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LARGE DEVIATIONS FOR THE ORNSTEIN–UHLENBECK PROCESS WITH SHIFT

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Abstract

We investigate the large deviation properties of the maximum likelihood estimators for the Ornstein–Uhlenbeck process with shift. We propose a new approach to establish large deviation principles which allows us, via a suitable transformation, to circumvent the classical nonsteepness problem. We estimate simultaneously the drift and shift parameters. On the one hand, we prove a large deviation principle for the maximum likelihood estimates of the drift and shift parameters. Surprisingly, we find that the drift estimator shares the same large deviation principle as the estimator previously established for the Ornstein–Uhlenbeck process without shift. Sharp large deviation principles are also provided. On the other hand, we show that the maximum likelihood estimator of the shift parameter satisfies a large deviation principle with a very unusual implicit rate function.

Keywords: Ornstein–Uhlenbeck process with shift; maximum likelihood estimate; large deviation

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1. Introduction

Consider the Ornstein–Uhlenbeck process with linear shift $\gamma \in \mathbb{R}$, observed over the time interval $[0, T]$,

$$dX_t = \theta X_t \, dt + \gamma \, dt + dB_t, \tag{1.1}$$

where the drift parameter $\theta < 0$, the initial state $X_0 = 0$, and the driven noise (B_t) is a standard Brownian motion. This process is widely used in financial mathematics and it is known as the Vasicek model; see, e.g. [10], [13]. The maximum likelihood estimates of the unknown parameters θ and γ are given by

$$\widehat{\theta}_T = \frac{T \int_0^T X_t \, dX_t - X_T \int_0^T X_t \, dt}{T \int_0^T X_t^2 \, dt - (\int_0^T X_t \, dt)^2} \tag{1.2}$$

and

$$\widehat{\gamma}_T = \frac{X_T \int_0^T X_t^2 \, dt - \int_0^T X_t \, dX_t \int_0^T X_t \, dt}{T \int_0^T X_t^2 \, dt - (\int_0^T X_t \, dt)^2}. \tag{1.3}$$

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A wide range of literature is available on the asymptotic behavior of $(\widehat{\theta}_T)$ and $(\widehat{\gamma}_T)$. It is well known (see [12]) that $\widehat{\theta}_T$ and $\widehat{\gamma}_T$ are both strongly consistent estimators of θ and γ, and their joint asymptotic normality is given by

$$\sqrt{T}\begin{pmatrix}\widehat{\theta}_T-\theta\\ \widehat{\gamma}_T-\gamma\end{pmatrix} \xrightarrow{\mathcal{L}} \mathcal{N}(0, L),$$

where the limiting matrix is

$$L = 2\begin{pmatrix}\theta & \gamma\\ \gamma & \kappa\end{pmatrix}$$

with $\kappa = (2\gamma^2+\theta)/2\theta$. Moreover, concentration inequalities for $(\widehat{\theta}_T)$ and $(\widehat{\gamma}_T)$ and moderate deviations were established by Gao and Jiang [8], while Jiang [11] recently obtained the joint law of an iterated logarithm as well as Berry–Esseen bounds. In the particular case $\gamma = 0$, Florens-Landais and Pham [7] proved the large deviation principle (LDP) for $(\widehat{\theta}_T)$, while sharp large deviation principles (SLDPs) were established in [1]. We also refer the reader to [3] for the sharp large deviations in the nonstationary case $\theta \geq 0$ and $\gamma = 0$.

Our goal is to extend these investigations by establishing the large deviations properties of the maximum likelihood estimators of the drift and shift parameters $\theta < 0$ and γ in the situation where θ and γ are estimated simultaneously. We shall propose a new approach to prove the LDP which allows us, via a suitable transformation, to circumvent the classical nonsteepness problem. In particular, it could be possible to apply the same approach for Jacobi or Cox–Ingersoll–Ross processes [5], [14].

The paper is organized as follows. In Section 2 we establish an LDP for the couple $(\widehat{\theta}_T, \widehat{\gamma}_T)$. Via the contraction principle, we realize that $(\widehat{\theta}_T)$ shares the same LDP as the one previously established for the Ornstein–Uhlenbeck process without shift. We also observe that $(\widehat{\gamma}_T)$ satisfies an LDP with a very unusual implicit rate function. An SLDP for the sequence $(\widehat{\theta}_T)$ is also provided. Section 3 is devoted to three key lemmas which are at the core of our analysis. All the technical proofs of Sections 2 and 3 are postponed to Appendices A, B, and C.

