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ORTHOGONALLY ADDITIVE HOLOMORPHIC FUNCTIONS OF BOUNDED TYPE OVER $C(K)$

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Abstract It is known that all k-homogeneous orthogonally additive polynomials P over $C(K)$ are of the form

$$P(x) = \int_K x^k \,\mathrm{d}\mu.$$

Thus, $x \mapsto x^k$ factors all orthogonally additive polynomials through some linear form μ. We show that no such linearization is possible without homogeneity. However, we also show that every orthogonally additive holomorphic function of bounded type f over $C(K)$ is of the form

$$f(x) = \int_K h(x) \,\mathrm{d}\mu$$

for some μ and holomorphic $h\colon C(K) \to L^1(\mu)$ of bounded type.

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1. Introduction

A holomorphic function $f\colon C(K) \to F$ is said to be orthogonally additive if $f(x+y) = f(x)+f(y)$ whenever x and y are mutually orthogonal (i.e. $xy = 0$ over K). In this paper we study orthogonal additivity of complex-valued bounded-type holomorphic functions over $C(K)$.

Recall [**7**] that $f\colon E \to F$ is of bounded type if it is bounded on all bounded subsets of E. The Taylor series $f = \sum_k P_k$ of such a function have infinite radii of uniform convergence, i.e. $\limsup \|P_k\|^{1/k} = 0$. We will denote the space of all such functions over E by $\mathcal{H}_{\mathrm{b}}(E, F)$, or simply $\mathcal{H}_{\mathrm{b}}(E)$ if $F = \mathbb{C}$.

It was proved in [**2, 8**] (see also [**3**]) that each orthogonally additive k-homogeneous polynomial P over $C(K)$ is represented by a regular Borel measure μ on K in the following sense:

$$P(x) = \int_K x^k \, \mathrm{d}\mu \quad \text{for all } x \in C(K).$$

This is a linearization result; the 'universal' polynomial $h(x) = x^k$ linearizes all orthogonally additive k-homogeneous polynomials P:

$$\begin{tikzcd} C(K) \arrow[r, "P"] \arrow[d, "h"'] & \mathbb{C} \\ C(K) \arrow[ur, "\mu"'] & \end{tikzcd}$$

The following lemma shows that f is orthogonally additive if and only if all polynomials in its Taylor-series expansion at zero are orthogonally additive. Note that this is not the case for expansions around other points.

Lemma 1.1. *Let $f\colon C(K) \to \mathbb{C}$ be holomorphic and let $f = \sum_{k=0}^{\infty} P_k$ be its Taylor series at zero. Then f is orthogonally additive if and only if all the P_k are orthogonally additive.*

Proof. Say x and y are orthogonal. If f is orthogonally additive, we may use the Cauchy integral formula [**7**, Equation (3.11)] to write

$$\begin{aligned} P_k(x+y) &= \frac{1}{2\pi \mathrm{i}} \int_{|\lambda|=1} \frac{f(\lambda(x+y))}{\lambda^{k+1}} \, \mathrm{d}\lambda \\ &= \frac{1}{2\pi \mathrm{i}} \int_{|\lambda|=1} \frac{f(\lambda x)}{\lambda^{k+1}} + \frac{f(\lambda y)}{\lambda^{k+1}} \, \mathrm{d}\lambda \\ &= P_k(x) + P_k(y). \end{aligned}$$

For the converse, if all P_k are orthogonally additive,

$$\begin{aligned} f(x+y) &= \sum_{k=0}^{\infty} P_k(x+y) \\ &= \sum_{k=0}^{\infty} (P_k(x) + P_k(y)) \\ &= \sum_{k=0}^{\infty} P_k(x) + \sum_{k=0}^{\infty} P_k(y) \\ &= f(x) + f(y). \end{aligned}$$

