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Revisiting Tietze–Nakajima: Local and Global Convexity for Maps

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Abstract. A theorem of Tietze and Nakajima, from 1928, asserts that if a subset X of $\mathbb{R}^n$ is closed, connected, and locally convex, then it is convex. We give an analogous "local to global convexity" theorem when the inclusion map of X to $\mathbb{R}^n$ is replaced by a map from a topological space X to $\mathbb{R}^n$ that satisfies certain local properties. Our motivation comes from the Condevaux–Dazord–Molino proof of the Atiyah–Guillemin–Sternberg convexity theorem in symplectic geometry.

1 Introduction

A theorem of Tietze and Nakajima from 1928 asserts that if a subset X of $\mathbb{R}^n$ is closed, connected, and locally convex, then it is convex [Ti, N]. There are many generalizations of this "local to global convexity" phenomenon in the literature; a partial list is [BF, C, Ka, KW, Kl, SSV, S, Ta].

This paper contains an analogous "local to global convexity" theorem when the inclusion map of X to $\mathbb{R}^n$ is replaced by a map from a topological space X to $\mathbb{R}^n$ that satisfies certain local properties. We define a map $\Psi\colon X \to \mathbb{R}^n$ to be convex if any two points in X can be connected by a path γ whose composition with Ψ parametrizes a straight line segment in $\mathbb{R}^n$ and this parametrization is monotone along the segment. See Definition 7. We show that, if X is connected and Hausdorff, Ψ is proper, and each point has a neighbourhood U such that $\Psi|_U$ is convex and open as a map to its image, then Ψ is convex and open as a map to its image. We deduce that the image of Ψ is convex and the level sets of Ψ are connected. See Theorems 15 and 10.

Our motivation comes from the Condevaux–Dazord–Molino proof [CDM, HNP] of the Atiyah–Guillemin–Sternberg convexity theorem in symplectic geometry [At, GS1]. See Section 7. A similar notion of convexity of momentum maps was studied in [Kn].

This paper is the result of an undergraduate research project that spanned the years 2004–2006. The senior author takes the blame for the delay in publication after posting our arXiv eprint. While preparing this paper we learned of the paper [BOR1] by Birtea, Ortega, and Ratiu, which achieves similar goals. In Section 8 we discuss relationships between our results and theirs. After [BOR1], our results are not essentially new, but the notion of "convex map" gives elegant statements, and our proofs are so elementary that they are accessible to undergraduate students with a basic topology background.

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2 The Tietze–Nakajima Theorem

Let $B(x, r)$ $[\overline{B}(x, r)]$ denote the open [closed] ball in $\mathbb{R}^n$ of radius r, centered at x. A closed subset X of $\mathbb{R}^n$ is *locally convex* if for every $x \in X$ there exists $\delta_x > 0$ such that $B(x, \delta_x) \cap X$ is convex. The Tietze–Nakajima theorem [Ti, N] asserts that "local convexity implies global convexity":

1 Theorem (Tietze–Nakajima) *Let X be a closed, connected, and locally convex subset of $\mathbb{R}^n$. Then X is convex.*

2 Example A disjoint union of two closed balls is closed and locally convex but is not connected. A punctured disk is connected and satisfies the locally convexity condition, but it is not closed.

A closed subset $X \subset \mathbb{R}^n$ is *uniformly locally convex* on a subset $A \subset X$ if there exists $\delta > 0$ such that $B(x, \delta) \cap X$ is convex for all $x \in A$.

3 Lemma (Uniform local convexity on compact sets) *Let X be a closed subset of $\mathbb{R}^n$. If X is locally convex and $A \subset X$ is compact, then X is uniformly locally convex on A.*

Proof Since X is locally convex, for every $x \in X$ there exists a $\delta_x > 0$ such that $B(x, \delta_x) \cap X$ is convex. By compactness there exist points $x_1, \dots, x_n$ such that $A \subset \bigcup_{i=1}^n B(x_i, \frac{1}{2}\delta_{x_i})$. Let $\delta = \min\{\frac{1}{2}\delta_{x_i}\}$. Then for every $x \in A$ there exists i such that $B(x, \delta) \subset B(x_i, \delta_{x_i})$. It follows that $B(x, \delta) \cap X$ is convex. ■

4 Definition Let X be a closed, connected, locally convex subset of $\mathbb{R}^n$. For two points x_0 and x_1 in X, their *distance in* X, denoted $d_X(x_0, x_1)$, is

$$d_X(x_0, x_1) = \inf\{l(\gamma) \mid \gamma\colon [0, 1] \to X,\ \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

where $l(\gamma)$ is the length of the path γ.

In this definition it does not matter if we take the infimum over continuous paths or polygonal paths: let $\gamma\colon [0, 1] \to X$ be a continuous path in X. Let δ be the radius associated with uniform local convexity on the compact set $\{\gamma(t), 0 \le t \le 1\}$. By uniform continuity of γ on the compact interval $[0, 1]$, there exist $0 = t_0 < t_1 < \cdots < t_k = 1$ such that $\|\gamma(t_{i-1}) - \gamma(t_i)\| < \delta$ for $i = 1, \dots, k$. The polygonal path through the points $\gamma(t_0), \dots, \gamma(t_k)$ is contained in X and has length $\le l(\gamma)$.

Also note that $d_X(x_0, x_1) \ge \|x_1 - x_0\|$, with equality if and only if the segment $[x_0, x_1]$ is contained in X.

5 Lemma (Existence of a midpoint) *Let X be a closed, connected, and locally convex subset of $\mathbb{R}^n$. Let x_0 and x_1 be in X. Then there exists a point $x_{1/2}$ in X such that*

$$d_X(x_0, x_{1/2}) = d_X(x_{1/2}, x_1) = \tfrac{1}{2} d_X(x_0, x_1). \tag{2.1}$$

Proof Let γ_j be paths in X connecting x_0 and x_1 such that $\{l(\gamma_j)\}$ converges to $d_X(x_0, x_1)$. Let $t_j \in [0, 1]$ be such that $\gamma_j(t_j)$ is the midpoint of the path γ_j:

$$l(\gamma_j|_{[0,t_j]}) = l(\gamma_j|_{[t_j,1]}) = \tfrac{1}{2} l(\gamma_j).$$

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Since the sequence of midpoints $\{\gamma_j(t_j)\}$ is bounded and X is closed, this sequence has an accumulation point $x_{1/2} \in X$. We will show that the point $x_{1/2}$ satisfies equation (2.1).

We first show that for every $\varepsilon > 0$ there exists a path γ connecting x_0 and $x_{\frac{1}{2}}$ such that $l(\gamma) < \frac{1}{2} d_X(x_0, x_1) + \varepsilon$.

Let $\delta > 0$ be such that $B(x_{\frac{1}{2}}, \delta) \cap X$ is convex. Let j be such that $\|\gamma_j(t_j) - x_{\frac{1}{2}}\| < \min(\delta, \frac{\varepsilon}{2})$ and such that $l(\gamma_j) < d_X(x_0, x_1) + \varepsilon$. The segment $[\gamma_j(t_j), x_{\frac{1}{2}}]$ is contained in X. Let γ be the concatenation of $\gamma_j|_{[0,t_j]}$ with this segment. Then γ is a path in X that connects x_0 and $x_{\frac{1}{2}}$, and $l(\gamma) < \frac{1}{2} d_X(x_0, x_1) + \varepsilon$.

Thus, $d_X(x_0, x_{\frac{1}{2}}) \le \frac{1}{2} d_X(x_0, x_1)$. By the same argument, $d_X(x_{\frac{1}{2}}, x_1) \le \frac{1}{2} d_X(x_0, x_1)$. If either of these were a strict inequality, then it would be possible to construct a path in X from x_0 to x_1 whose length is less than $d_X(x_0, x_1)$, which contradicts the definition of $d_X(x_0, x_1)$. ■

Proof of the Tietze-Nakajima theorem Fix x_0 and x_1 in X.