2. The results for large deviations

Our large deviations results are as follows.

Theorem 2.1. *The couple $(\widehat{\theta}_T, \widehat{\gamma}_T)$ satisfies an LDP with good rate function*

$$I_{\theta,\gamma}(c,d) = \begin{cases} -\dfrac{(\theta-c)^2}{4c} + \dfrac{1}{2}\Big(\gamma - \dfrac{d\theta}{c}\Big)^2 & \text{if } c \leq \dfrac{\theta}{3},\\ (2c-\theta) + \dfrac{1}{2}\Big(\gamma - \dfrac{d\theta}{c}\Big)^2 & \text{if } c \geq \dfrac{\theta}{3} \text{ and } c \neq 0,\\ -\theta & \text{if } (c,d) = (0,0),\\ +\infty & \text{if } c = 0 \text{ and } d \neq 0. \end{cases} \tag{2.1}$$

A direct application of the contraction principle [4] immediately leads to the two following corollaries.

Corollary 2.1. *The sequence $(\widehat{\theta}_T)$ satisfies an LDP with good rate function*

$$I_{\theta}(c) = \begin{cases} -\dfrac{(c-\theta)^2}{4c} & \text{if } c \leq \dfrac{\theta}{3},\\ 2c-\theta & \text{if } c \geq \dfrac{\theta}{3}. \end{cases} \tag{2.2}$$

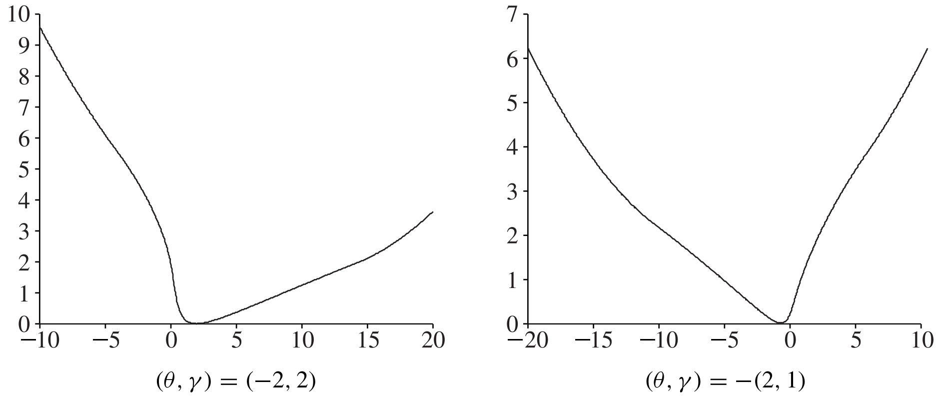

FIGURE 1: Rate functions for the drift parameter in the special cases $(\theta, \gamma) = (-2, 2)$ *(left)* and $(\theta, \gamma) = -(2, 1)$ *(right)*.

Corollary 2.2. *The sequence $(\widehat{\gamma}_T)$ satisfies an LDP with good rate function*

$$I_\gamma(d) = \inf\{I_{\theta,\gamma}(c, d) \mid c \in \mathbb{R}\}.$$

Proof. The proofs are given in Section 4.

Remark 2.1. On the one hand, we observe that $(\widehat{\theta}_T)$ shares exactly the same LDP as the one previously established by Florens-Landais and Pham [7] for the Ornstein–Uhlenbeck without shift $\gamma = 0$. On the other hand, $(\widehat{\gamma}_T)$ satisfies an LDP with a very unusual rate function. Unfortunately, an explicit expression for this rate function is quite complicated. Its very particular form in the special cases $(\theta, \gamma) = (-2, 2)$ and $(\theta, \gamma) = -(2, 1)$ is shown in Figure 1.

Our goal now is to improve Corollary 2.1 by a first-order SLDP for $(\widehat{\theta}_T)$. It is of course possible to establish an SLDP of any order for the sequence $(\widehat{\theta}_T)$. However, for clarity, we have chosen to restrict ourself to a first-order expansion.

