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A NOTE ON THE PRODUCT OF $\mathscr{F}$-SUBGROUPS IN A FINITE GROUP

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Saturated formations are closed under the product of subgroups which are connected by certain permutability properties.

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All groups we consider are finite. It is well known that the product of supersolvable normal subgroups is not supersolvable in general (see Huppert [3]).

In [1], Asaad and Shaalan proved the following result:

Let $G=G_1G_2$ be a group such that G_1 and G_2 are supersolvable subgroups. If every subgroup of G_1 is permutable with every subgroup of G_2, then G is supersolvable.

If G_1 and G_2 are subgroups of a group G such that every subgroup of G_1 is permutable with every subgroup of G_2, we say that G_1 and G_2 are *totally permutable.*

In [6], Maier proved that Asaad and Shaalan's result is a special case of a general completeness property of all saturated formations which contain the class of supersolvable groups. In [6], the following theorem is proved:

Let $G=G_1G_2$ be a group such that G_1 and G_2 are totally permutable subgroups. Let $\mathscr{F}$ be a saturated formation which contains the class of supersolvable groups. If G_1 and G_2 lie in $\mathscr{F}$, then so does G.

In this paper we give a generalization for an arbitrary number of factors of Maier's result. We prove:

Theorem 1. *Let $G=G_1G_2\ldots G_r$ be a group such that $G_1,G_2,\ldots,G_r$ are pairwise totally permutable subgroups of G. Let $\mathscr{F}$ be a saturated formation which contains the class of supersolvable group. If for all $i\in\{1,2,\ldots,r\}$ the subgroups G_i are in $\mathscr{F}$, then $G\in\mathscr{F}$.*

If G_1 and G_2 are totally permutable subgroups of a group G, then $\langle x,y\rangle=\langle x\rangle\langle y\rangle=\langle y\rangle\langle x\rangle$ is a supersolvable subgroup, for each $x\in G_1$ and $y\in G_2$, by ([4, p. 722, Th. 10.1]). If G_1 and G_2 are subgroups of a group G and $\mathscr{L}$ is a non-empty class of groups,

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we say that G_1 and G_2 are *$\mathcal{L}$-connected*, whenever for each $x \in G_1$ and $y \in G_2$ we have $\langle x, y\rangle \in \mathcal{L}$.

Assuming this definition, we prove the following:

Theorem 2. *Let $G = G_1G_2, \ldots G_r$ be a group such that $G_1, G_2, \ldots, G_r$ are pairwise permutable subgroups of G. Let $\mathcal{L} = \mathcal{N}$ be the class of nilpotent groups and let $\mathcal{F}$ be a saturated formation such that $\mathcal{N} \subseteq \mathcal{F}$. If for every pair $i,j \in \{1,2,\ldots,r\}$, $i \neq j$, the subgroups G_i and G_j are $\mathcal{N}$-connected $\mathcal{F}$-groups, then $G \in \mathcal{F}$.*

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Proofs of our Theorems

To prove theorem 1, we first generalize Lemma 2 of [6]:

Lemma 1. *Let the group $G = G_1G_2 \ldots G_r$ be the product of the pairwise totally permutable subgroups $G_1, G_2, \ldots, G_r$ of G.*

(a) *If $|G| > 1$, then there exists $i \in \{1,2,\ldots,r\}$ such that G_i contains a nonidentity normal subgroup of G.*

(b) *For every pair $i,j \in \{1,2,\ldots,r\}$, $i \neq j$, we have that $G_i \cap G_j \leqq \mathbf{F}(G_iG_j)$, where $\mathbf{F}(G_iG_j)$ denotes the Fitting subgroup of G_iG_j.*

Proof. (a) Let p denote the largest prime divisor of $|G|$. Certainly p divides at least one of $|G_1|, |G_2|, \ldots, |G_r|$. Let x be a p-element of the union set $G_1 \cup G_2 \cup \ldots \cup G_r$ of maximal order and suppose $x \in G_1$. Let R be the subgroup of order p in $\langle x\rangle$. As in the proof of Lemma 2 in [6], we conclude that G_i normalizes R for all $i \in \{2,\ldots,r\}$. Therefore the normal closure

$$R^G = R^{G_2G_3\ldots G_rG_1} = R^{G_1} \leqq G_1$$

is a nonidentity normal subgroup of G contained in G_1.

(b) This is (b) of Lemma 2 in [6].

Theorem 1. *Let $G = G_1G_2 \ldots G_r$ be a group such that $G_1, G_2, \ldots, G_r$ are pairwise totally permutable subgroups of G. Let $\mathcal{F}$ be a saturated formation which contains the class of supersolvable groups. If for all $i \in \{1,2,\ldots,r\}$ the subgroups G_i are in $\mathcal{F}$, then $G \in \mathcal{F}$.*

Proof. Suppose the theorem is false and let G be a counterexample of smallest order with r least possible. Then $1 < G_i < G$ for all $i \in \{1,2,\ldots,r\}$. We will prove a series of items under this assumption. They will lead us to a contradiction. Certainly the hypothesis is inherited by factor groups.