By Lemma 5, there exists a point $x_{1/2}$ such that

$$d_X(x_0, x_{1/2}) = d_X(x_{1/2}, x_1) = \frac{1}{2} d_X(x_0, x_1).$$

Likewise, there exists a point $x_{1/4}$ that satisfies

$$d_X(x_0, x_{1/4}) = d_X(x_{1/4}, x_{1/2}) = \frac{1}{2} d_X(x_0, x_{1/2}).$$

By iteration, we get a map $\frac{j}{2^m} \mapsto x_{\frac{j}{2^m}}$, for nonnegative integers j and m where $0 \le j \le 2^m$, such that

$$d_X(x_{\frac{j-1}{2^m}}, x_{\frac{j}{2^m}}) = d_X(x_{\frac{j}{2^m}}, x_{\frac{j+1}{2^m}}) = \frac{1}{2} d_X(x_{\frac{j-1}{2^m}}, x_{\frac{j+1}{2^m}}). \tag{2.2}$$

Let $r > d_X(x_0, x_1)$. For all $0 \le j \le 2^m$,

$$\|x_{\frac{j}{2^m}} - x_0\| \le d_X(x_{\frac{j}{2^m}}, x_0) \le \sum_{i=1}^{j} d_X(x_{\frac{i-1}{2^m}}, x_{\frac{i}{2^m}}) = \frac{j}{2^m} d_X(x_0, x_1) < r.$$

Thus, $x_{\frac{j}{2^m}}$ belongs to the compact set $\overline{B}(x_0, r) \cap X$. Let δ denote the radius associated with uniform local convexity on this compact set. Choose m large enough such that $\frac{1}{2^m} d_X(x_0, x_1) < \delta$. Since the intersection $B(x_{\frac{j}{2^m}}, \delta) \cap X$ is convex and $x_{\frac{j-1}{2^m}} \in B(x_{\frac{j}{2^m}}, \delta) \cap X$,

$$\left[x_{\frac{j-1}{2^m}}, x_{\frac{j}{2^m}}\right] \subset X \quad \text{for each} \quad 1 \le j \le 2^m. \tag{2.3}$$

Since also $x_{\frac{j+1}{2^m}} \in B(x_{\frac{j}{2^m}}, \delta)$,

$$\left[x_{\frac{j-1}{2^m}}, x_{\frac{j+1}{2^m}}\right] \subset X \quad \text{for each} \quad 1 \le j < 2^m.$$

It follows that

$$d_X(x_{\frac{j-1}{2^m}}, x_{\frac{j}{2^m}}) = \|x_{\frac{j-1}{2^m}} - x_{\frac{j}{2^m}}\| \quad \text{and} \quad d_X(x_{\frac{j-1}{2^m}}, x_{\frac{j+1}{2^m}}) = \|x_{\frac{j-1}{2^m}} - x_{\frac{j+1}{2^m}}\|.$$

Thus, equation (2.2) can be rewritten as

$$\|x_{\frac{j-1}{2^m}} - x_{\frac{j}{2^m}}\| = \|x_{\frac{j}{2^m}} - x_{\frac{j+1}{2^m}}\| = \frac{1}{2}\|x_{\frac{j-1}{2^m}} - x_{\frac{j+1}{2^m}}\|,$$

which implies, by the triangle inequality, that the points $x_{\frac{j-1}{2^m}}, x_{\frac{j}{2^m}}, x_{\frac{j+1}{2^m}}$ are collinear. This and (2.3) imply that $[x_0, x_1] \subset X$. ■

3 Local and Global Convexity of Maps

The Tietze–Nakajima theorem involves subsets of $\mathbb{R}^n$. We will now consider spaces with maps to $\mathbb{R}^n$ that are not necessarily inclusion maps.

Consider a continuous path $\overline{\gamma}\colon [0,1] \to \mathbb{R}^n$. Its length, which is denoted $l(\overline{\gamma})$, is the supremum, over all natural numbers N and all partitions $0 = t_0 < t_1 < \cdots < t_N = 1$, of $\sum_{i=1}^{N} \|\overline{\gamma}(t_i) - \overline{\gamma}(t_{i-1})\|$. We have $l(\overline{\gamma}) \geq \|\overline{\gamma}(1) - \overline{\gamma}(0)\|$ with equality if and only if one of two cases occurs:

(i) the path $\overline{\gamma}$ is constant;

(ii) the image of $\overline{\gamma}$ is the segment $[\overline{\gamma}(0), \overline{\gamma}(1)]$, and $\overline{\gamma}$ is a weakly monotone parametrization of this segment: if $0 \leq t_1 < t_2 < t_3 \leq 1$, then the point $\overline{\gamma}(t_2)$ lies on the segment $[\overline{\gamma}(t_1), \overline{\gamma}(t_3)]$.

6 Definition The path $\overline{\gamma}\colon [0,1] \to \mathbb{R}^n$ is *monotone straight* if it satisfies (i) or (ii).

7 Definition Let X be a Hausdorff topological space. A continuous map Ψ from X to $\mathbb{R}^n$, or to a subset of $\mathbb{R}^n$, is called *convex* if every two points x_0 and x_1 in X can be connected by a continuous path $\gamma\colon [0,1] \to X$ such that

(3.1) $\quad \gamma(0) = x_0, \quad \gamma(1) = x_1, \quad$ and $\quad \Psi \circ \gamma$ is monotone straight.

Warning For a function ψ from $\mathbb{R}$ to $\mathbb{R}$, the condition in Definition 7 is equivalent to $\psi\colon \mathbb{R} \to \mathbb{R}$ being weakly monotone. This is different from the usual notion of a convex function (that $\psi(ta + (1-t)b) \leq t\psi(a) + (1-t)\psi(b)$ for all a, b and for all $0 \leq t \leq 1$). In the usual notion of a convex function, the domain X must be an affine space, and the target space must be $\mathbb{R}$. In Definition 7, the domain X is only a topological space, and the target space can be $\mathbb{R}^n$. In this paper, "convex map" is always in the sense of Definition 7.

8 Remark If $\Psi(x_0) = \Psi(x_1)$, condition (3.1) means that the path γ lies entirely within a level set of Ψ. If $\Psi(x_0) \neq \Psi(x_1)$, the condition implies that the image of $\Psi \circ \gamma$ is the segment $[\Psi(x_0), \Psi(x_1)]$.

9 Example Consider the two-sphere $S^2 = \{x \in \mathbb{R}^3 \mid \|x\|^2 = 1\}$. The height function $\Psi\colon S^2 \to \mathbb{R}$, given by $(x_1, x_2, x_3) \mapsto x_3$, is convex. The projection $\Psi\colon S^2 \to \mathbb{R}^2$, given by $(x_1, x_2, x_3) \mapsto (x_1, x_2)$, is not convex.

We shall prove the following generalization of the Tietze–Nakajima theorem:

10 Theorem *Let X be a connected Hausdorff topological space, let $\mathcal{T} \subset \mathbb{R}^n$ be a convex subset, and let $\Psi\colon X \to \mathcal{T}$ be a continuous and proper map. Suppose that for every point $x \in X$ there exists an open neighbourhood $U \subset X$ of x such that the map $\Psi|_U\colon U \to \Psi(U)$ is convex and open. Then*

(i) *the image of Ψ is convex;*
(ii) *the level sets of Ψ are connected;*
(iii) *the map $\Psi\colon X \to \Psi(X)$ is open.*

11 Remark The Tietze–Nakajima theorem is the special case of Theorem 10 in which $\mathcal{T} = \mathbb{R}^n$, the space X is a subset of $\mathbb{R}^n$, and the map $\Psi\colon X \to \mathbb{R}^n$ is the inclusion map.

The convexity of a map has the following immediate consequences.

12 Lemma *If $\Psi\colon X \to \mathbb{R}^n$ is a convex map, then, for any convex subset $A \subset \mathbb{R}^n$, the restriction of Ψ to the preimage $\Psi^{-1}(A)$ is also a convex map.*

Proof Let $A \subset \mathbb{R}^n$ be convex, and let $x_0, x_1 \in \Psi^{-1}(A)$. Let $\gamma\colon [0,1] \to X$ be a path from x_0 to x_1 whose composition with Ψ is monotone straight. The image of $\Psi \circ \gamma$ is the (possibly degenerate) segment $[\Psi(x_0), \Psi(x_1)]$. Because A is convex and contains the endpoints of this segment, it contains the entire segment, so γ is a path in $\Psi^{-1}(A)$. Thus, x_0 and x_1 are connected by a path in $\Psi^{-1}(A)$ whose composition with Ψ is monotone straight. ■

13 Lemma (Global properties imply convexity) *If $\Psi\colon X \to \mathbb{R}^n$ is a convex map, then its image, $\Psi(X)$, is convex, and its level sets, $\Psi^{-1}(w)$, for $w \in \Psi(X)$, are connected.*

Proof Take any two points in $\Psi(X)$; write them as $\Psi(x_0)$ and $\Psi(x_1)$ where x_0 and x_1 are in X. Because the map Ψ is convex, there exists a path γ in X that connects x_0 and x_1 and such that the image of $\Psi \circ \gamma$ is the segment $[\Psi(x_0), \Psi(x_1)]$. In particular, the segment $[\Psi(x_0), \Psi(x_1)]$ is contained in the image of Ψ. This shows that the image of Ψ is convex.

