On the Integration of the Canonic Equations and the Principle of Duality

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§1. The object of this paper is to essay an analytical statement of the reduction of the integration of a canonic system of differential equations (into which time does not enter explicitly) to that of the partial differential equation of Jacobi and Hamilton; and to illustrate the principle of duality by an outline of the solution for the problem of two bodies both by the standard form of the equation referred to and by the analogous form which that principle involves. Most statements of the reduction are verifications and somewhat obscure the symmetry of the canonic form. The shortest procedure, of course, is by means of the well known theorem of Jacobi, and this verificatory method is followed by Tisserand, Charlier and Appell. Poincaré gives a proof depending on a simple form given by him to the conditions for a canonical change of variables, but again the statement lacks analytical form. The essentials of this proof will be given here, but in an entirely different way. An analytical treatment of the subject has been given by Professor L. Becker in his class lectures at Glasgow, but it has not been published.

§2. *On the Conditions for a Canonical Change of Variables.*

To transform the canonic system

$$\dot{p}_r = -\frac{\partial H(q,\,p)}{\partial q_r},\quad \dot{q}_r = +\frac{\partial H(q,\,p)}{\partial p_r},\quad r = 1,\,2,\,\ldots n, \quad \ldots\ldots\ldots(1)$$

p, q being functions of ξ, η.

$$\frac{\partial H(\xi\,;\,\eta)}{\partial \xi_r}=\sum_\kappa\left(\frac{\partial H(q,\,p)}{\partial p_\kappa}\cdot\frac{\partial p_\kappa}{\partial \xi_r}+\frac{\partial H(q,\,p)}{\partial q_\kappa}\cdot\frac{\partial q_\kappa}{\partial \xi_r}\right)$$
$$=\sum_\kappa\left(\dot{q}_\kappa\frac{\partial p_\kappa}{\partial \xi_r}-\dot{p}_\kappa\frac{\partial q_\kappa}{\partial \xi_r}\right)$$
$$=\sum_\kappa\left\{\frac{\partial p_\kappa}{\partial \xi_r}\sum_s\left(\frac{\partial q_\kappa}{\partial \xi_s}\dot{\xi}_s+\frac{\partial q_\kappa}{\partial \eta_s}\dot{\eta}_s\right)-\frac{\partial q_\kappa}{\partial \xi_r}\sum_s\left(\frac{\partial p_\kappa}{\partial \xi_s}\dot{\xi}_s+\frac{\partial p_\kappa}{\partial \eta_s}\dot{\eta}_s\right)\right\}$$
$$=\sum_{\kappa s}\left\{\dot{\xi}_s\frac{\partial(p_\kappa,\,q_\kappa)}{\partial(\xi_r,\,\xi_s)}+\dot{\eta}_s\frac{\partial(p_\kappa,\,q_\kappa)}{\partial(\xi_r,\,\eta_s)}\right\}.$$
$$\frac{\partial H(\xi\,;\,\eta)}{\partial \eta_r}=\sum_{\kappa s}\left\{\dot{\xi}_s\frac{\partial(p_\kappa,\,q_\kappa)}{\partial(\eta_r,\,\xi_s)}+\dot{\eta}_s\frac{\partial(p_\kappa,\,q_\kappa)}{\partial(\eta_r,\,\eta_s)}\right\}.$$

The conditions that the new system takes the canonic form

$$\dot{\eta}_r=-\frac{\partial H(\xi\,;\,\eta)}{\partial \xi_r},\quad \dot{\xi}_r=+\frac{\partial H(\xi\,;\,\eta)}{\partial \eta_r},\quad r=1,\,2,\,\ldots n,\quad \ldots\ldots\text{(2)}$$

are therefore

$$\sum_\kappa\frac{\partial(p_\kappa,\,q_\kappa)}{\partial(\xi_r,\,\xi_s)}=0,\ \text{for any values of } r,\,s,\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}$$

$$\sum_\kappa\frac{\partial(p_\kappa,\,q_\kappa)}{\partial(\eta_r,\,\eta_s)}=0,\qquad ,, \qquad ,, \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)}$$

$$\left.\begin{aligned}\sum_\kappa\frac{\partial(p_\kappa,\,q_\kappa)}{\partial(\xi_r,\,\eta_s)}&=1,\ \text{if } r=s,\ \text{for any value of } s\\ &=0,\ \text{if } r\neq s,\ \text{for any values of } r,\,s\end{aligned}\right\}\qquad\text{(iii)}$$