(i) *The group G has a unique minimal normal subgroup N and the Frattini subgroup $\Phi(G)=1$.*

Since the hypothesis is inherited by quotient groups, by the minimality of $|G|$, every proper quotient group G/M ($1\neq M\trianglelefteq G$) lies in $\mathscr{F}$. Since $\mathscr{F}$ is a saturated formation, the minimal normal subgroup N of G is unique and $\Phi(G)=1$.

(ii) *There exists $i\in\{1,2,\dots,r\}$, such that $N\leqq G_i$.*

This is Lemma 1 (a).

We suppose $N\leqq G_1$.

(iii) *N is an elementary abelian p-group, for some prime number p.*

Otherwise N is the direct product of nonabelian simple groups. Because of the uniqueness of N we have $\mathbf{C}_G(N)=1$, the centralizer of N in G. Put $H=G_1G_2$. By Lemma 1 (b), we have that $G_1\cap G_2\leqq\mathbf{F}(H)$. So, $N\cap G_2=N\cap G_1\cap G_2\leqq N\cap\mathbf{F}(H)\leqq\mathbf{F}(N)=1$.

Let $X\leqq N$. Since G_1 and G_2 are totally permutable, we have that $G_2X=XG_2$. So $X=X(N\cap G_2)=N\cap XG_2\trianglelefteq G_2X$. Hence, G_2 normalizes every subgroup of N. By ([**2**], Th. 2.2.1.]), the commutator group $[G_2,N]$ is in the centre of N. Therefore, $G_2\leqq\mathbf{C}_G(N)=1$, a contradiction.

(iv) *There is a complement V of N in G, $|N|>p$ and $\mathbf{C}_G(N)=N=\mathbf{F}(G)$.*

This is shown in the same way as the items (iv)/(v) in [**6**].

It is clear that, if $N\leqq G_i$, then $U_i=G_i\cap V$ is a complement of N in G_i.

(v) *There exists $i\in\{2,3,\dots,r\}$, such that $N\nleqq G_i$.*

Suppose $N\leqq\bigcap_{i=1}^r G_i$. Then $U_i=G_i\cap V$ is a complement of N in G_i. Let $X\leqq N$. Since $r>1$, we have $XU_i=U_iX$ and $X=X(U_i\cap N)=N\cap U_iX\trianglelefteq XU_i$. Since N is abelian, we have $X\trianglelefteq G_i$ for all $i\in\{1,2,\dots,r\}$. Therefore, by the minimality of N, we conclude $|N|=p$, against (iv).

We renumber the indices in such a way that $N\leqq G_i$ for all $i\in\{1,2,\dots,s\}$ and $N\nleqq G_j$ for all $j\in\{s+1,\dots,r\}$. Let $K=G_{s+1}G_{s+2}\dots G_r$.

(vi) *For all $j\in\{s+1,\dots,r\}$ we have $N\cap G_j=1$.*

Let $j\in\{s+1,\dots,r\}$ and $D=N\cap G_j$. *Suppose* $D>1$. Let $i\in\{1,2,\dots,s\}$. We have that $N\leqq G_i$ and $U_i=V\cap G_i$ is a complement of N in G_i. Since $D\leqq G_j$ and $i\neq j$, we have $DU_i=U_iD$. Hence $D\trianglelefteq U_iD$. Therefore $D\trianglelefteq G_i$ for all $i\in\{1,2,\dots,s\}$. It follows that $N=D^G=D^{G_{s+1}\dots G_r}\leqq G_{s+1}\dots G_r=K$.

Since K is the product of pairwise totally permutable subgroups, once more by Lemma 1 (a) there exists $1\neq L\trianglelefteq K$ such that, for example $L\leqq G_{s+1}$.

Consider $J=N\cap L\trianglelefteq K$. Since $J\leqq G_{s+1}$ we have that the subgroups G_i normalize J, for all $i\in\{1,2,\dots,s\}$. Hence $J\trianglelefteq G$. By the minimality of N we conclude $J=N$ or $J=1$. Since $N\nleqq G_{s+1}$, we have $J=1$. Therefore, the commutator $[N,L]\leqq L\cap N=J=1$. So $L\leqq\mathbf{C}_G(N)=N$. Since $L\leqq G_{s+1}$, we have that G_i normalizes L, for all $i\in\{1,2,\dots,s\}$. Hence, $L=N\leqq G_{s+1}$, a contradiction.