Now let x_0 and x_1 be any two points in $\Psi^{-1}(w)$. Because the map Ψ is convex, there exists a path γ that connects x_0 and x_1 and such that the curve $\Psi \circ \gamma$ is constant. Thus, this curve is entirely contained in the level set $\Psi^{-1}(w)$. This shows that the level set $\Psi^{-1}(w)$ is connected. ■

14 Remark Suppose that the map $\Psi\colon X \to \mathbb{R}^n$ has the *path lifting property, i.e.,* for every path $\overline{\gamma}\colon [0,1] \to \mathbb{R}^n$ and every point $x \in \Psi^{-1}(\overline{\gamma}(0))$ there exists a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$ and $\Psi \circ \gamma = \overline{\gamma}$. Then the converse of Lemma 13 holds: if the image $\Psi(X)$ is convex and the level sets $\Psi^{-1}(w)$, $w \in \Psi(X)$, are path connected, then the map $\Psi\colon X \to \mathbb{R}^n$ is convex.

The main ingredient in the proof of Theorem 10 is the following theorem, which we shall prove in Section 6.

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15 Theorem (Local convexity and openness imply global convexity and openness) *Let X be a connected Hausdorff topological space, let $\mathfrak{T}$ be a convex subset of $\mathbb{R}^n$, and let $\Psi\colon X \to \mathfrak{T}$ be a continuous proper map. Suppose that for every point $x \in X$ there exists an open neighbourhood U of x such that the map $\Psi|_U\colon U \to \Psi(U)$ is convex and open.*

Then the map $\Psi\colon X \to \Psi(X)$ is convex and open.

Following [HNP], one may call Theorem 15 a *Lokal-Global-Prinzip*.

16 Remark In Theorem 15, we assume that each point is contained in an open set on which the map is convex and is open as a map to its image, but we do not insist that these open sets form a basis of the topology. This requirement would be too restrictive, as is illustrated in the following two examples.

(i) Consider the map $(x, y) \mapsto -y + \sqrt{x^2 + y^2}$ from $\mathbb{R}^2$ to $\mathbb{R}$. One level set is the nonnegative y-axis $\{(0, y) \mid y \geq 0\}$; the other level sets are the parabolas $y = \frac{1}{2\alpha}x^2 - \frac{\alpha}{2}$ for $\alpha > 0$. This map is convex, but its restrictions to small neighbourhoods of individual points on the positive y-axis are not convex. (These restrictions have disconnected fibres.)

(ii) Consider the map $(t, e^{i\theta}) \mapsto te^{i\theta}$ from $\mathbb{R} \times S^1$ to $\mathbb{C} \cong \mathbb{R}^2$. This map is convex, but its restrictions to small neighbourhoods of individual points on the zero section $\{0\} \times S^1$ are not convex. (These restrictions have a nonconvex image.)

Proof of Theorem 10, assuming Theorem 15 By Theorem 15, the map Ψ is convex, and it is open as a map to its image. By Lemma 13, the level sets of Ψ are connected and the image of Ψ is convex. ∎

The bulk of this paper is devoted to proving Theorem 15.

4 Convexity for Components of Preimages of Neighbourhoods

We first set some notation.

Let X be a Hausdorff topological space and $\Psi\colon X \to \mathbb{R}^n$ a continuous map. For $x \in X$ with $\Psi(x) = w$, we denote by $[x]$ the path connected component of x in $\Psi^{-1}(w)$, and, for $\varepsilon > 0$, we denote by $U_{[x],\varepsilon}$ the path connected component of x in $\Psi^{-1}(B(w, \varepsilon))$. Note that $U_{[x],\varepsilon}$ does not depend on the particular choice of x in $[x]$.

Remark Suppose that every point in X has an open neighbourhood U on which the restriction $\Psi|_U$ is convex. Then, in the definitions of $[x]$ and $U_{[x],\varepsilon}$, the term *path connected component* can be replaced by *connected component*. Indeed, let $Y = \Psi^{-1}(B(w, \varepsilon))$ or $Y = \Psi^{-1}(w)$. If $\Psi|_U$ is convex, so is $\Psi|_{U\cap Y}$; in particular, $U \cap Y$ is path connected. Thus, every point in Y has a path connected open neighbourhood with respect to the relative topology on Y. So the connected components of Y coincide with its path connected components.

A crucial step in the proof of Theorem 15 is that the neighbourhoods U such that $\Psi|_U\colon U \to \Psi(U)$ is convex and open can be taken to be the entire connected components $U_{[x],\varepsilon}$.

17 Proposition (Properties for connected components) *Let X be a Hausdorff topological space, $\mathcal{T} \subset \mathbb{R}^n$ a convex subset, and $\Psi\colon X \to \mathcal{T}$ a continuous proper map. Suppose that for every point $x \in X$ there exists an open neighbourhood U of x such that the map $\Psi|_U\colon U \to \Psi(U)$ is convex and open.*

Then for every point $x \in X$ there exists an $\varepsilon > 0$ such that the map $\Psi|_{U_{[x],\varepsilon}}\colon U_{[x],\varepsilon} \to \Psi(U_{[x],\varepsilon})$ is convex and open.

We digress to recall standard consequences of the properness of a map.

18 Lemma *Let X be a Hausdorff topological space, $\mathcal{T} \subset \mathbb{R}^n$ a subset, and $\Psi\colon X \to \mathcal{T}$ a continuous proper map. Let $w_0 \in \mathcal{T}$.*

(i) *Let U be an open subset of X that contains the level set $\Psi^{-1}(w_0)$. Then there exists $\varepsilon > 0$ such that the preimage $\Psi^{-1}(B(w_0,\varepsilon))$ is contained in U.*

(ii) *Suppose that every point of $\Psi^{-1}(w_0)$ has a connected open neighbourhood in $\Psi^{-1}(w_0)$ with respect to the relative topology. Then there exists $\varepsilon > 0$ such that, whenever $[x]$ and $[y]$ are distinct connected components of $\Psi^{-1}(w_0)$, the sets $U_{[x],\varepsilon}$ and $U_{[y],\varepsilon}$ are disjoint.*

Proof of part (i) Suppose otherwise. Then for every $\varepsilon > 0$ there exists $x_\varepsilon \in X \smallsetminus U$ such that $\|\Psi(x_\varepsilon) - w_0\| < \varepsilon$.

Let ε_j be a sequence such that $\varepsilon_j \to 0$ as $j \to \infty$. Then $x_{\varepsilon_j} \in X \smallsetminus U$ for all j, and $\Psi(x_{\varepsilon_j}) \to w_0$ as $j \to \infty$.

The set $\{\Psi(x_{\varepsilon_j})\}_{j=1}^{\infty} \cup \{w_0\}$ is compact. By properness, its preimage,

$$\bigcup_{j=1}^{\infty} \Psi^{-1}(\Psi(x_{\varepsilon_j})) \cup \Psi^{-1}(w_0),$$

is compact. The sequence $\{x_{\varepsilon_j}\}_{j=1}^{\infty}$ is in this preimage. So there exists a point x_∞ such that every neighbourhood of x_∞ contains x_{ε_j} for infinitely many values of j.

