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LETTER TO THE EDITOR

Dear Editor,

An intuitive insight into a result of Cowan

Motivated by a problem in DNA replication, Cowan (2001) considered a sequence of water springs distributed along a straight road out of town according to a Poisson process of rate λ. An infinite team of workers leaves the town at constant speed r; on reaching any spring, one worker peels off and builds a pipe back towards town at rate c. The first worker stops when he reaches town, later ones stop when they reach the previous spring.

Given $t > 0$, let N_t be the number of workers building at time t, and let $g_j(t) = \mathrm{P}(N_t = j)$, $g_j = \lim_{t\to\infty} g_j(t)$, and $\phi(s) = \sum_{j\geq 0} g_j s^j$, the corresponding probability generating function of the asymptotic number actually building. Cowan showed that $\phi(s) = \prod_{n\geq 1}(1 + (s-1)b^n)$, where $b = r/(c+r)$, and noted that, if $\{I_n\}$ are independent Bernoulli(b^n) variables, then ϕ is the probability generating function of $\sum_{n\geq 1} I_n$. We give here an intuitive explanation of this elegant representation.

Suppose that $\{X_i : i = 1, 2, \ldots\}$ are independent Exp(λ) variables, i.e. exponential with mean $1/\lambda$, representing the distance between successive springs, and let $S_n = \sum_{i=1}^{n} X_i$, so that $S_1 < S_2 < \cdots$ denote the positions of the springs. Let $A_t = \max\{n : S_n < rt\}$ be the label of the last spring found before time t. When $A_t \geq 1$, define $Y_0(t) = rt - S_{A_t}$ and write $Y_n(t) = X_{A_t+1-n}$ for each $n = 1, 2, \ldots, A_t$. Then $N_t = \sum_{n\geq 1} I_n(t)$, where

$$I_n(t) = \begin{cases} 1 & \text{if } Y_n(t) > \dfrac{c(Y_0(t) + Y_1(t) + \cdots + Y_{n-1}(t))}{r}, \\ 0 & \text{otherwise.} \end{cases}$$

We seek the asymptotic distribution of N_t. Note that the variates $\{Y_i(t) : i \geq 0\}$ are not independent and have a complicated joint distribution. As $t \to \infty$, however, it is intuitively clear that these variates are asymptotically independent and Exp(λ) distributed. Formal passage to the limit of $I_n(t)$ is difficult but, by using the intuitively clear asymptotics for the Y variates, we can describe the limiting *properties* of $\{I_n(t)\}$: let $Y_0, Y_1, \ldots$ be independent Exp(λ) variables, and write $N = \sum_{n\geq 1} I_n$, where

$$I_n = \begin{cases} 1 & \text{if } Y_n > \dfrac{c(Y_0 + Y_1 + \cdots + Y_{n-1})}{r}, \\ 0 & \text{otherwise.} \end{cases}$$

It is straightforward to show that $\mathrm{P}(I_n = 1) = b^n$, as Y_n is independent of $Y_0 + Y_1 + \cdots + Y_{n-1}$, which has a gamma density. For independence, it is sufficient to show that $\mathrm{P}(I_1 = 1, I_2 = 1, \ldots, I_n = 1) = \mathrm{P}(I_1 = 1)\,\mathrm{P}(I_2 = 1) \cdots \mathrm{P}(I_n = 1)$ for all $n \geq 1$. We do this explicitly for the case $n = 3$: the general case follows by the same method. Plainly, we may take $\lambda = 1$. Now, $\mathrm{P}(I_1 = 1, I_2 = 1, I_3 = 1)$ is the same as

$$\mathrm{P}\biggl(Y_1 > \frac{cY_0}{r}, Y_2 > \frac{c(Y_0 + Y_1)}{r}, Y_3 > \frac{c(Y_0 + Y_1 + Y_2)}{r}\biggr),$$

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which can be written as an integral and then evaluated as

$$\begin{aligned}
&\int_0^\infty \mathrm{e}^{-y_0} \int_{cy_0/r}^\infty \mathrm{e}^{-y_1} \int_{c(y_0+y_1)/r}^\infty \mathrm{e}^{-y_2} \int_{c(y_0+y_1+y_2)/r}^\infty \mathrm{e}^{-y_3}\,\mathrm{d}y_3\,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}y_0 \\
&\quad = \int_0^\infty \mathrm{e}^{-y_0/b} \int_{cy_0/r}^\infty \mathrm{e}^{-y_1/b} \int_{c(y_0+y_1)/r}^\infty \mathrm{e}^{-y_2/b}\,\mathrm{d}y_2\,\mathrm{d}y_1\,\mathrm{d}y_0 \\
&\quad = b \int_0^\infty \mathrm{e}^{-y_0/b^2} \int_{cy_0/r}^\infty \mathrm{e}^{-y_1/b^2}\,\mathrm{d}y_1\,\mathrm{d}y_0 \\
&\quad = b \cdot b^2 \int_0^\infty \mathrm{e}^{-y_0/b^3}\,\mathrm{d}y_0 \\
&\quad = b \cdot b^2 \cdot b^3 \\
&\quad = \mathrm{P}(I_1 = 1)\,\mathrm{P}(I_2 = 1)\,\mathrm{P}(I_3 = 1).
\end{aligned}$$

Reference

Cowan, R. (2001). A new discrete distribution arising in a model of DNA replication. *J. Appl. Prob.* **38,** 754–760.

Yours sincerely,

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