the size of the parent bird—*i.e.* by the stress placed upon it during laying. Romanoff and Romanoff [20] have stated: "The egg would tend to be a sphere if it were not subjected to external forces while still in a plastic condition", and Thompson [21] that: "It can be shewn . . . that those eggs which are most unsymmetrical, or most tapered off posteriorly, are also eggs of a large size relatively to the parent bird . . . (they) are those which are subject to the greatest pressure while being forced along." This size ratio, although the chief, is not the only factor which influences egg shape, and, for example, influences of adaptive evolution have been suggested as significant controls in some instances. However, the relationships between egg shape and the size of the bird which are shown in Fig. 5 [22] are for birds having a somewhat similar marsh habitat, and in this instance it seems less likely that differential adaptive evolutionary trends would be present strongly enough to mask the more common relationship between egg shape and the ratio of size between egg and bird. It would seem that, by analogy therefore, elliptical drumlins might occur where some factor, other than the mere stress of the

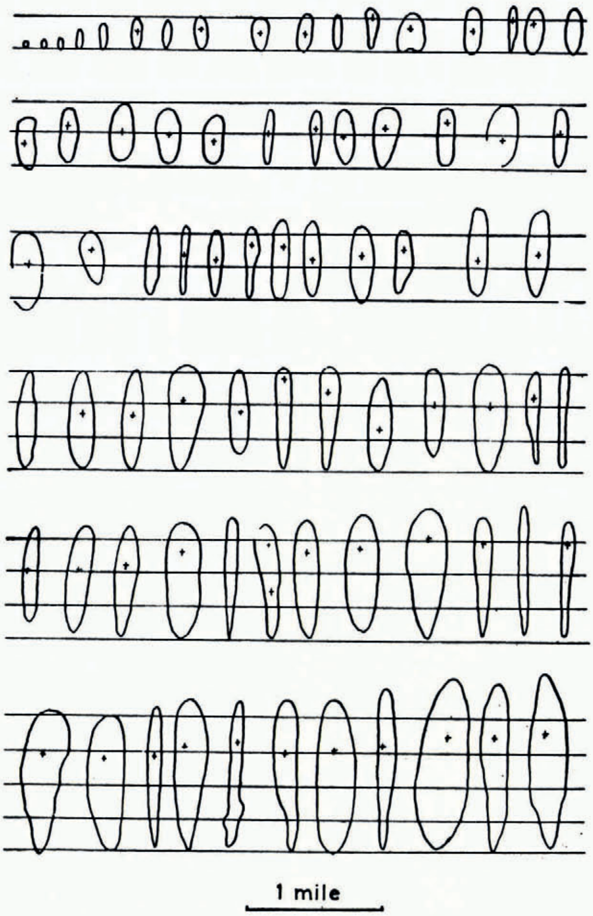

Fig. 6. Outlines of drumlin bases of the Green Bay Glacier, Wisconsin (after Alden)

moving ice, limits the amount of till being deposited and that they are not true limiting equilibrium forms. This would explain why smaller drumlins often tend to exhibit the highest degree of two-fold symmetry about a centrally-located high point. Returning to Fig. 5, it seems apparent that eggs laid under greatest stress show the greatest difference between the rounding of each end, and, consequently, can be approximated by lemniscate loops of lower values of k. It would thus appear that the factor k gives an inverse measure of the relative resistance presented by the equilibrium form (whether egg or drumlin), either because of the strength of the material itself or because of the low stress of the moulding medium, expressed in terms of the velocity of flow.

Fig. 6 shows the basal outlines and high points (marked with crosses) of a selection of drumlins representative of those formed by the Green Bay Glacier in Wisconsin,[23] and in Fig. 7 appropriate lemniscate loops have been fitted to the twenty-three largest ones making up the bottom two rows in Fig. 6. It would seem that, besides providing a simple and genetically defensible standard for the quantitative estimation of drumlin shape, comparison with