By continuity, $\Psi(x_\infty) = w_0$. Since U contains $\Psi^{-1}(w_0)$ and is open, U is a neighbourhood of x_∞, so there exist arbitrarily large values of j such that $x_{\varepsilon_j} \in U$. This contradicts the assumption $x_{\varepsilon_j} \in X \smallsetminus U$. ■

Proof of part (ii) Because Ψ is proper, the level set $\Psi^{-1}(w_0)$ is compact. Because $\Psi^{-1}(w_0)$ is compact and is covered by connected open subsets with respect to the relative topology, it has only finitely many components $[x]$. Because these components are compact and disjoint and X is Hausdorff, there exist open subsets $\mathcal{O}_{[x]}$ in X such that $[x] \subset \mathcal{O}_{[x]}$ for each component $[x]$ of $\Psi^{-1}(w_0)$ and such that for $[x]$ and $[y]$ in $\Psi^{-1}(w_0)$, if $[x] \neq [y]$ then $\mathcal{O}_{[x]} \cap \mathcal{O}_{[y]} = \varnothing$. The union of the sets $\mathcal{O}_{[x]}$ is an open subset of X that contains the fibre $\Psi^{-1}(w_0)$. By part (i), this open subset contains $\Psi^{-1}(B(w_0,\varepsilon))$ for every sufficiently small ε. For such an ε, because each $U_{[x],\varepsilon}$ is contained in $\mathcal{O}_{[x]}$ and the sets $\mathcal{O}_{[x]}$ are disjoint, the sets $U_{[x],\varepsilon}$ are disjoint. ■

We now prepare for the proof of Proposition 17. In the remainder of this section, let X be a Hausdorff topological space, $\mathcal{T} \subset \mathbb{R}^n$ a subset, and $\Psi\colon X \to \mathcal{T}$ a continuous map. Fix a point $w_0 \in \mathcal{T}$. Let $\{U_i\}$ be a collection of open subsets of X whose union contains $\Psi^{-1}(w_0)$.

19 Lemma *Let $[x]$ be a connected component of $\Psi^{-1}(w_0)$. If $U_k \cap [x] \neq \varnothing$ and $U_l \cap [x] \neq \varnothing$, then there exists a sequence $k = i_0, i_1, \dots, i_s = l$ such that*

$$U_{i_{q-1}} \cap U_{i_q} \cap [x] \neq \varnothing \quad \text{for} \quad q = 1, \dots, s. \tag{4.1}$$

Proof Let I_k denote the set of indices j for which one can get from U_k to U_j through a sequence of sets $U_{i_0}, U_{i_1}, \dots, U_{i_s}$ with the property (4.1). If $j \in I_k$ and $j' \notin I_k$, then $U_j \cap [x]$ and $U_{j'} \cap [x]$ are disjoint. Thus,

$$[x] = \Big(\bigcup_{j \in I_k} U_j \cap [x] \Big) \cup \Big(\bigcup_{j' \notin I_k} U_{j'} \cap [x] \Big)$$

expresses $[x]$ as the union of two disjoint open subsets, of which the first is nonempty. Because $[x]$ is connected, the second set in this union must be empty. So $U_l \cap [x] \neq \varnothing$ implies $l \in I_k$. ■

Now assume, additionally, that the covering $\{U_i\}$ is finite and that, for each i, the map $\Psi|_{U_i} \colon U_i \to \Psi(U_i)$ is open. Let

$$W_i := \Psi(U_i). \tag{4.2}$$

20 Lemma *Let $[x]$ be a connected component of $\Psi^{-1}(w_0)$. For sufficiently small $\varepsilon > 0$, the following is true.*

(i) *For any i and j, if $U_i \cap U_j \cap [x] \neq \varnothing$, then*

$$W_i \cap B(w_0, \varepsilon) = \Psi(U_i \cap U_j) \cap B(w_0, \varepsilon).$$

(ii) *For any k and l, if $U_k \cap [x]$ and $U_l \cap [x]$ are nonempty, then*

$$W_k \cap B(w_0, \varepsilon) = W_l \cap B(w_0, \varepsilon).$$

Proof Suppose that $U_i \cap U_j \cap [x] \neq \varnothing$. Then the set $\Psi(U_i \cap U_j)$ contains w_0. Since $U_i \cap U_j$ is open in U_i, and since the restriction of Ψ to U_i is an open map to its image, the set $\Psi(U_i \cap U_j)$ is open in W_i. Let $\varepsilon_{ij} > 0$ be such that the set $\Psi(U_i \cap U_j)$ contains $W_i \cap B(w_0, \varepsilon_{ij})$. Because we also have $\Psi(U_i \cap U_j) \subset \Psi(U_i) = W_i$,

$$W_i \cap B(w_0, \varepsilon_{ij}) = \Psi(U_i \cap U_j) \cap B(w_0, \varepsilon_{ij}).$$

Let ε be any positive number that is smaller than or equal to ε_{ij} for all the pairs U_i, U_j for which $U_i \cap U_j \cap [x] \neq \varnothing$. Then, for every such pair U_i, U_j,

$$W_i \cap B(w_0, \varepsilon) = \Psi(U_i \cap U_j) \cap B(w_0, \varepsilon).$$

This proves (i).

Now suppose that $U_k \cap [x] \neq \varnothing$ and $U_l \cap [x] \neq \varnothing$. By Lemma 19, one can get from U_k to U_l by a sequence of sets $U_k = U_{i_0}, \dots, U_{i_s} = U_l$ such that $U_{i_{q-1}} \cap U_{i_q} \cap [x] \neq \varnothing$ for $q = 1, \dots, s$. Part (i) then implies that the intersections $W_{i_q} \cap B(w_0, \varepsilon)$ are the same for all the elements in the sequence. Because the sequence begins with U_k and ends with U_l, it follows that

$$W_k \cap B(w_0, \varepsilon) = W_l \cap B(w_0, \varepsilon).$$

This proves (ii). ■

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Let $[x]$ be a connected component of $\Psi^{-1}(w_0)$. Fix an $\varepsilon > 0$ that satisfies the conditions of Lemma 20. Let

$$W_{[x],\varepsilon} := W_i \cap B(w_0, \varepsilon) \text{ when } U_i \cap [x] \neq \varnothing. \tag{4.3}$$

By part (ii) of Lemma 20, this is independent of the choice of such i. Also, define

$$\tilde{U}_{[x],\varepsilon} := \bigcup_{U_i \cap [x] \neq \varnothing} U_i \cap \Psi^{-1}(B(w_0, \varepsilon)). \tag{4.4}$$

We have

$$\begin{aligned} \Psi(\tilde{U}_{[x],\varepsilon}) &= \bigcup_{U_i \cap [x] \neq \varnothing} \Psi(U_i) \cap B(w_0, \varepsilon) \quad \text{by (4.4)} \\ &= W_{[x],\varepsilon} \quad \text{by (4.2) and (4.3).} \end{aligned} \tag{4.5}$$

21 Lemma *Suppose that, for each i, the level sets of $\Psi|_{U_i} \colon U_i \to W_i$ are path connected. Then the level sets of $\Psi|_{\tilde{U}_{[x],\varepsilon}} \colon \tilde{U}_{[x],\varepsilon} \to W_{[x],\varepsilon}$ are path connected.*

Proof Let $w \in W_{[x],\varepsilon}$ and let $x_0, x_1 \in \tilde{U}_{[x],\varepsilon} \cap \Psi^{-1}(w)$. By (4.4) there exist i and k such that $x_0 \in U_i$, $x_1 \in U_k$, $U_i \cap [x] \neq \varnothing$, and $U_k \cap [x] \neq \varnothing$. Fix such i and k. By Lemma 19, there exists a sequence $i = i_0, i_1, \dots, i_s = k$ such that $U_{i_{l-1}} \cap U_{i_l} \cap [x] \neq \varnothing$ for $l = 1, \dots, s$. Part (i) of Lemma 20 and the definition (4.3) of $W_{[x],\varepsilon}$ imply that $\Psi(U_{i_{l-1}} \cap U_{i_l}) \cap B(w_0, \varepsilon) = W_{[x],\varepsilon}$, and thus $U_{i_{l-1}} \cap U_{i_l} \cap \Psi^{-1}(w)$ is nonempty for each $1 \leq l \leq s$. Since each $U_{i_l} \cap \Psi^{-1}(w)$ is path connected, this implies that x_0 and x_1 can be connected by a path in $\tilde{U}_{[x],\varepsilon} \cap \Psi^{-1}(w)$. ■

22 Lemma *Suppose that, for each i, the restriction of Ψ to U_i is a convex map. Then the map*

$$\Psi|_{\tilde{U}_{[x],\varepsilon}} \colon \tilde{U}_{[x],\varepsilon} \to \Psi(\tilde{U}_{[x],\varepsilon}) \tag{4.6}$$

is convex and open.

Proof Let x_0 and x_1 be in $\tilde{U}_{[x],\varepsilon}$. Let i be such that $x_0 \in U_i$ and $U_i \cap [x] \neq \varnothing$. By (4.5), $\Psi(x_1) \in W_{[x],\varepsilon}$. By (4.3) and (4.2), there exists $y \in U_i$ such that $\Psi(y) = \Psi(x_1)$.