□

Given Lemma 1.1, one might expect that in the holomorphic setting one will have a linearizing factorization, where the polynomial $h(x) = x^k$ is replaced by $h(x) = \Phi \circ x$, with $\Phi\colon \mathbb{C} \to \mathbb{C}$ holomorphic and $\Phi(0) = 0$ (for example, $h(x) = \mathrm{e}^x - 1$). The first objection to such a factorization is that it would produce a linearization of the space of orthogonally additive holomorphic functions through $C(K)$, and thus a Banach predual of the space; but spaces of holomorphic functions are seldom Banach. A second objection to this line of thought is that, for any fixed $\Phi(z) = \sum_{k=1}^{\infty} a_k z^k$, the factorization can only take place for functions f with limited growth: indeed, if

$$f(x) = \int_K \Phi \circ x \,\mathrm{d}\mu = \int_K \sum_{k=1}^{\infty} a_k x^k \,\mathrm{d}\mu = \sum_{k=1}^{\infty} a_k \int_K x^k \,\mathrm{d}\mu,$$

then

$$|f(x)| \leqslant \sum_{k=1}^{\infty} |a_k|\, |\mu|(K) \|x\|^k = |\mu|(K) \sum_{k=1}^{\infty} |a_k|\, \|x\|^k,$$

so the growth of f is bounded by that of Φ.

We will show (Example 2.2) that there are stronger, algebraic, obstructions and that such a factorization cannot be obtained even for non-homogeneous polynomials of degree 2.

In § 2 we study the relationship between orthogonal additivity and integrality, and prove that all orthogonally additive functions of bounded type are integral. We also show an orthogonally additive polynomial of degree 2 that does not factor through $C(K)$.

In § 3 we characterize orthogonally additive holomorphic functions of bounded type as those which may be written as

$$f(x) = \int_K h(x) \,\mathrm{d}\mu \quad \text{for all } x \text{ in } C(K),$$

for some measure μ on K and $h \colon C(K) \to L^1(\mu)$ holomorphic and of bounded type.

2. Orthogonal additivity and integrality

Recall [**6**] that a k-homogeneous polynomial P over a Banach space E is said to be integral if it can be represented by a Borel measure ν over the unit ball $B_{E'}$ of the dual space (with the weak* topology) in the following way:

$$P(x) = \int_{B_{E'}} \hat{x}^k \,\mathrm{d}\nu \quad \text{for all } x \in E$$

(where $\hat{x}(\gamma) = \gamma(x)$). Recall also [**5**] that an integral holomorphic function $f \colon B_E^\circ \to \mathbb{C}$ on the open unit ball of E is one which may be written as

$$f(x) = \int_{B_{E'}} \frac{1}{1 - \hat{x}} \,\mathrm{d}\nu \quad \text{for all } x \in B_E^\circ$$

and for some Borel measure ν on $B_{E'}$.

Note that when $E = C(K)$ the dual E' is the space of regular Borel measures on K, and that K may be identified with part of the unit ball $B_{E'}$: the point measures δ_a, $a \in K$.

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We begin with some observations on the case of orthogonally additive k-homogeneous polynomials. Consider the closed subspace X_k of $C(B_{E'})$ spanned by $\{\hat{x}^k : x \in E\}$. The space X_k is then the symmetric ϵ-tensor product

$$\widehat{\bigotimes}_{\epsilon,k,s} E$$

and integral k-homogeneous polynomials are those which linearize through X_k [**7**]:

$$\begin{tikzcd} x \arrow[d, mapsto] & E \arrow[r, "P"] \arrow[d] & \mathbb{C} \\ \hat{x}^k & X_k \arrow[ur, "\nu"] & \end{tikzcd}$$