Theorem 2.2. *Consider the Ornstein–Uhlenbeck process with shift given by (1.1) where the drift parameter $\theta < 0$.*

(i) *For all $\theta < c < \theta/3$, we have, for large enough T,*

$$\mathbb{P}\{\widehat{\theta}_T \geq c\} = \frac{e^{-TI_\theta(c)+J(c)}}{a_c\sigma_c\sqrt{2\pi T}}(1 + o(1)), \tag{2.3}$$

while, for $c < \theta$,

$$\mathbb{P}\{\widehat{\theta}_T \leq c\} = -\frac{e^{-TI_\theta(c)+J(c)}}{a_c\sigma_c\sqrt{2\pi T}}(1 + o(1)), \tag{2.4}$$

where

$$a_c = \frac{c^2 - \theta^2}{2c}, \qquad \sigma_c^2 = -\frac{1}{2c}, \tag{2.5}$$

$$J(c) = -\frac{1}{2}\log\left(\frac{\theta^2(c+\theta)(3c-\theta)}{4c^4}\right) + \gamma^2\frac{(c-\theta)^2}{4c\theta^2}.$$

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(ii) *For all $c > \theta/3$ with $c \neq 0$, we have, for large enough T,*

$$\mathbb{P}\{\widehat{\theta}_T \geq c\} = \frac{\mathrm{e}^{-TI_\theta(c)+K(c)}}{a_c\sigma_c\sqrt{2\pi T}}(1+o(1)), \tag{2.6}$$

where

$$a_c = 2(c-\theta), \qquad \sigma_c^2 = \frac{c^2}{2(2c-\theta)^3}, \tag{2.7}$$
$$K(c) = -\frac{1}{2}\log\biggl(\frac{\theta^2(c-\theta)(3c-\theta)}{4c^2(2c-\theta)^2}\biggr) - \frac{\gamma^2}{\theta^2}(2c-\theta).$$

(iii) *For $c = \theta/3$, we have, for large enough T,*

$$\mathbb{P}\{\widehat{\theta}_T \geq c\} = \frac{\mathrm{e}^{-TI_\theta(c)+\gamma^2 b_\theta}}{6\pi T^{1/4}}\frac{\Gamma(1/4)}{\sqrt{2}a_\theta^{3/4}\sigma_\theta}(1+o(1)), \tag{2.8}$$

where

$$a_\theta = -\frac{4\theta}{3}, \qquad b_\theta = \frac{1}{3\theta}, \qquad \sigma_\theta^2 = -\frac{3}{2\theta}. \tag{2.9}$$

(iv) *For $c = 0$, we have, for large enough T,*

$$\mathbb{P}\{\widehat{\theta}_T \geq c\} = \frac{\sqrt{2}\mathrm{e}^{-TI_\theta(c)+\gamma^2/\theta+2}}{\sqrt{2\pi T}\sqrt{-\theta}}(1+o(1)).$$

3. Three key lemmas

Firstly, let us recall some elementary properties of the Ornstein–Uhlenbeck process with linear shift [10], [12]. We observe that the process (X_T) can be rewritten as $X_T = Y_T + m_T$, where

$$m_T = \mathbb{E}\{X_T\} = -\frac{\gamma}{\theta}(1-\mathrm{e}^{\theta T})$$

and (Y_T) is the Ornstein–Uhlenbeck process without shift

$$Y_T = \mathrm{e}^{\theta T}\int_0^T \mathrm{e}^{-\theta t}\,\mathrm{d}B_t,$$

By the same token, if

$$\overline{X}_T = \frac{1}{T}\int_0^T X_t\,\mathrm{d}t, \qquad \overline{Y}_T = \frac{1}{T}\int_0^T Y_t\,\mathrm{d}t,$$

we clearly have $\overline{X}_T = \overline{Y}_T + \mu_T$, where

$$\mu_T = \mathbb{E}\{\overline{X}_T\} = -\frac{\gamma}{\theta}\biggl(1 + \frac{1}{\theta T}(1-\mathrm{e}^{\theta T})\biggr).$$