By assumption, the restriction of Ψ to U_i is a convex map. By Lemma 12, the restriction of Ψ to $U_i \cap \Psi^{-1}(B(w_0, \varepsilon))$ is also convex. Let γ' be a path in $U_i \cap \Psi^{-1}(B(w_0, \varepsilon))$ from x_0 to y such that $\psi \circ \gamma'$ is monotone straight. By Lemma 21 there exists a path γ'' in $\tilde{U}_{[x],\varepsilon}$ from y to x_1 whose composition with Ψ is constant. Let γ be the concatenation of γ' with γ''; then γ is a path from x_0 to x_1 and $\Psi \circ \gamma$ is monotone straight.

Thus, the map (4.6) is convex. To show that this map is open, we want to show that given any open set $\Omega \subset \tilde{U}_{[x],\varepsilon}$, its image $\Psi(\Omega)$ is open in $W_{[x],\varepsilon}$. By (4.4), $\Psi(\Omega) = \cup_i \Psi(\Omega \cap U_i)$ for i such that $U_i \cap [x] \neq \varnothing$, and each $\Psi(\Omega \cap U_i)$ is contained in $B(w_0, \varepsilon)$. Since $\Psi|_{U_i} \colon U_i \to W_i$ is open, $\Psi(\Omega \cap U_i)$ is open in W_i. By (4.3), each $\Psi(\Omega \cap U_i)$ is open in $W_{[x],\varepsilon}$. ■

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Proof of Proposition 17 Let $w_0 = \Psi(x)$. For each $x' \in \Psi^{-1}(w_0)$, let $U_{x'}$ be an open neighbourhood of x' such that the map $\Psi|_{U_{x'}} : U_{x'} \to \Psi(U_{x'})$ is convex and open. The sets $U_{x'}$, for $x' \in \Psi^{-1}(w_0)$, cover $\Psi^{-1}(w_0)$. Because $\Psi^{-1}(w_0)$ is compact, there exists a finite subcovering; let $\{U_i\}_{i=1}^n$ be a finite subcovering.

Because $\Psi^{-1}(w_0)$ is compact and each point has a connected neighbourhood with respect to the relative topology, $\Psi^{-1}(w_0)$ has only finitely many components $[x]$. Let $\varepsilon > 0$ satisfy the conditions of Lemma 20 for all these components. By Lemma 18, after possibly shrinking ε, we may assume that $\Psi^{-1}(B(w_0, \varepsilon)) \subset \cup_i U_i$ and that, whenever $[x]$ and $[y]$ are distinct connected components of $\Psi^{-1}(w_0)$, the sets $U_{[x],\varepsilon}$ and $U_{[y],\varepsilon}$ are disjoint.

Let $\tilde{U}_{[x],\varepsilon}$ and $W_{[x],\varepsilon}$ be the sets defined in (4.4) and (4.3). Then the preimage $\Psi^{-1}(B(w_0, \varepsilon))$ is the union of the sets $\tilde{U}_{[x],\varepsilon}$, for components $[x]$ of $\Psi^{-1}(w_0)$. Because each $\tilde{U}_{[x],\varepsilon}$ is connected and contains $[x]$, it is contained in the connected component $U_{[x],\varepsilon}$ of x in $\Psi^{-1}(B(w_0, \varepsilon))$. Because the sets $U_{[x],\varepsilon}$ are disjoint and the union of the sets $\tilde{U}_{[x],\varepsilon}$ is the entire preimage $\Psi^{-1}(B(w_0, \varepsilon))$, each $\tilde{U}_{[x],\varepsilon}$ is *equal* to $U_{[x],\varepsilon}$. This and Lemma 22 give Proposition 17. ■

5 Distance with Respect to a Locally Convex Map

Let X be a Hausdorff topological space and $\Psi\colon X \to \mathbb{R}^n$ a continuous map. Let x_0 and x_1 be two points in X. We define their Ψ-distance to be

$$d_\Psi(x_0, x_1) = \inf\{l(\Psi \circ \gamma) \mid \gamma\colon [0,1] \to X,\ \gamma(0) = x_0,\ \gamma(1) = x_1\}.$$

Note that the Ψ-distance can take any value in $[0, \infty]$. Also note that $d_\Psi(x_0, x_1) = 0$ if and only if x_0 and x_1 are in the same path-component of a level set of Ψ.

Remark In practice, we will work with a space X that is connected and in which each point has a neighbourhood U such that the restriction of Ψ to U is a convex map. For such a space, in the above definition of Ψ-distance we may take the infimum to be over the set of paths γ such that $\Psi \circ \gamma$ is polygonal:

Indeed, let $\gamma\colon [0,1] \to X$ be any path such that $\gamma(0) = x_0$ and $\gamma(1) = x_1$. By our assumption on X, for every $\tau \in [0,1]$ there exists an open interval J_τ containing τ and an open subset $U_\tau \subset X$ such that the restriction of Ψ to U_τ is a convex map and such that $\gamma(J_\tau \cap [0,1]) \subset U_\tau$.

The open intervals $\{J_\tau\}$ form an open covering of $[0,1]$. Because the interval $[0,1]$ is compact, there exists a finite subcovering; denote it $J_1, \dots, J_s$. Let

$$\varepsilon = \min\{\,\text{length}(J_i \cap J_k) \mid i, k \in \{1, \dots, s\} \text{ and } J_i \cap J_k \neq \varnothing\}.$$

Any subinterval $[\alpha, \beta] \subset [0,1]$ of length $< \varepsilon$ is contained in one of the J_is. Indeed, given such a subinterval $[\alpha, \beta]$, consider those intervals of $J_1, \dots, J_s$ that contain α; let J_i be the one whose upper bound is maximal; then J_i also contains β.

Thus, for any subinterval $[\alpha, \beta] \subset [0,1]$ of length $< \varepsilon$ there exists an open subset $U \subset X$ such that the restriction of Ψ to U is a convex map and such that $\gamma(\alpha)$ and $\gamma(\beta)$ are both contained in U.

Partition $[0,1]$ into m intervals $0 = t_0 < \cdots < t_m = 1$ such that $|t_j - t_{j-1}| < \varepsilon$ for each j. By the previous paragraph, for every $1 \leq j \leq m$ there exists $U \subset X$ such that the restriction of Ψ to U is a convex map and such that $\gamma(t_{j-1})$ and $\gamma(t_j)$ are both contained in U. Because the restriction of Ψ to U is convex, there exists a path γ_j in X connecting $\gamma(t_{j-1})$ and $\gamma(t_j)$ such that the image of $\Psi \circ \gamma$ is a (possibly degenerate) segment with a weakly monotone parametrization. The path γ' that is formed by concatenating $\gamma_1, \ldots, \gamma_m$ has the following properties: it connects x_0 and x_1, the composition $\Psi \circ \gamma'$ is polygonal, and $l(\Psi \circ \gamma') \leq l(\Psi \circ \gamma)$.

6 Proof that Local Convexity and Openness Imply Global Convexity and Openness

23 Lemma (Existence of midpoints) *Let X be a compact, connected, Hausdorff topological space and $\Psi\colon X \to \mathbb{R}^n$ a continuous map. Suppose that for every point $x \in X$ there exists an open neighbourhood U such that the restriction of Ψ to U is a convex map.*

Let x_0 and x_1 be in X. Then there exists a point $x_{1/2} \in X$ such that

$$d_\Psi(x_0, x_{1/2}) = d_\Psi(x_{1/2}, x_1) = \tfrac{1}{2} d_\Psi(x_0, x_1). \tag{6.1}$$

Proof Choose paths γ_n connecting x_0 and x_1 such that the sequence $\{l(\Psi \circ \gamma_n)\}$ converges to $d_\Psi(x_0, x_1)$.

Let $t_j \in [0,1]$ be such that $\gamma_j(t_j)$ is the midpoint of the path γ_j:

$$l(\Psi \circ \gamma_j|_{[0,t_j]}) = l(\Psi \circ \gamma_j|_{[t_j,1]}) = \tfrac{1}{2} l(\Psi \circ \gamma_j).$$

Because X is compact, there exists a point $x_{1/2}$ such that every neighbourhood of $x_{1/2}$ contains $\gamma_j(t_j)$ for infinitely many values of j. We will show that the point $x_{1/2}$ satisfies equation (6.1).

We first show that $d_\Psi(x_0, x_{\frac{1}{2}}) \leq \frac{1}{2} d_\Psi(x_0, x_1)$, or, equivalently, that for every $\varepsilon > 0$ there exists a path γ connecting x_0 and $x_{\frac{1}{2}}$ such that $l(\Psi \circ \gamma) < \frac{1}{2} d_\Psi(x_0, x_1) + \varepsilon$.