(vii) *For all $j\in\{s+1,\dots,r\}$, we have that G_j normalizes every subgroup of N.*

If $X\leqq N$, then $XG_j=G_jX$ and $X=X(G_j\cap N)=G_jX\cap N\trianglelefteq G_jX$ by (vi).

(viii) *We have $s=1$.*

Suppose $N \leqq G_i$ and $i > 1$. Let $X \leqq N$ and $U_1 = G_1 \cap V$. We have that $U_1 X = X U_1$, (because $i > 1$). So, G_1 normalizes X. Similarly G_i normalizes X, for all G_i such that $N \leqq G_i$. Hence $X \trianglelefteq G$, by (vii). So, $|N| = p$, against (ii).

By (vii) and (viii) we have that N is a minimal normal subgroup of G_1.

(ix) *For all $j \in \{2, \ldots, r\}$ we have $G_1 \cap G_j = 1$.*

Let $D = G_1 \cap G_j$. By Lemma 2 (b), we have $D \leqq \mathbf{F}(G_1 G_j)$. Since N is minimal normal in G_1, ([**4**], p. 277, Th. 4.2 (e)), we have $\mathbf{F}(G_1) \leqq \mathbf{C}_G(N) = N$. It follows that $D \leqq N \cap G_j = 1$, by (vi).

By (vi) and (vii), we have that for all $j \in \{2, \ldots, r\}$ the subgroups G_j are faithfully represented on the vector space N by scalar transformations. Let U_1^x be a conjugate of $U_1 = G_1 \cap V$ in G_1.

(x) *For all $j \in \{2, \ldots, r\}$ we have that G_j centralizes U_1^x.*

Clearly, $G_j U_1^x = U_1^x G_j$ and $U_1^x G_j \cap N = U_1^x G_j \cap G_1 \cap N = U_1^x (G_j \cap G_1) \cap N = U_1^x \cap N = 1$. So, $U_1^x G_j$ is faithfully represented on N. Since G_j is represented by scalar transformations, we have that G_j centralizes U_1^x.

(xi) *The contradiction*

Clearly, $N \neq G_1$, so U_1 is a non-normal subgroup of G_1. It follows that $G_1 = \langle U_1^x / x \in G_1 \rangle$. Hence, for all $j \in \{2, \ldots, r\}$, G_j centralizes G_1, by (x). Therefore $G_j \leqq \mathbf{C}_G(N) = N$ and $G_j = 1$, by (vi).

Theorem 2. *Let $G = G_1 G_2 \ldots G_r$ be a group such that $G_1, G_2, \ldots, G_r$ are pairwise permutable subgroups of G. Let $\mathscr{L} = \mathscr{N}$ be the class of nilpotent groups and let $\mathscr{F}$ be a saturated formation such that $\mathscr{N} \subseteq \mathscr{F}$. If for every pair $i, j \in \{1, 2, \ldots, r\}$, $i \neq j$, the subgroups G_i and G_j are $\mathscr{N}$-connected $\mathscr{F}$-groups, then $G \in \mathscr{F}$.*

Proof. Suppose the theorem is false and let G be a counterexample of smallest order. Since the hypothesis is inherited by quotients, we conclude that G has a unique minimal normal subgroup N. Since $\mathscr{F}$ is saturated, we have $\Phi(G) = 1$.

Let p be a prime number and $i, j \in \{1, 2, \ldots, r\}$, such that $i \neq j$. Let $x \in G_i$ be a p-element and $y \in G_j$ a p'-element. Since $\langle x, y \rangle$ is nilpotent, we have that y centralizes x. Let $P_i \in Syl_p(G_i)$. Since $\mathbf{O}^p(G_j)$ is generated by all p'-elements of G_j, we have $\mathbf{O}^p(G_j) \leqq \mathbf{C}_G(P_i)$. For the definition of $\mathbf{O}^p(G_j)$ see ([7, p. 142]).

Set $G_j^* = \bigcap_p \mathbf{O}^p(G_j)$. The above consideration implies that $G_i \leqq \mathbf{C}_G(G_j^*)$. Since our argument is true for all $i \in \{1, 2, \ldots r\}$, such that $i \neq j$, we have that $G_j^* \trianglelefteq G$.

(I) Suppose $G_j^* \neq 1$, for some $j \in \{1, 2, \ldots, r\}$.

Because of the uniqueness of N we have $N \leqq G_j^*$.

(a) If N is solvable, then $N = \mathbf{C}_G(N)$ and $G_i \leqq N \leqq G_j^*$, for all $i \in \{1, 2, \ldots, r\}$, with $i \neq j$. It follows that $G = G_j \in \mathscr{F}$.

(b) If N is not solvable, then $\mathbf{C}_G(N) = G_i = 1$ for all $i \in \{1, 2, \ldots, r\}$ with $i \neq j$. Again we have $G = G_j \in \mathscr{F}$.