Therefore, the random vector

$$\begin{pmatrix} X_T \\ \overline{X}_T \end{pmatrix} \sim \mathcal{N}\biggl(\begin{pmatrix} m_T \\ \mu_T \end{pmatrix}, \Gamma_T(\theta)\biggr),$$

where the covariance matrix $\Gamma_T(\theta)$ is given by

$$\Gamma_T(\theta) = \begin{pmatrix} a_T(\theta) & b_T(\theta) \\ b_T(\theta) & c_T(\theta) \end{pmatrix}, \tag{3.1}$$

with

$$a_T(\theta) = \frac{1}{2\theta}(\mathrm{e}^{2\theta T} - 1), \qquad b_T(\theta) = \frac{1}{2\theta^2 T}(\mathrm{e}^{\theta T} - 1)^2,$$
$$c_T(\theta) = \frac{1}{\theta^2 T^2}\bigg(\frac{1}{2\theta}(\mathrm{e}^{2\theta T} - 1) - \frac{2}{\theta}(\mathrm{e}^{\theta T} - 1) + T\bigg).$$

Denote by Λ_T the normalized cumulant generating function of the triplet

$$\bigg(\frac{X_T}{\sqrt{T}}, \frac{1}{T}\int_0^T X_t^2 \,\mathrm{d}t, \frac{1}{T}\int_0^T X_t \,\mathrm{d}t\bigg)$$

defined, for all $(a, b, c) \in \mathbb{R}^3$, by

$$\Lambda_T(a, b, c) = \frac{1}{T}\log \mathbb{E}\bigg\{\exp\bigg(a\sqrt{T}X_T + b\int_0^T X_t^2 \,\mathrm{d}t + c\int_0^T X_t \,\mathrm{d}t\bigg)\bigg\}.$$

Our first lemma deals with the extended real function Λ defined as the pointwise limit of Λ_T.

Lemma 3.1. *Let $\mathcal{D}_\Lambda$ be the effective domain of Λ,*

$$\mathcal{D}_\Lambda = \bigg\{(a, b, c) \in \mathbb{R}^3 \;\bigg|\; b < \frac{\theta^2}{2}\bigg\},$$

and set $\varphi(b) = \sqrt{\theta^2 - 2b}$. Then, for all $(a, b, c) \in \mathcal{D}_\Lambda$, we have

$$\Lambda(a, b, c) = -\frac{1}{2}(\theta + \varphi(b) + \gamma^2) + \frac{1}{2}\bigg(\frac{a^2}{\varphi(b) - \theta}\bigg) + \frac{1}{2}\bigg(\frac{c - \theta\gamma}{\varphi(b)}\bigg)^2.$$

Proof. The proof in given in Appendix A.

A direct calculation shows that the function Λ is steep, which means that the norm of its gradient goes to ∞ for any sequence in the interior of $\mathcal{D}_\Lambda$, converging to a boundary point of $\mathcal{D}_\Lambda$. This is the reason why we are able to deduce an LDP for the couple $(\widehat{\theta}_T, \widehat{\gamma}_T)$. In order to establish the SLDP for the drift parameter $\widehat{\theta}_T$, it is necessary to modify our strategy. To be more precise, we shall now focus our attention on the normalized cumulant generating function $\mathcal{L}_T$ of the couple

$$\bigg(\frac{1}{T}\int_0^T (X_t - \overline{X}_T)\,\mathrm{d}X_t, \frac{1}{T}S_T\bigg),$$

where

$$S_T = \int_0^T (X_t - \overline{X}_T)^2 \,\mathrm{d}t,$$

which is given, for all $(a, b) \in \mathbb{R}^2$, by

$$\mathcal{L}_T(a, b) = \frac{1}{T}\log \mathbb{E}\bigg\{\exp\bigg(a\int_0^T (X_t - \overline{X}_T)\,\mathrm{d}X_t + bS_T\bigg)\bigg\}.$$

The reason for this is twofold. On the one hand, it is not possible to deduce an SLDP for $(\widehat{\theta}_T)$ via Λ_T. On the other hand, it immediately follows from (1.2) that

$$\widehat{\theta}_T = \frac{\int_0^T (X_t - \overline{X}_T)\,\mathrm{d}X_t}{S_T}.$$