Now if $E = C(K)$, we have the linear map $R_k : X_k \to C(K)$ given by $R_k(\hat{x}^k) = x^k$. It is easy to see that, since $K \subset B_{E'}$, this map is well defined and has norm 1. The diagram

$$\begin{tikzcd} x \arrow[d, mapsto] & C(K) \arrow[r, "P"] \arrow[d] & \mathbb{C} \\ \hat{x}^k & X_k \arrow[ur, "\nu"] \arrow[d, "R_k"'] & \\ & C(K) \arrow[uur, "\mu"'] & \end{tikzcd}$$

then shows (applying [**2**, **8**]) that all orthogonally additive k-homogeneous polynomials are integral. Indeed, if $P(x) = \mu(x^k)$, it also equals $\nu(\hat{x}^k)$. This shows the well-known fact that the 'integrating' polynomial

$$P(x) = \int_K x(t)^k \, \mathrm{d}\mu(t) \quad \text{on } C(K)$$

is integral.

It is also true that orthogonally additive holomorphic functions of bounded type are integral. For the proof, we will need the following. Recall [**1**] that any $f \in \mathcal{H}_{\mathrm{b}}(E)$ has a canonical extension to the bidual E''. This extension, $\bar{f}$, called the Aron–Berner extension, is also holomorphic and of bounded type.

Given a Borel subset A of K, and $f : C(K) \to \mathbb{C}$ a bounded-type holomorphic function, we define

$$f_A(x) = \bar{f}(1_A x),$$

where $\bar{f}$ is the Aron–Berner extension of f.

Proposition 2.1. *If $f \in \mathcal{H}_{\mathrm{b}}(C(K))$ is orthogonally additive, then it is integral.*

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Proof. Let $f = \sum_{k=1}^{\infty} P_k$ be the Taylor-series expansion of f at zero and note that each P_k is orthogonally additive and hence integral. By [**5**] we need only verify that

$$\sum_{k=1}^{\infty} \|P_k\|_I < \infty.$$

Since f is of bounded type, we have $\limsup \|P_k\|^{1/k} = 0$, so $\sum_{k=1}^{\infty} \|P_k\| < \infty$. But if P is an orthogonally additive k-homogeneous polynomial, then its natural and integral norms are equal: in general, one has $\|P\| \leqslant \|P\|_I$; to see $\|P\|_I \leqslant \|P\|$, given $\varepsilon > 0$, we find a measure ν on $B_{E'}$ representing P and such that $|\nu| < \|P\| + \varepsilon$. Since P is orthogonally additive, consider a measure μ on K such that

$$P(x) = \int_K x^k \, \mathrm{d}\mu,$$

and take $\nu = \mu \circ R_k$:

$$\begin{tikzcd}
x \arrow[d, mapsto] & C(K) \arrow[r, "P"] \arrow[d] & \mathbb{C} \\
\hat{x}^k & X_k \arrow[ur, "\nu"] \arrow[d, "R_k"'] & \\
 & C(K) \arrow[uur, "\mu"'] &
\end{tikzcd}$$

Since $\|R_k\| = 1$,

$$|\nu| = \|\mu \circ R_k\| \leqslant |\mu| \, \|R_k\| = |\mu| < \sum_{i=1}^{n} |\mu(A_i)| + \varepsilon$$

for some sequence (A_i) of disjoint closed subsets of K. Hence,

$$\begin{aligned}
|\nu| &< \sum_{i=1}^{n} \left| \int_K 1_{A_i} \, \mathrm{d}\mu \right| + \varepsilon \\
&= \sum_{i=1}^{n} |P_{A_i}(1)| + \varepsilon \\
&\leqslant \sum_{i=1}^{n} \|P_{A_i}\| + \varepsilon \\
&\leqslant \|P\| + \varepsilon;
\end{aligned}$$

this last inequality is obtained by [**3**, Lemma 1.2]. □

Note that in the proof we have shown that the natural and integral norms coincide on the space of orthogonally additive k-homogenous polynomials on $C(K)$. We have also shown that any orthogonally additive function in $\mathcal{H}_{\mathrm{b}}(C(K))$ is, in fact, in $\mathcal{H}_{\mathrm{b}I}(C(K))$, which means that its restriction to $nB_{C(K)}$ belongs to $\mathcal{H}_I(nB_{C(K)})$ for all n [**5**].