Let U be a neighbourhood of $x_{1/2}$ such that the restriction of Ψ to U is a convex map. Let j be such that the following facts are true:

(i) $\gamma_j(t_j) \in U$ and $\|\Psi(\gamma_j(t_j)) - \Psi(x_{1/2})\| < \frac{\varepsilon}{2}$.
(ii) $l(\Psi \circ \gamma_j) < d_\Psi(x_0, x_1) + \varepsilon$.

By (i) and since $\Psi|_U$ is a convex map, there exists a path μ connecting $\gamma_j(t_j)$ and $x_{1/2}$ such that $l(\Psi \circ \mu) < \frac{\varepsilon}{2}$. Let γ be the concatenation of $\gamma_j|_{[0,t_j]}$ and μ. Then γ is a path connecting x_0 and $x_{1/2}$, and $l(\Psi \circ \gamma) < \frac{1}{2} d_\Psi(x_0, x_1) + \varepsilon$.

Thus, $d_\Psi(x_0, x_{1/2}) \leq \frac{1}{2} d_\Psi(x_0, x_1)$. Likewise, $d_\Psi(x_{1/2}, x_1) \leq \frac{1}{2} d_\Psi(x_0, x_1)$. If either of these were a strict inequality, then it would be possible to construct a path from x_0 to x_1 whose image has length less than $d_\Psi(x_0, x_1)$, which contradicts the definition of $d_\psi(x_0, x_1)$. Thus, $d_\Psi(x_0, x_{1/2}) = d_\Psi(x_{1/2}, x_1) = \frac{1}{2} d_\Psi(x_0, x_1)$. ∎

To prove Theorem 15, we need to have some uniform control on the sizes of ε such that the restrictions of Ψ to the connected components $U_{[x],\varepsilon}$ of $\Psi^{-1}(B(w_0, \varepsilon))$ are convex. The precise result that we will use is established in the following proposition:

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24 Proposition *Let X be a Hausdorff topological space and let $\Psi\colon X \to \mathbb{R}^n$ be a continuous map. Suppose that for each $x \in X$ there exists an $\varepsilon > 0$ such that the restriction of Ψ to the set $U_{[x],\varepsilon}$ is a convex map.*

Then for every compact subset $A \subset X$ there exists $\varepsilon > 0$ such that for every $x \in A$ and $x' \in X$, if $d_\Psi(x, x') < \varepsilon$, then there exists a path $\gamma\colon [0, 1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = x'$, and $\Psi \circ \gamma$ is monotone straight.

Proof For each $x \in X$, let $\varepsilon_x > 0$ be such that the restriction of Ψ to the set $U_{[x],\varepsilon_x}$ is a convex map. The sets $U_{[x],\varepsilon_x/2}$, for $x \in A$, form an open covering of the compact set A. Choose a finite subcovering: let $x_1, \dots, x_k$ be points of A and $\varepsilon_1, \dots, \varepsilon_k$ be positive numbers such that, for each $1 \le i \le k$, the restriction of Ψ to the set $U_{[x_i],\varepsilon_i}$ is a convex map, and such that the sets $U_{[x_i],\varepsilon_i/2}$ cover A.

Let

$$\varepsilon = \min_{1 \le i \le k} \frac{\varepsilon_i}{2}.$$

Let $x \in A$, and let $1 \le i \le k$ be such that $x \in U_{[x_i],\varepsilon_i/2}$.

Because $U_{[x_i],\varepsilon_i/2}$, by its definition, is contained in $\Psi^{-1}(B(\Psi(x_i), \varepsilon_i/2))$, we have $\|\Psi(x) - \Psi(x_i)\| < \varepsilon_i/2$.

Because x and x_i are also contained in the larger set $U_{[x_i],\varepsilon_i}$, and the restriction of Ψ to this set is a convex map, there exists a path γ' from x_i to x such that $\Psi \circ \gamma'$ is monotone straight; in particular, $l(\Psi \circ \gamma') = \|\Psi(x) - \Psi(x_i)\|$, so $l(\Psi \circ \gamma') < \varepsilon_i/2$.

Let $x' \in X$ be such that $d_\Psi(x, x') < \varepsilon$. Then, by the definition of d_Ψ, there exists a path γ'' from x to x' such that $l(\Psi \circ \gamma'') < \varepsilon$.

Let $\hat{\gamma}$ be the concatenation of γ' and γ''. Then $\hat{\gamma}$ is a path from x_i to x', and $l(\Psi \circ \hat{\gamma}) \le l(\Psi \circ \gamma') + l(\Psi \circ \gamma'') < \varepsilon_i/2 + \varepsilon \le \varepsilon_i$. Therefore, $\Psi \circ \hat{\gamma} \subset B(\Psi(x_i), \varepsilon_i)$. Thus, x' and x_i are in the same connected component of $\Psi^{-1}(B(\Psi(x_i), \varepsilon_i))$; that is, x' is in the set $U_{[x_i],\varepsilon_i}$. Because x is also in the set $U_{[x_i],\varepsilon_i}$, and because the restriction of Ψ to this set is a convex map, there exists a path γ from x to x' such that $\Psi \circ \gamma$ is monotone straight. ■

25 Proposition *Let X be a compact, connected, Hausdorff topological space, and let $\Psi\colon X \to \mathbb{R}^n$ be a continuous map. Suppose that there exists $\varepsilon > 0$ such that, for every x and x' in X, if $d_\Psi(x, x') < \varepsilon$, then there exists a path $\gamma\colon [0, 1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = x'$, and $\Psi \circ \gamma$ is monotone straight.*

Then $\Psi\colon X \to \mathbb{R}^n$ is a convex map.

Proof Fix x_0 and x_1 in X.

By Lemma 23, there exists a point $x_{1/2}$ such that

$$d_\Psi(x_0, x_{1/2}) = d_\Psi(x_{1/2}, x_1) = \frac{1}{2} d_\Psi(x_0, x_1).$$

Likewise, there exists a point $x_{1/4}$ that satisfies

$$d_\Psi(x_0, x_{1/4}) = d_\Psi(x_{1/4}, x_{1/2}) = \frac{1}{2} d_\Psi(x_0, x_{1/2}).$$

By iteration, we get a map $\frac{j}{2^m} \mapsto x_{\frac{j}{2^m}}$, for nonnegative integers j and m with $0 \le j \le 2^m$, such that

$$d_\Psi(x_{\frac{j-1}{2^m}}, x_{\frac{j}{2^m}}) = d_\Psi(x_{\frac{j}{2^m}}, x_{\frac{j+1}{2^m}}) = \frac{1}{2} d_\Psi(x_{\frac{j-1}{2^m}}, x_{\frac{j+1}{2^m}}). \tag{6.2}$$

Let $\varepsilon > 0$ be as in the assumption of the proposition.
Choose m large enough such that for every $1 \le j \le 2^m$,

$$d_\Psi(x_{\frac{j-1}{2^m}}, x_{\frac{j}{2^m}}) < \frac{\varepsilon}{2}.$$

By the assumption, there exists a path γ_j from $x_{(j-1)/2^m}$ to $x_{j/2^m}$ such that $\Psi \circ \gamma_j$ is monotone straight. Thus,

$$d_\Psi(x_{\frac{j-1}{2^m}}, x_{\frac{j}{2^m}}) = \|\Psi(x_{\frac{j-1}{2^m}}) - \Psi(x_{\frac{j}{2^m}})\| \quad \text{for each } 1 \le j \le 2^m.$$

Similarly,

$$d_\Psi(x_{\frac{j-1}{2^m}}, x_{\frac{j+1}{2^m}}) = \|\Psi(x_{\frac{j-1}{2^m}}) - \Psi(x_{\frac{j+1}{2^m}})\| \quad \text{for each } 1 \le j < 2^m.$$

Thus, equation (6.2) can be rewritten as

$$\|\Psi(x_{\frac{j-1}{2^m}}) - \Psi(x_{\frac{j}{2^m}})\| = \|\Psi(x_{\frac{j}{2^m}}) - \Psi(x_{\frac{j+1}{2^m}})\| = \frac{1}{2}\|\Psi(x_{\frac{j-1}{2^m}}) - \Psi(x_{\frac{j+1}{2^m}})\|,$$

which implies, by the triangle inequality, that the points

$$\Psi(x_{\frac{j-1}{2^m}}), \quad \Psi(x_{\frac{j}{2^m}}), \quad \Psi(x_{\frac{j+1}{2^m}})$$

are collinear. The concatenation of the paths γ_j, for $1 \le j \le 2^m$, is a path from x_0 to x_1 whose composition with Ψ is monotone straight. ■

26 Corollary *Let X be a compact Hausdorff topological space and $\Psi\colon X \to \mathbb{R}^n$ a continuous map. Suppose that for every point $x \in X$ there exists an open neighbourhood U such that the map $\Psi|_U\colon U \to \Psi(U)$ is convex and open. Then the map $\Psi\colon X \to \mathbb{R}^n$ is convex.*

Proof By Proposition 17, for every point $x \in X$ there exists an $\varepsilon > 0$ such that the map

$$\Psi|_{U_{[x],\varepsilon}}\colon U_{[x],\varepsilon} \to \Psi(U_{[x],\varepsilon})$$

is convex and open.