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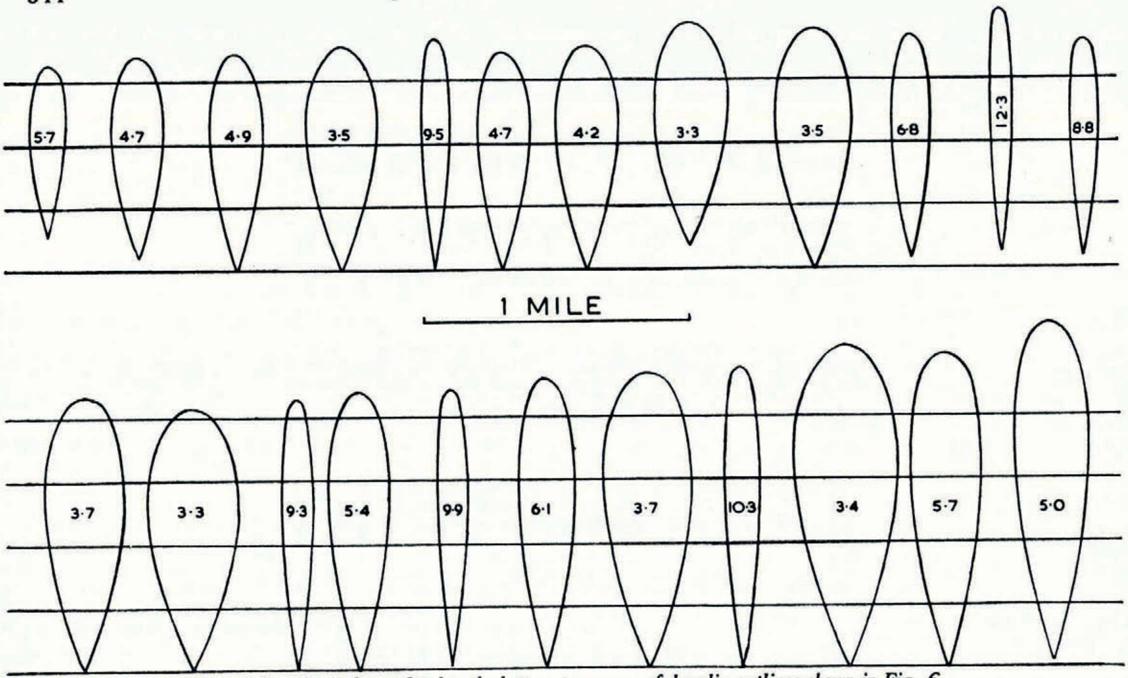

Fig. 7. Lemniscate loops fitted to the bottom two rows of drumlin outlines shown in Fig. 6

the lemniscate form might be used to investigate the mechanics of drumlin formation—particularly with respect to the three factors controlling drumlin shape: their size, the resistance of the till and the velocity of the moulding ice.

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REFERENCES

1. Alden, W. C. The drumlins of southeastern Wisconsin. *U.S. Geological Survey. Bulletin* 273, 1905, p. 18.
2. ——— op. cit., p. 22.
3. Flint, R. F. *Glacial and Pleistocene geology.* New York, John Wiley, 1957, p. 66.
4. ——— op. cit., p. 72.
5. Charlesworth, J. K. *The Quaternary era with special reference to its glaciation. Vol. I.* London, Edward Arnold, 1957, p. 394-95.
6. ——— op. cit., p. 399.
7. Von Mises, R. *Theory of flight.* New York, McGraw-Hill, 1945, p. 102.
8. ——— op. cit., p. 117, 102.
9. Alden, W. C. op. cit., p. 42.
10. Charlesworth, J. K. op. cit., p. 394.
11. Seligman, G. *Snow structure and ski fields.* London, Macmillan, 1936, p. 213.
12. Chorley, R. J., Malm, D. E. G., *and* Pogorzelski, H. A. A new standard for estimating drainage basin shape. *American Journal of Science,* Vol. 255, 1957, p. 138-41.
13. Horton, R. E. Sheet erosion—past and present. *Transactions of the American Geophysical Union,* 1941, Pt. 2, p. 303.
14. ——— op. cit., p. 304.
15. ——— Erosional development of streams and their drainage basins. *Bulletin of the Geological Society of America,* Vol. 56, 1945, p. 365.
16. Alden, W. C. op. cit., p. 20.
17. Von Mises, R. op. cit., p. 102.
18. Thompson, D'Arcy W. *On growth and form. New edition.* Cambridge, University Press, 1942, p. 941.
19. ——— op. cit., p. 938.
20. Romanoff, A. L., *and* Romanoff, A. J. *The avian egg.* New York, John Wiley, 1949, p. 88.
21. Thompson, D'Arcy W. op. cit., p. 938-39.
22. Romanoff, A. L., *and* Romanoff, A. J. op. cit., p. 89.
23. Alden, W. C. op. cit., p. 20.