The converse of Proposition 2.1 does not hold even for homogeneous polynomials: the condition for an integral polynomial P to be orthogonally additive is that it factor through R_k, i.e.

$$\sum_i a_i x_i^k = 0 \quad \text{on } K \quad \Longrightarrow \quad \sum_i a_i P(x_i) = 0.$$

Now consider the analogous situation for an integral holomorphic map $f\colon B^{\circ}_{C(K)} \to \mathbb{C}$. Take X, the closed subspace of $C(B_{C(K)'})$ spanned by $\{\hat{x}/(1-\hat{x})\colon \|x\| < 1\}$, and $R\colon X \to C(K)$ given by

$$R\left(\frac{\hat{x}}{1-\hat{x}}\right) = \frac{x}{1-x}.$$

Then R is again well defined and continuous, and by composition with R all maps of the form

$$f(x) = \int_K \frac{x}{1-x}\, \mathrm{d}\mu$$

are integral (and $f(0) = 0$). However, not all orthogonally additive bounded-type holomorphic functions are of this form. To obtain such a representation, one would have to construct a measure over K, and here the obstruction is algebraic: the spaces $R_k(X_k)$ are not 'independent' in $C(K)$. The following example shows that for non-homogeneous polynomials there is no such representation with any holomorphic $\Phi\colon \mathbb{C} \to \mathbb{C}$.

Example 2.2. Take K to be the closed unit disc in $\mathbb{C}$. There is no entire function $\Phi\colon \mathbb{C} \to \mathbb{C}$ such that $h(x) = \Phi \circ x$ factors all degree-2 orthogonally additive polynomials over $C(K)$.

Proof. Suppose there were such a function Φ, i.e. given an orthogonally additive polynomial P, there is a measure μ such that

$$P(x) = \int_K \Phi \circ x \, \mathrm{d}\mu \quad \text{for } x \in C(K).$$

Write the Taylor series of Φ at 0,

$$\Phi(z) = \sum_{k=0}^{\infty} c_k z^k,$$

and choose a and b in K such that $c_2 a^2 \neq c_1 b^2$. Define $P\colon C(K) \to \mathbb{C}$ by $P(x) = x(a) + x(b)^2$. Clearly, P is orthogonally additive, so let μ be as above. Now, for any λ,

$$x(a)\lambda + x(b)^2\lambda^2 = P(\lambda x) = \int_K \Phi \circ \lambda x \, \mathrm{d}\mu = \sum_{k=0}^{\infty} c_k \int_K x^k \, \mathrm{d}\mu \lambda^k$$

and thus

$$x(a) = c_1 \int_K x \, \mathrm{d}\mu \quad \text{and} \quad x(b)^2 = c_2 \int_K x^2 \, \mathrm{d}\mu \quad \text{for } x \in C(K).$$

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However, if we take $x(z) = (c_2/c_1)z^2$ and $y(z) = z$ (and thus $c_1 x = c_2 y^2$), then

$$\frac{c_2}{c_1}a^2 = x(a) = c_1 \int_K x \, \mathrm{d}\mu = c_2 \int_K y^2 \, \mathrm{d}\mu = y(b)^2 = b^2,$$

which contradicts our choice of a and b. $\square$

3. Characterization of orthogonally additive holomorphic functions of bounded type

For any Borel measure μ on K, we will write $f \ll \mu$, if $\mu(A) = 0$ implies $f_A = 0$. Also, we will say that $h \colon C(K) \to L^1(\mu)$ is a power-series function if

$$h(x) = \sum_{k=1}^{\infty} g_k x^k,$$

with g_ks in $L^1(\mu)$. Note that this is a very special type of holomorphic function as its definition uses the algebra structure of $C(K)$. We then have the following.