§3. *Poincaré's Form of Condition.*

The form of condition used by Poincaré is

$$\sum_\kappa(q_\kappa dp_\kappa-\xi_\kappa d\eta_\kappa)=dS\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(3)}$$

This is an adaptation of a transformation theorem of Jacobi (quoted on p. 15 of *Nouvelles Methodes*, I.), and its equivalence with the Jacobian form of §2 is most easily shown as follows :

$$\sum_\kappa(q_\kappa dp_\kappa-\xi_\kappa d\eta_\kappa)=\sum_\kappa\left[q_\kappa\sum_i\left(\frac{\partial p_\kappa}{\partial \xi_i}d\xi_i+\frac{\partial p_\kappa}{\partial \eta_i}d\eta_i\right)-\xi_\kappa d\eta_\kappa\right].$$

Coefficient of $d\xi_r=\sum_\kappa q_\kappa\dfrac{\partial p_\kappa}{\partial \xi_r}$.

Coefficient of $d\eta_s=\sum_\kappa\left(q_\kappa\dfrac{\partial p_\kappa}{\partial \eta_s}-\xi_s\right)$;

$$\therefore \begin{cases} \dfrac{\partial}{\partial \xi_s}\sum_\kappa q_\kappa \dfrac{\partial p_\kappa}{\partial \xi_r} = \dfrac{\partial}{\partial \xi_r}\sum_\kappa q_\kappa \dfrac{\partial p_\kappa}{\partial \xi_s}, \\ \dfrac{\partial}{\partial \eta_s}\sum_\kappa\left(q_\kappa \dfrac{\partial p_\kappa}{\partial \eta_r} - \xi_r\right) = \dfrac{\partial}{\partial \eta_r}\sum_\kappa\left(q_\kappa \dfrac{\partial p_\kappa}{\partial \eta_s} - \xi_s\right), \\ \dfrac{\partial}{\partial \eta_s}\sum_\kappa q_\kappa \dfrac{\partial p_\kappa}{\partial \xi_r} = \dfrac{\partial}{\partial \xi_r}\sum_\kappa\left(q_\kappa \dfrac{\partial p_\kappa}{\partial \eta_s} - \xi_s\right). \end{cases}$$

On differentiating out, these at once reduce to (i), (ii), (iii) of §2.

A corollary is

$$\sum_\kappa(pd_\kappa q_\kappa - \eta_\kappa d\xi_\kappa) = d\sum_\kappa(p_\kappa q_\kappa) - \sum_\kappa(q_\kappa dp_\kappa - \xi_\kappa d\eta_\kappa)$$
$$= dS'.$$

Poincaré's form is elegant, easy to remember, and in general more convenient to apply than the alternative form. The object of introducing it here, however, is to discuss the integration of a canonic system by transforming it to a new and integrable canonic system.

§4. *Integration of the Canonic Equations.*

The form of $S(p, \eta)$ required to satisfy Poincaré's condition is arbitrary. Let us suppose some particular form written down, and let the change of variables be defined by

$$q_\kappa = \frac{\partial S(p, \eta)}{\partial p_\kappa}, \quad \xi_\kappa = -\frac{\partial S(p, \eta)}{\partial \eta_\kappa}, \quad \kappa = 1, 2, \ldots n. \qquad \text{(4)}$$

This definition is suggested by (3) as immediately giving a canonic change, but in order to be valid we must have

$$\frac{\partial S(p, \eta)}{\partial p_\kappa} \neq f(p), \quad \frac{\partial S(p, \eta)}{\partial \eta_\kappa} \neq \phi(\eta). \qquad \text{(5)}$$

By solving (4) we find $p_\kappa(\xi, \eta)$, $q_\kappa(\xi, \eta)$, and hence $H(\xi\,;\,\eta)$; but the success of the transformation depends on the suitableness of the $H(\xi\,;\,\eta)$ produced, that is, on the choice of $S(p, \eta)$.