However, we observe that, for all $c \in \mathbb{R}$, $\mathbb{P}\{\widehat{\theta}_T \geq c\} = \mathbb{P}\{\mathcal{Z}_T(1, -c) \geq 0\}$, where, for all $(a, b) \in \mathbb{R}^2$, $\mathcal{Z}_T(a, b)$ stands for the random variable

$$\mathcal{Z}_T(a, b) = a \int_0^T (X_t - \overline{X}_T)\,\mathrm{d}X_t + bS_T. \tag{3.2}$$

Our second lemma provides the full asymptotic expansion for $\mathcal{L}_T$. Denote by $\mathcal{L}$ the extended real function defined as the pointwise limit of $\mathcal{L}_T$.

Lemma 3.2. *Let $\mathcal{D}_{\mathcal{L}}$ be the effective domain of $\mathcal{L}$,*

$$\mathcal{D}_{\mathcal{L}} = \{(a, b) \in \mathbb{R}^2 \mid \theta^2 - 2b > 0 \text{ and } a + \theta < \sqrt{\theta^2 - 2b}\},$$

and set $\varphi(b) = \sqrt{\theta^2 - 2b}$, $\tau(a, b) = \varphi(b) - (a + \theta)$. Then, for all $(a, b) \in \mathcal{D}_{\mathcal{L}}$ and large enough T, we have

$$\mathcal{L}_T(a, b) = \mathcal{L}(a, b) + \frac{1}{T}\mathcal{H}(a, b) + \frac{1}{T^2}\mathcal{R}_T(a, b),$$

where

$$\mathcal{L}(a, b) = -\tfrac{1}{2}(a + \theta + \sqrt{\theta^2 - 2b}),$$

$$\mathcal{H}(a, b) = -\frac{1}{2}\log\left(\frac{\tau(a, b)\theta^2}{2\varphi^3(b)}\right) - \frac{\gamma^2(a + \theta + \sqrt{\theta^2 - 2b})}{2\theta^2}.$$

Moreover, the remainder $\mathcal{R}_T(a, b)$ may be explicitly calculated as a rational function of a, b, T and $\exp(-\varphi(b)T)$. In addition, $\mathcal{R}_T$ can be extended to the two-dimensional complex plane, and it is a bounded analytic function as soon as the real parts of its arguments belong to the interior of $\mathcal{D}_{\mathcal{L}}$.

Proof. The proof is given in Appendix B.

Our third lemma relies on the Karhunen–Loève expansion of the process (X_T). Denote by $\mathcal{F}$ the class of all real-valued continuous functions f such that $f(x) = x^2 h(x)$, where h is continuous. Moreover, let g be the spectral density of the stationary Ornstein–Uhlenbeck process without shift $\gamma = 0$ given, for all $x \in \mathbb{R}$, by

$$g(x) = \frac{1}{\theta^2 + x^2}. \tag{3.3}$$

Lemma 3.3. *We can find two sequences of real numbers (α_k^T) and (β_k^T) both in $\ell^2(\mathbb{N})$ such that*

$$\mathcal{Z}_T(a, b) = \mathbb{E}\{\mathcal{Z}_T(a, b)\} + \sum_{k=1}^{\infty} \alpha_k^T(\varepsilon_k^2 - 1) + \beta_k^T \varepsilon_k,$$

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where (ε_k) are independent standard $\mathcal{N}(0,1)$ random variables. Moreover, for all $(a,b) \in \mathcal{D}_{\mathcal{L}}$, there exist two constants $A > 0$ and $B > 0$ that do not depend on T such that, for large enough T, $\alpha_k^T \in [-A, A]$ for all $k \geq 1$ and

$$\sum_{k=1}^{+\infty} (\beta_k^T)^2 < B. \tag{3.4}$$

Consequently, for all $(a,b) \in \mathcal{D}_{\mathcal{L}}$ and $x \in \mathbb{R}$ such that $|x| < \frac{1}{2}A$, and for large enough T, we have

$$\mathcal{L}_T(xa, xb) = \frac{1}{T}\mathbb{E}\{\mathcal{Z}_T(xa, xb)\} - \frac{1}{2T}\sum_{k=1}^{\infty}(\log(1 - 2x\alpha_k^T) + 2x\alpha_k^T) + \frac{1}{2T}\sum_{k=1}^{\infty}\frac{(x\beta_k^T)^2}{1 - 2x\alpha_k^T}.$$