Theorem 3.1. *Given $f \in \mathcal{H}_{\mathrm{b}}(C(K))$, orthogonally additive, $f \ll \mu$ if and only if there exists a power-series function $h \in \mathcal{H}_{\mathrm{b}}(C(K), L^1(\mu))$ such that*

$$f(x) = \int_K h(x)(t) \, \mathrm{d}\mu(t) \quad \text{for all } x \in C(K).$$

Proof. Consider the Taylor series of f about 0: $f = \sum_k P_k$. Note that $\bar{f} = \sum_k \bar{P}_k$ in $C(K)''$. Thus, for any Borel set A,

$$f_A(x) = \bar{f}(1_A x) = \sum_{k=0}^{\infty} \bar{P}_k(1_A x) = \sum_{k=0}^{\infty} P_{kA}(x).$$

Since $\|P_{kA}\| \leqslant \|P_k\|$, by [**3**, Lemma 1.2] $f_A \in \mathcal{H}_{\mathrm{b}}(C(K))$.

($\Longrightarrow$) Now suppose that $f \ll \mu$. Since $f_A = 0$ if and only if $P_{kA} = 0$ for all k, we have $P_k \ll \mu$ for all k. Also, since f is orthogonally additive, P_k is also orthogonally additive for all k. Therefore, by [**2**, **8**], there is a measure μ_k such that

$$P_k(x) = \int_K x^k \, \mathrm{d}\mu_k,$$

but, since $P_k \ll \mu$, we have $\mu_k \ll \mu$; indeed, if $\mu(A) = 0$,

$$\mu_k(A) = \int_K 1_A \, \mathrm{d}\mu_k = \bar{P}_k(1_A) = P_{kA}(1) = 0$$

(the second equality is obtained by [**3**, Corollary 2.1]). Now, by the Radon–Nikodým Theorem there is a $g_k \in L^1(\mu)$ such that

$$P_k(x) = \int_K x^k g_k \, \mathrm{d}\mu.$$

We prove that $\|g_k\|_{L^1(\mu)} \leqslant \|P_k\|$ for each k: choose a representative g_k and consider the function

$$t \mapsto \begin{cases} \left(\dfrac{\bar{g}_k(t)}{|g_k(t)|}\right)^{1/k} & \text{if } g_k(t) \neq 0, \\ 0 & \text{if } g_k(t) = 0, \end{cases}$$

where in taking the kth root we have chosen any branch. This function, though not continuous, is Borel measurable and has its image in the closed unit disc. It is an element of $C(K)''$. Using [**3**, Corollary 2.1] again we have

$$\begin{aligned} \|g_k\|_{L^1(\mu)} &= \int_K |g_k| \,\mathrm{d}\mu = \int_K \left[\left(\frac{\bar{g}_k(t)}{|g_k(t)|}\right)^{1/k}\right]^k g_k \,\mathrm{d}\mu \\ &= \bar{P}_k\left(\left(\frac{\bar{g}_k(t)}{|g_k(t)|}\right)^{1/k}\right) \leqslant \|\bar{P}_k\| = \|P_k\|; \end{aligned}$$

this last equality is obtained by [**4**]. Now define $h\colon C(K) \to L^1(\mu)$ by

$$h(x) = \sum_{k=1}^{\infty} g_k x^k.$$

This series converges absolutely:

$$\begin{aligned} \left\|\sum_{k=0}^{\infty} g_k x^k\right\|_{L^1(\mu)} &\leqslant \sum_{k=0}^{\infty} \|g_k x^k\|_{L^1(\mu)} \\ &= \sum_{k=0}^{\infty} \int_K |g_k|\,|x|^k \,\mathrm{d}\mu \\ &\leqslant \sum_{k=0}^{\infty} \|g_k\|_{L^1(\mu)} \|x\|_{\infty}^k \\ &\leqslant \sum_{k=0}^{\infty} \|P_k\| \|x\|^k < \infty \quad \text{for all } x. \end{aligned}$$