Problem. Can a canonic change of variables be found which will convert the function $H(q, p)$ into an assigned form $f(\xi, \eta)$?

Not in general. For, if possible, we must have

$$\left.\begin{aligned} H(q, p) &= f(\xi, \eta)\,; \\ \therefore\ H\left(\frac{\partial S}{\partial p}, p\right) &= f\left(-\frac{\partial S}{\partial \eta}, \eta\right). \end{aligned}\right\} \qquad \text{(6)}$$

$S(p, \eta)$ must therefore be the integral of a partial differential equation whose form resolves it into

$$\left.\begin{aligned} H\left(\frac{\partial S}{\partial p}, p\right) = C, \\ f\left(-\frac{\partial S}{\partial \eta}, \eta\right) = C. \end{aligned}\right\} \qquad (7)$$

But these two equations give just that form of S which is excluded by (5). There are two exceptional cases however: when $H(q, p)$ does not contain q, and when $f(\xi, \eta)$ does not contain ξ; for, one of the equations (7) is no longer differential, and one set of the variables may be taken as the integration constants. If $H(q, p)$ does not contain q, the original system becomes immediately integrable. We are thus narrowed down to $f(\xi, \eta) \equiv \phi(\eta)$, and now the new system

$$\dot{\eta}_\kappa = -\frac{\partial \phi(\eta)}{\partial \xi_\kappa} = 0, \quad \dot{\xi}_\kappa = +\frac{\partial \phi(\eta)}{\partial \eta_\kappa}$$

is at once integrable, giving

$$\eta_\kappa = a_\kappa, \ \xi_\kappa = \frac{\partial \phi(a)}{\partial a_\kappa} \cdot t + \beta_\kappa \qquad (8)$$

$\phi(\eta)$ may be put equal to η_1, since it is arbitrary. This gives

$$\xi_1 = t + \beta_1 = -\frac{\partial S(p, \eta)}{\partial \eta_1},$$

$$\xi_\kappa = \beta_\kappa \quad = -\frac{\partial S(p, \eta)}{\partial \eta_\kappa}, \ \kappa \neq 1.$$

These are the integrals of the original system.

§5. *The Principle of Duality.*

The alternative form of Poincaré's condition shows that the change of variables may also be assumed in the form

$$p_\kappa = \frac{\partial S(q, \xi)}{\partial q_\kappa}, \quad \eta_\kappa = -\frac{\partial S(q, \xi)}{\partial \xi_\kappa}.$$

In other words, the integration is reduced to that of either of the equations

$$H\left(q, \frac{\partial S}{\partial q}\right) = \text{const.}, \qquad (9)$$

$$H\left(\frac{\partial S'}{\partial q}, p\right) = \text{const.} \qquad (9)a$$

This follows also from considerations of symmetry. H is of arbitrary form with respect to both p and q, and the canonic system does not change its form when p, q are interchanged and the sign of H altered.

When the system referred to is dynamical $H(p, q)$ is, in general, quadratic with reference to the p's (the momenta) but arbitrary with reference to the q's (the coordinates); and this restriction probably accounts for the fact that little mention is made of the Principle of Duality in the literature of the subject. Both methods remain valid, of course, in dynamics, but in general (9) is more easily integrated than (9)a. Elementary cases, however, are not wanting in which there is little to choose between the alternatives. As it will not generally be known that the unpromising case of the problem of two bodies can be treated by (9)a, it seems worth while to indicate the steps of its integration.

§6. *Procedure for Dynamical Problems.*

Form the K.E. function $T(q, \dot{q})$, and express in terms of q, p by means of the relations $p_\kappa = \dfrac{\partial T(q, \dot{q})}{\partial \dot{q}_\kappa}$. Form the force function $U(q)$.

