ON THE NATURAL ORDERING OF $\mathscr{J}$-CLASSES AND OF IDEMPOTENTS IN A REGULAR SEMIGROUP

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1. Introduction and summary. In this paper we prove the following:

MAIN THEOREM. *Let S be a regular semigroup and a, b any elements of S such that $J_b \leqq J_a$. Then, for each idempotent $e \in J_a$, there exists an idempotent $f \in J_b$ such that $f \leqq e$.*

This makes easy a conceptual proof of the result (see Theorem 6.39 [**1**], Theorem 3 [**2**] or Theorem 1 [**4**]) that a semigroup is primitive and regular if and only if it is a 0-direct union of completely 0-simple semigroups.

The above theorem can also be used in an obvious manner to simplify the statements of the results of Section 3, Chapter II of Lallement [**3**], in particular Theorem 2.17 [**3**].

We use wherever possible notations and conventions from Clifford and Preston [**1**].

2. The natural orderings. We shall prove in fact (see Theorem 1) a stronger result than the one above. The idea for the proof of this theorem came from the proof of the lemma in Warne's paper [**6**]. In fact Warne effectively proves Theorem 1 for the special case of a regular semigroup with only two principal factors, both completely 0-simple.

We prove first the following lemma.

LEMMA 1. *Let a, b be elements of a semigroup S such that $J_b \leqq J_a$ and such that the principal factor $J(b)/I(b)$ is 0-simple or simple. Then $(a\,J_b) \cap J_b \neq \square$.*

Proof. Since $J(b)/I(b)$ is not a null semigroup, there exist elements x, $y \in J_b$ for which $xy \in J_b$. Now $J_x = J_b \leqq J_a$, so there exist elements u, $v \in S^1$ such that $x = uav$; whence $xy = uavy \in J_b$. It follows that vy, $avy \in J_b$ (because $v \in J_b$ implies that avy, $vy \in J(b)$ and avy, $vy \in I(b)$ would imply that $xy = uavy \in I(b)$).

The lemma may also be deduced from example 5, part (a), p. 36, [**1**] volume II.

THEOREM 1. *Let S be a semigroup and a, b any elements of S such that $J_b \leqq J_a$ and such that each element of J_b is regular. Then for each idempotent $e \in J_a$ (if such exists) there is an idempotent $f \in J_b$ such that $f \leqq e$.*

Proof. Take any idempotent $e \in J_a$. From Lemma 1, $(e\,J_b) \cap J_b \neq \square$. Take any element $y \in (e\,J_b) \cap J_b$ and any inverse, y' say, of y. By a routine calculation, we find that $y'e$ is an inverse of y; whence $yy'e(=f$ say) is an idempotent, $f \in J_b$ and $f \leqq e$.

REMARK 1. Proposition 3.1 of [**5**] is similar to Theorem 1. Its applications in common with Theorem 1 would include the two mentioned in paragraphs 3 and 4.

REMARK 2. It is clear that, for any idempotents e, f in a semigroup S, $f \leqq e$ implies that $J_f \leqq J_e$. Theorem 1 and this converse remain true when J is replaced by L or R (only minor modifications of the proof are necessary to show this). It is then easy to show that the set of L- (or R- but not J-) classes of an inverse semigroup, under the natural ordering, is always a semilattice.

COROLLARY 1. (Due to Lallement and Petrich [2] Theorem 3, and Preston [4] Theorem 1. See also [1] Theorem 6.39.) *A semigroup $S = S^0$ is primitive and regular if and only if it is a 0-direct union of completely 0-simple semigroups.*

Proof. Suppose that S is primitive and regular. Let a, b be any non-zero elements of S such that $J_a \neq J_b$.

Now, for any element $x \in I(a)$, we have $J_x < J_a$; whence, from Theorem 1, $0 \in J_x$ and $I(a) = \{0\}$. Therefore $S^1aS^1 = J_a \cup \{0\}$ for each non-zero element $a \in S$.

Also $J_{ab} \leqq J_a$ and $J_{ab} \leqq J_b$; whence either $J_{ab} < J_a$ or $J_{ab} < J_b$. In either case $ab = 0$, and so S is the 0-direct union of the subsemigroups $\{J_x \cup \{0\} : x \in S \backslash \{0\}\}$. But for each $x \in S \backslash \{0\}$, $J_x \cup \{0\}$ is isomorphic to the principal factor $J(x)/I(x)$ and hence is 0-simple and thence completely 0-simple.

Conversely, since a completely 0-simple semigroup is primitive and regular, it is clear that a 0-direct union of completely 0-simple semigroups is also primitive and regular.

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