(II) Suppose $G_j^* = 1$ for all $j \in \{1, 2, \ldots, r\}$. Now G_j is nilpotent for all $j \in \{1, 2, \ldots, r\}$. Hence, $G_j = P_j \times \mathbf{O}^p(G_j)$, for every prime number p.

Let $i, j \in \{1, 2, \ldots, r\}$ such that $i \neq j$. By ([**4**, p. 676, Th. 4.7]) we have that $P_i P_j \in Syl_p(G_i G_j)$. Hence $P_1 P_2 \ldots P_r \in Syl_p(G)$. Since for all $i \in \{1, 2, \ldots, r\}$ we have

$\mathbf{O}^p(G_i) \leqq \mathbf{C}_G(P_1P_2 \dots P_r)$, we conclude that $G_i \leqq \mathbf{N}_G(P_1P_2 \dots P_r)$ and therefore $P_1P_2 \dots P_r \trianglelefteq G$. It follows that $G \in \mathscr{N} \subseteq \mathscr{F}$.

The following example shows that Theorem 2 is not true when $\mathscr{N} \subsetneqq \mathscr{L} \subseteq \mathscr{F}$, without additional hypothesis (see also the Example given in [**6**]):

Example. Let $G = S_4$ be the symmetric group of degree 4. Let G_1 be the normal subgroup of order 4 of G and let G_2 be a subgroup of order 6 of G. Clearly, $G = G_1G_2$. Let $\mathscr{L} = \mathscr{N}\mathscr{A} = \mathscr{F}$ be the class of all groups whose commutator subgroups are nilpotent. Clearly, G_1 and G_2 are $\mathscr{N}\mathscr{A}$-connected $\mathscr{F}$-groups, but $G \notin \mathscr{F}$.

In view of the fact that the finite simple groups are 2-generated, the following seems to be reasonable:

Conjecture. *Let $\mathscr{S}$ be the class of solvable groups. If the group $G = G_1G_2 \dots G_r$ is the product of the pairwise permutable and pairwise $\mathscr{S}$-connected $\mathscr{S}$-subgroups G_i, then G is solvable.*

To mention the solution of a particular case of this conjecture, we introduce the following notation: Let $\mathscr{T}$ be the class of groups having Sylow-tower for the prime numbers arranged in decreasing order.

Proposition. *Let $G = G_1G_2 \dots G_r$ be a group such that $G_1, G_2, \dots G_r$ are pairwise permutable and pairwise $\mathscr{T}$-connected supersolvable subgroups of G. Then G is a $\mathscr{T}$-group. In particular, G is solvable.*

Proof. Suppose the proposition is false. Let G be a counterexample of smallest order with r least possible. Every quotient group of G satisfies the hypothesis of the proposition. Because of the minimality of $|G|$, every proper quotient group is a $\mathscr{T}$-group.

Let p denote the largest prime number divisor of $|G|$. We may assume that p divides $|G_1|$. We have to produce a nonidentity normal p-subgroup N of G.

Because of the supersolvability of G_1, we can choose $\langle x \rangle$ a normal subgroup of G_1, with $|\langle x \rangle| = p$. We show $\langle x \rangle$ is subnormal in G. Then $N = \langle x \rangle^G$ is a normal p-subgroup of G.

First we show that $r \leqq 2$. If $r \geqq 3$, then $H = G_1G_2 \dots G_{r-1}$ and $K = G_1G_2 \dots G_{r-2}G_r$ are $\mathscr{T}$-groups, since r is least possible. Hence $\langle x \rangle$ is subnormal in H and K. By ([**5**, p. 239, Th. 7.7.1]) we have that $\langle x \rangle$ is subnormal in $HK = G$. So $G = G_1G_2$.

Let $g \in G$. Write $g = g_1g_2$ with $g_1 \in G_1$ and $g_2 \in G_2$. Since $\langle x \rangle \trianglelefteq G_1$, we have that $x^{g_1} = x^i$ with $1 \leqq i \leqq p$. By hypothesis $\langle x, g_2 \rangle$ is a $\mathscr{T}$-group, thus $\langle x, g_2 \rangle_p \trianglelefteq \langle x, g_2 \rangle$, where $\langle x, g_2 \rangle_p$ denotes the Sylow-p-subgroup of $\langle x, g_2 \rangle$. Therefore $x, x^{g_2} \in \langle x, g_2 \rangle_p$ and $x^g = x^{g_1g_2} = (x^i)^{g_2} = (x^{g_2})^i \in \langle x, g_2 \rangle_p$. It follows that $\langle x, x^g \rangle$ is a p-group, for all $g \in G$. By ([**7**, p. 195, Th. 4.8]) we have that $\langle x \rangle$ is subnormal in G.

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