Then $$H(q, p) \equiv T(q, p) - U(q).$$

(i) Solve $$H\left(q, \frac{\partial S}{\partial q}\right) = a_1.$$

Then the integrals are

$$t + \beta_1 = + \frac{\partial S(q, a)}{\partial a_1},$$

$$\beta_\kappa = + \frac{\partial S(q, a)}{\partial a_\kappa}, \quad \kappa \neq 1;$$

or (ii) Solve $$H\left(\frac{\partial S'}{\partial p}, p\right) = a_1.$$

Then the integrals are

$$t + \beta_1 = - \frac{\partial S(p, a)}{\partial a_1},$$

$$\beta_\kappa = - \frac{\partial S(p, a)}{\partial a_\kappa}, \quad \kappa \neq 1.$$

In each case the remaining variables can be obtained as functions of t by elimination.

§7. *Outline of the Solutions for the Problem of Two Bodies.*

(For elliptic motion a_1 is negative: the sign has been changed throughout for convenience).

If $r=q$ and $\theta=q_2$, we find

$$2H(q,p)=p_1^2+\frac{1}{q_1^2}p_2^2-\frac{\mu}{q_1}=-2a_1(\mu,\ a_1 \text{ positive}) \quad\text{.........}(10)$$

q_2 is an ignorable coordinate, so that $p_2=c$ (const.)

First method.

Dropping suffixes, we have

$$\frac{\partial S}{\partial q}=\sqrt{\left(-2a+\frac{\mu}{q}-\frac{c^2}{q^2}\right)};$$

$$\therefore\ t+\beta=-\frac{\partial S}{\partial a}=\int\frac{dq}{\sqrt{\left(-2a+\frac{\mu}{q}-\frac{c^2}{q^2}\right)}} \quad\text{..............................}(11)$$

$$=-\frac{1}{4a}.\int\frac{(\mu-4aq)dq}{\sqrt{(-2aq^2+\mu q-c^2)}}+\frac{\mu}{4a\sqrt{2a}}.\int\frac{dq}{\sqrt{(q-k_1)(k_2-q)}},$$

where $$k_1+k_2=\frac{\mu}{2a} \text{ and } k_1k_2=\frac{c_2}{2a}; \quad\text{....................}(12)$$

$$\therefore\ t+\beta=-\frac{1}{\sqrt{2a}}\sqrt{(q-k_1(k_2-q)}+\frac{\mu}{2a\sqrt{2a}}\sin^{-1}\sqrt{\left(\frac{q-k_1}{k_2-k_1}\right)} \quad\text{.......}(13)$$

If we put $$q-k_1=(k_2-k_1)\sin^2\frac{E}{2},$$

we derive Kepler's equation

$$n(t+\beta)=E-e\sin E,$$

where $$n=\frac{4a\sqrt{2a}}{\mu} \text{ and } e=\frac{2a}{\mu}(k_2-k_1).$$

Another form of the second integral in (13) is

$$\frac{\mu}{4a\sqrt{2a}}.\sin^{-1}\frac{2q-(k_1+k_2)}{k_2-k_1}.$$

Second method.

From (10) we have to integrate

$$\left(\frac{\partial S'}{\partial p}\right)^2(p^2+2a)-\mu\frac{\partial S'}{\partial p}+c^2=0.$$

Solving as a quadratic,

$$\frac{\partial S'}{\partial p} = \frac{\mu}{2A} \pm \frac{\sqrt{B}}{2A},$$

where

$$\left.\begin{aligned} A &\equiv p^2 + 2a = \frac{\mu q - c^2}{q^2}, \\ B &\equiv \mu^2 - 4c^2 A = \left(\mu - \frac{2c^2}{q}\right)^2. \end{aligned}\right\} \quad \text{..................(14)}$$