By Proposition 24, there exists $\varepsilon > 0$ such that for every x and x' in X, if $d_\Psi(x, x') < \varepsilon$, then there exists a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x$, $\gamma(1) = x'$, and $\Psi \circ \gamma$ is monotone straight.

By Proposition 25, the map $\Psi\colon X \to \mathbb{R}^n$ is convex. ■

Proof of Theorem 15 Let X be a connected Hausdorff topological space, $\mathcal{T} \subset \mathbb{R}^n$ a convex subset, and $\Psi\colon X \to \mathcal{T}$ a continuous proper map. Suppose that for every point $x \in X$ there exists an open neighbourhood U such that the map $\Psi|_U\colon U \to \Psi(U)$ is convex and open.

Fix any two points $x_0, x_1 \in X$. Let γ' be a path in X connecting x_0 to x_1; let $K = \operatorname{conv}(\operatorname{image}(\Psi \circ \gamma'))$ be the convex hull of its image in $\mathbb{R}^n$. Then K is a compact subset of $\mathcal{T}$. Let A be the component of $\Psi^{-1}(K)$ that contains the point x_0. Then A also contains x_1, and A is compact and connected.

For every $x \in A$, if U is a neighbourhood of x in X such that $\Psi|_U\colon U \to \Psi(U)$ is convex and open, then $U \cap \Psi^{-1}(K)$ is a neighbourhood of x in $\Psi^{-1}(K)$ such that $\Psi|_{U\cap\Psi^{-1}(K)}\colon U\cap\Psi^{-1}(K) \to \Psi(U)\cap K$ is convex and open. In particular, $U\cap\Psi^{-1}(K)$ is connected, so it is contained in A. So $\Psi|_A\colon A \to \mathbb{R}^n$ satisfies the assumptions of Corollary 26. By this corollary, $\Psi|_A\colon A \to \mathbb{R}^n$ is convex.

So there exists a path $\gamma\colon [0,1] \to X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$, and $\Psi \circ \gamma$ is monotone straight. Because $x_0, x_1 \in X$ were chosen arbitrarily, this shows that $\Psi\colon X \to \mathbb{R}^n$ is convex.

To show that the map $\Psi\colon X \to \Psi(X)$ is open, it is enough to show that for each $w_0 \in \mathbb{R}^n$ there exists $\varepsilon > 0$ such that the restriction of Ψ to $\Psi^{-1}(B(w_0,\varepsilon))$ is open as a map to its image.

Fix $w_0 \in \mathbb{R}^n$.

Because the map $\Psi\colon X \to \mathbb{R}^n$ is convex, the level set $\Psi^{-1}(w_0)$ is connected. Thus, this level set consists of a single connected component, $[x]$.

By Proposition 17, for sufficiently small ε, the restriction of Ψ to the set $U_{[x],\varepsilon}$ is open as a map to its image. The set $U_{[x],\varepsilon}$ is an open set that contains $\Psi^{-1}(w_0)$. Because Ψ is proper, there exists an $\varepsilon' > 0$ such that the set $U_{[x],\varepsilon}$ contains the preimage $\Psi^{-1}(B(w_0,\varepsilon'))$; see Lemma 18. Thus, the restriction of Ψ to the preimage $\Psi^{-1}(B(w_0,\varepsilon'))$ is open as a map to its image. ■

7 Applications to Moment Maps

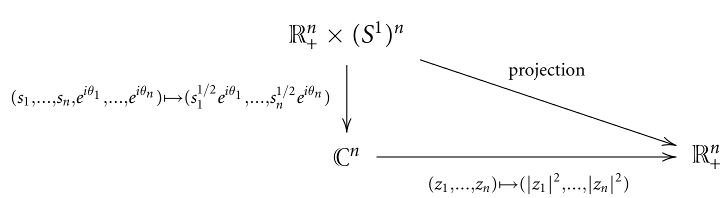
27 Example The map from $\mathbb{C}^n$ to $\mathbb{R}^n$ given by

$$(z_1,\dots,z_n) \mapsto (|z_1|^2,\dots,|z_n|^2) \tag{7.1}$$

is convex, and it is open as a map from $\mathbb{C}^n$ to the positive orthant $\mathbb{R}^n_+$.

Moreover, the restriction of the map (7.1) to any ball $B_\rho = \{z \in \mathbb{C}^n \mid \|z\| < \rho\}$ is convex, and it is open as a map to its image.

Proof Consider the following commuting diagram of continuous maps:



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Because the projection map is convex and open and the map on the left is onto, the bottom map is convex and open.

Because the ball B_ρ is the preimage of a convex set (namely, it is the preimage of the set $\{(s_1,\dots,s_n) \mid s_1+\dots+s_n<\rho^2\}$), the restriction of the map (7.1) to B_ρ is also convex and open as a map to its image. ■

28 Example Let $\alpha_1,\dots,\alpha_n$ be any vectors. Then the map

$$\Phi_H\colon (z_1,\dots,z_n) \mapsto \sum_{j=1}^{n} |z_j|^2\alpha_j \tag{7.2}$$

is convex, and it is open as a map to its image.

Moreover, the restriction of Φ_H to any ball $B_\rho = \{z \in \mathbb{C}^n \mid \|z\| < \rho\}$ is convex, and it is open as a map to its image.

Proof Because the map (7.1) is convex, so is its composition with the linear map $(s_1,\dots,s_n) \mapsto (s_1\alpha_1+\dots+s_n\alpha_n)$. Because the restriction of a linear map to the positive orthant $\mathbb{R}^n_+$ is open as a map to its image[1], so is this composition. Because the map (7.2) is open as a map to its image, so is its restriction to the open ball B_ρ. Because this restriction is the composition of a convex map with a linear projection, it is convex. ■

We proceed with applications to symplectic geometry. Relevant definitions can be found, for example, in the original paper [GS1] of Guillemin and Sternberg. We first describe local models for Hamiltonian torus actions.

Let $T \cong (S^1)^k$ be a torus, $\mathfrak{t} \cong \mathbb{R}^k$ its Lie algebra, and $\mathfrak{t}^* \cong \mathbb{R}^k$ the dual space. Let $H \subset T$ be a closed subgroup, $\mathfrak{h} \subset \mathfrak{t}$ its Lie algebra, and $\mathfrak{h}^0 \subset \mathfrak{t}^*$ the annihilator of $\mathfrak{h}$ in $\mathfrak{t}^*$. Let H act on $\mathbb{C}^n$ through a group homomorphism $H \to (S^1)^n$ followed by coordinatewise multiplication. The corresponding quadratic moment map, $\Phi_H\colon \mathbb{C}^n \to \mathfrak{h}^*$, has the form $z \mapsto \sum_{j=1}^n |z_j|^2\alpha_j$, where $\alpha_1,\dots,\alpha_n$ are elements of $\mathfrak{h}^*$ (namely, they are the weights of the H action on $\mathbb{C}^n$, times $\frac{1}{2}$).