Thus, h is a power-series function and is clearly bounded on bounded subsets of $C(K)$. So $h \in \mathcal{H}_{\mathrm{b}}(C(K), L^1(\mu))$, and

$$f(x) = \sum_{k=0}^{\infty} P_k(x) = \sum_{k=0}^{\infty} \int_K g_k x^k \,\mathrm{d}\mu = \int_K \sum_{k=0}^{\infty} g_k x^k \,\mathrm{d}\mu = \int_K h(x) \,\mathrm{d}\mu.$$

($\Longleftarrow$) If $\mu(A) = 0$, then

$$f_A(x) = \bar{f}(1_A x) = \int_K \bar{h}(1_A x) \,\mathrm{d}\mu = \int_K h(x) 1_A \,\mathrm{d}\mu = \int_A h(x) \,\mathrm{d}\mu = 0,$$

so $f \ll \mu$. The second equality holds because h is a power-series function of bounded type. Indeed,

$$f = \sum_k P_k = \int_K h \,\mathrm{d}\mu.$$

Since h is a power series,

$$P_k(x) = \int_K g_k x^k \, \mathrm{d}\mu.$$

The proof of [**3**, Corollary 2.1] then shows that

$$\bar{P}_k(1_A x) = \int_A g_k x^k \, \mathrm{d}\mu,$$

and the result follows. □

Proposition 3.2. *If f is orthogonally additive, then, for some Borel measure μ, $f \ll \mu$.*

Proof. Say $f = \sum_k P_k$ is the Taylor-series expansion of f at zero. Since f is orthogonally additive, all the P_k are orthogonally additive. Thus, there are Borel measures μ_k for which

$$P_k(x) = \int_K x^k \, \mathrm{d}\mu_k.$$

Now,

$$P_{kA}(x) = \bar{P}_k(1_A x) = \int_A x^k \, \mathrm{d}\mu_k,$$

so $P_{kA}(1) = \mu_k(A)$ and $|\mu_k| \leqslant \|P_k\|$. Therefore, $(|\mu_k|)$ is summable, and we can define $\mu = \sum_k |\mu_k|$. For any A, $\mu(A) = \sum_k |\mu_k|(A)$, so $\mu_k \ll \mu$ for all k. This implies that $P_k \ll \mu$ for all k, and hence $f \ll \mu$. □

We have the following characterization of orthogonally additive holomorphic functions of bounded type.

Theorem 3.3. *$f \in \mathcal{H}_\mathrm{b}(C(K))$ is orthogonally additive if and only if there is a Borel measure μ and a power-series function $h \in \mathcal{H}_\mathrm{b}(C(K), L^1(\mu))$ such that*

$$f(x) = \int_K h(x)(t) \, \mathrm{d}\mu(t) \quad \textit{for all } x.$$

Proof. ($\Longrightarrow$) This is immediate from the preceding results.

($\Longleftarrow$) Note that all power-series functions are orthogonally additive, for if x and y are orthogonal, $(x+y)^k = x^k + y^k$ for all $k \geqslant 1$, so

$$h(x+y) = \sum_{k=1}^{\infty} g_k (x+y)^k = \sum_{k=1}^{\infty} g_k (x^k + y^k) = \sum_{k=1}^{\infty} g_k x^k + \sum_{k=1}^{\infty} g_k y^k = h(x) + h(y).$$

Now if

$$f(x) = \int_K h(x) \, \mathrm{d}\mu,$$

orthogonal additivity of f follows from linearity of the integral. Also, if $\|x\| \leqslant c$, then

$$|f(x)| \leqslant \sum_{k=1}^{\infty} \|g_k\|_{L^1(\mu)} c^k,$$

so f is bounded on bounded sets. □

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