If we take $\sqrt{B} = +\left(\mu - \frac{2c^2}{q}\right)$, only the upper sign for $\frac{\partial S'}{\partial p}$ can be taken, since

$$\mu + \sqrt{B} = \frac{2(\mu q - c^2)}{q} = 2Aq \text{ by (14).}$$

Corresponding to (11) we have

$$t + \beta = \frac{\partial S'}{\partial a} = \int \frac{\partial}{\partial a}\left(\frac{\mu}{2A} + \frac{\sqrt{B}}{2A}\right) dp$$

$$= \int \left(-\frac{\mu}{A^2} - \frac{\sqrt{B}}{A^2} - \frac{2c^2}{A\sqrt{B}}\right) dp$$

$$= \int \left(-\frac{\mu}{A^2} + \frac{2c^2}{A\sqrt{B}} - \frac{\mu^2}{A^2\sqrt{B}}\right) dp \text{...........(15)}$$

The first and third terms are reduced by the relations

$$\frac{d}{dp}\left(\frac{p}{A}\right) = -\frac{1}{A} + \frac{4a}{A^2},$$

$$\frac{d}{dp}\left(\frac{p\sqrt{B}}{A}\right) = -\frac{(\mu^2 - 8c^2 a)}{A\sqrt{B}} + \frac{4a\mu^2}{A^2\sqrt{B}}.$$

Hence we find

$$\int \frac{\mu}{A^2} dp = \frac{\mu}{4a} \cdot \frac{p}{A} + \frac{\mu}{4a} \int \frac{dp}{p^2 + 2a}$$

$$= \frac{\mu}{4a} \cdot \frac{p}{A} + \frac{\mu}{4a\sqrt{2a}} \tan^{-1} \frac{p}{\sqrt{2a}} \text{...........(16)}$$

and

$$\int \frac{\mu^2}{A^2\sqrt{B}} dp = \frac{1}{4a} \cdot \frac{p\sqrt{B}}{A} + \frac{\mu^2 - 8c^2 a}{4a} \int \frac{dp}{A\sqrt{B}} \text{. ......(17)}$$

The remaining integral $\int \frac{dp}{A\sqrt{B}}$ is of standard form, and is easily shown to be

$$-\frac{1}{\mu\sqrt{2a}}\sin^{-1}\sqrt{\frac{2a}{\mu^2-8c^2a}\cdot\frac{B}{A}}$$

for a positive.

On substituting this with (16) and (17) in (15), the result is

$$\left.\begin{aligned} t+\beta = & -\frac{p}{4aA}(\mu+\sqrt{B}) \\ & +\frac{\mu}{4a\sqrt{2a}}\left(\sin^{-1}\sqrt{\frac{2a}{\mu^2-8c^2a}\cdot\frac{B}{A}}-\tan^{-1}\frac{p}{\sqrt{2a}}\right). \end{aligned}\right\} \ldots\ldots(18)$$

The last term is simplified by putting

$$\tan^{-1}\frac{p}{\sqrt{2a}}=\sin^{-1}\frac{p}{\sqrt{A}}$$

and applying the formula

$$\sin^{-1}x-\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}-y\sqrt{1-x^2}).$$

This gives $\qquad \sin^{-1}\frac{2}{A\sqrt{\mu^2-8c^2a}}\left(a\sqrt{B}-\frac{\mu p^2}{2}\right).$

Now apply (14) to eliminate p from the whole expression in (18).

$$\begin{aligned} a\sqrt{B}-\frac{\mu p^2}{2} &= a\left(\mu-\frac{2c^2}{q}\right)-\frac{\mu}{2}\left(-2a+\frac{\mu}{q}-\frac{c^2}{q^2}\right) \\ &= \frac{1}{2q^2}(4aq-\mu)(\mu q-c^2) \\ &= (4aq-\mu)\cdot\frac{A}{2}; \end{aligned}$$

$$\therefore \quad -\frac{p}{4aA}(\mu+\sqrt{B})=-\frac{pq}{2a}=\frac{1}{2a}\sqrt{(q-k_1)(k_2-q)}$$

and $\qquad \dfrac{2}{A\sqrt{(\mu^2-8c^2a)}}\cdot\left(a\sqrt{B}-\dfrac{\mu p^2}{2}\right)=\dfrac{4aq-\mu}{\sqrt{(\mu^2-8c^2a)}}$

$$=\frac{2q-(k_1+k_2)}{k_2-k_1}, \text{ by } (12)$$

Hence (13) is again found (with the second form of the second integral).

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