Consider the model

$$Y = T \times_H \mathbb{C}^n \times \mathfrak{h}^0;$$

its elements are represented by triples $[a,z,\nu]$ with $a \in T, z \in \mathbb{C}^n$, and $\nu \in \mathfrak{h}^0$, with $[ab,z,\nu] = [a, b\cdot z,\nu]$ for all $b \in H$. Fix a splitting $\mathfrak{t}^* = \mathfrak{h}^* \oplus \mathfrak{h}^0$, and consider the map

$$\Phi_Y\colon T \times_H \mathbb{C}^n \times \mathfrak{h}^0 \to \mathfrak{t}^*, \quad [a,z,\nu] \mapsto \Phi_H(z)+\nu. \tag{7.3}$$

[1] This is a consequence of the following lemma: *For any vectors $\alpha_1,\dots,\alpha_n \in \mathbb{R}^\ell$ there exists $\varepsilon > 0$ such that for every $\beta = \sum s_j\alpha_j$ with all $s_j \geq 0$, if $\|\beta\| < \varepsilon$ then there exists $s' = (s'_1,\dots,s'_n)$ such that $\beta = \sum s'_j\alpha_j$ and $\|s'\| < 1$.* Proof: Let $\beta = \sum s_j\alpha_j$ with all $s_j \geq 0$. Then there exist s'_j such that $\beta = \sum s'_j\alpha_j$, all $s'_j \geq 0$, and the vectors $\{\alpha_j \mid s'_j \neq 0\}$ are linearly independent; *cf.* Carathéodory's theorem in convex geometry. Let $J = \{j \mid s'_j \neq 0\}$. The map $s \mapsto \sum s_j\alpha_j$ from $\mathbb{R}^J$ to $\operatorname{span}\{\alpha_j \mid j \in J\}$ is a linear isomorphism; denote its inverse by L_J. Then $s' = L_J(\beta)$, so $\|s'\| \leq \|L_J\|\|\beta\|$ where $\|L_J\|$ is the operator norm. The lemma holds with any $\varepsilon < \min_J\{\frac{1}{\|L_J\|}\}$ where J runs over the subsets of $\{1,\dots,n\}$ for which $\{\alpha_j \mid j \in J\}$ are linearly independent.

The map Φ_Y is convex and is open as a map to its image. This follows from the commuting diagram

$$\begin{array}{ccc} T \times \mathbb{C}^n \times \mathfrak{h}^0 & \xrightarrow{(a,z,\nu)\mapsto(\Phi_H(z),\nu)} & \mathfrak{h}^* \times \mathfrak{h}^0 \\ \downarrow & & \cong \downarrow \\ T \times_H \mathbb{C}^n \times & \xrightarrow{\Phi_Y} & \mathfrak{t}^* \end{array}$$

because the top map is convex and is open as a map to its image, the map on the left is onto, and the map on the right is a linear isomorphism.

Similarly, if $D \subset \mathbb{C}^n$ and $D' \subset \mathfrak{h}^0$ are disks centered at the origin, the restriction of Φ_Y to the subset $T \times_H D \times D'$ of $T \times_H \mathbb{C}^n \times \mathfrak{h}^0$ is convex and is open as a map to its image. This follows from the following diagram:

$$\begin{array}{ccc} T \times D \times D' & \xrightarrow{(a,z,\nu)\mapsto(\Phi_H(z),\nu)} & \mathfrak{h}^* \times \mathfrak{h}^0 \\ \downarrow & & \cong \downarrow \\ T \times_H D \times D' & \xrightarrow{\Phi_Y} & \mathfrak{t}^* \end{array}$$

29 Proposition *Let T act on a symplectic manifold with a moment map $\Phi\colon M \to \mathfrak{t}^*$. Then each point of M is contained in an open set $U \subset M$ such that the restriction of Φ to U is convex and is open as a map to its image, $\Phi(U)$.*

Proof Fix a point $x \in M$.

There exists a T-invariant neighbourhood U of x and an equivariant diffeomorphism $f\colon U \to T \times_H D \times D'$ that carries $\Phi|_U$ to a map that differs from Φ_Y by a constant in $\mathfrak{t}^*$, where the model $T \times_H D \times D'$ and the map Φ_Y are as in (7.3). This follows from the local normal form theorem for Hamiltonian torus actions [GS2]. Because the restriction of Φ_Y to $T \times_H D \times D'$ is convex and is open as a map to its image, so is $\Phi|_U$. ■

We can now recover the convexity theorem of Atiyah, Guillemin, and Sternberg along the lines given by Condevaux, Dazord, and Molino.

30 Theorem *Let M be a manifold equipped with a symplectic form and a torus action, and let $\Phi\colon M \to \mathfrak{t}^*$ be a corresponding moment map. Suppose that Φ is proper as a map to some convex subset of $\mathfrak{t}^*$. Then the image of Φ is convex, its level sets are connected, and the moment map is open as a map to its image.*

Proof By Proposition 29, every point in M is contained in an open set U such that the map $\Phi|_U$ is convex and is open as a map to its image, $\Phi(U)$. The conclusion then follows from Theorem 10. ■

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8 The Results of Birtea, Ortega, and Ratiu

The paper [BOR1] of Birtea, Ortega, and Ratiu contains results that are similar to ours. For the benefit of the reader, we present their results here.

31 Theorem ([BOR1, Thm. 2.28]) *Let X be a topological space that is connected, locally connected, first countable, and normal. Let V be a finite dimensional vector space. Let $f: X \to V$ be a map that satisfies the following conditions.*

(i) *The map f is continuous and is closed.*
(ii) *The map f has* local convexity data*: for each $x \in X$ and each sufficiently small neighbourhood U of x there exist a convex cone $C \subset V$ with vertex at $f(x)$ such that the restriction $f|_U : U \to C$ is an open map with respect to the subset topology on $C \subset V$.*
(iii) *The map f is* locally fiber connected*: for each $x \in X$, any open neighbourhood of x contains a neighbourhood U of x that does not intersect two connected components of the fiber $f^{-1}(f(x'))$ for any $x' \in U$.*

Then the fibers of f are connected, the map f is open onto its image, and the image $f(X)$ is a closed convex set.

32 Remark The paper [BOR2] contains a more general convexity result; in particular, it contains a more liberal definition of having local convexity data: for each $x \in X$, there exist arbitrarily small neighbourhoods U of x such that $f(U)$ is convex [BOR2, Def. 2.8]. Here, openness of the maps $f|_U : U \to f(U)$ is not part of the definition of "local convexity data", but it is assumed separately.

Birtea, Ortega, and Ratiu also sketch a proof of the following infinite dimensional version:

33 Theorem ([BOR1, Thm. 2.31]) *Let X be a topological space that is connected, locally connected, first countable, and normal. Let $(V, \| \|)$ be a Banach space that is the dual of another Banach space. Let $f: X \to V$ be a map that satisfies the following conditions.*

(i) *The map f is continuous with respect to the norm topology on V and is closed with respect to the weak-star topology on V.*
(ii) *The map f has* local convexity data *(see above).*
(iii) *The map f is* locally fiber connected *(see above).*

Then the fibers of f are connected, the map f is open onto its image with respect to the weak-star topology, and the image $f(X) \subset V$ is convex and is closed in the weak-star topology.

34 Remark

- We work with a convex subset of V; they similarly note that their theorem remains true when V is replaced by a convex subset of V [BOR1, Rem. 2.29].
- In [BOR2] they allow more general target spaces, which are not vector spaces.

- We assume that the domain is Hausdorff and the map is proper (in the sense that the preimage of a compact set is compact); they assume that the domain is first countable and normal and that the map is closed. We are not aware of non-artificial examples where one of these assumptions holds and the other does not.
- We assume that each point is contained in an open set on which the map is a *convex map*, a condition that we define in Definition 7. They assume that the map *has local convexity data* (defined in [BOR1, Def. 2.7] and re-defined in [BOR2, Def. 2.8]) and satisfies the *locally fiber connected condition* (defined in [BOR1, Def. 2.15] as a slight generalization of [Be, § 3.4, after Def. 3.6]).

  Example The inclusion map of a closed ball into $\mathbb{R}^n$ is a convex map in our sense. It does not have local convexity data in the sense of [BOR1], but it does have local convexity data in the sense of [BOR2].

- If a map is convex, then it has local convexity data (in the broader sense of [BOR2]) and it is locally fiber connected. Thus, our "convexity/connectedness" assumptions are stricter than those of [BOR1], but our conclusion is stronger.
- Both we and [BOR1] allow a broad interpretation of "local":
  - In [BOR1], the "locally fiber connected condition" on a subset A of X with respect to a map $f\colon X \to V$ depends not only on the restriction of the map f to the set A but also on which points in A belong to the same fiber *in* X. (This is where the definition of [BOR1] differs from that of Benoist.)
  - In our paper, we assume that each point is contained in an open set on which the map is convex and is open as a map to its image, but we do not insist that these open sets form a basis to the topology (*cf.* Remark 16).

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