Finally, if $b \neq 0$, the empirical spectral measure

$$\nu_T = \frac{1}{T}\sum_{k=1}^{\infty}\delta_{\alpha_k^T}$$

satisfies, for any continuous $f \in \mathcal{F}$ with compact support,

$$\lim_{T\to\infty}\langle \nu_T, f\rangle = \lim_{T\to\infty}\frac{1}{T}\sum_{k=1}^{\infty} f(\alpha_k^T) = \langle \nu, f\rangle = \frac{1}{2\pi}\int_{\mathbb{R}} f(bg(x))\,\mathrm{d}x. \tag{3.5}$$

Proof. The proof is given in Appendix C.

4. Proofs of the large deviations results

Firstly,

$$\widehat{V}_T = \begin{pmatrix}\widehat{\theta}_T \\ \widehat{\gamma}_T\end{pmatrix}, \qquad \widetilde{V}_T = \begin{pmatrix}\widetilde{\theta}_T \\ \widetilde{\gamma}_T\end{pmatrix},$$

where

$$\widetilde{\theta}_T = \frac{\int_0^T X_t\,\mathrm{d}X_t}{S_T}, \qquad \widetilde{\gamma}_T = -\widetilde{\theta}_T\overline{X}_T. \tag{4.1}$$

In the following lemma we show that the sequences $(\widehat{V}_T)$ and $(\widetilde{V}_T)$ share the same LDP. We refer the reader to [4] for the classical notion of exponential approximation.

Lemma 4.1. *The sequences of random vectors $(\widehat{V}_T)$ and $(\widetilde{V}_T)$ are exponentially equivalent, that is to say, for all $\varepsilon > 0$,*

$$\lim_{T\to+\infty}\frac{1}{T}\log\mathbb{P}\{\|\widehat{V}_T - \widetilde{V}_T\| > \varepsilon\} = -\infty.$$

In particular, if $(\widetilde{V}_T)$ satisfies an LDP with good rate function I, then the same LDP holds for $(\widehat{V}_T)$.

Proof. It is easy to see from the definition of our estimates given in (1.2), (1.3), and (4.1) that

$$\widehat{\theta}_T - \widetilde{\theta}_T = \frac{X_T}{T}\left(\frac{\overline{X}_T}{\Sigma_T}\right), \qquad \widehat{\gamma}_T - \widetilde{\gamma}_T = \frac{X_T}{T}\left(1 - \frac{(\overline{X}_T)^2}{\Sigma_T}\right),$$

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where $\Sigma_T = S_T/T$. On the event $\{|\overline{X}_T| \le \xi,\ \Sigma_T \ge \xi^{-1}\}$ with $\xi > 1$, we have

$$\|\widehat{V}_T - \widetilde{V}_T\| \le \sqrt{3}\xi^3 \frac{|X_T|}{T}.$$

Hence, for all $\varepsilon > 0$,

$$\mathbb{P}\{\|\widehat{V}_T - \widetilde{V}_T\| > \varepsilon\} \le \mathbb{P}\left\{\frac{|X_T|}{T} \ge \frac{\varepsilon\xi^{-3}}{\sqrt{3}}\right\} + \mathbb{P}\{|\overline{X}_T| \ge \xi\} + \mathbb{P}\{\Sigma_T \le \xi^{-1}\}. \tag{4.2}$$

On the one hand, it is not hard to see that for all $c > 0$,

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\left\{\frac{|X_T|}{T} \ge c\right\} = -\infty. \tag{4.3}$$

We recall that X_T is a Gaussian $\mathcal{N}(m_T, a_T(\theta))$ random variable. Consequently, for all $c > 0$,

$$\lim_{T\to\infty} \frac{1}{T^2} \log \mathbb{P}\{|X_T| \ge cT\} = \theta c^2,$$

which immediately leads to (4.3), as $\theta < 0$. By the same token, from the beginning of Section 3, since we know that $\overline{X}_T$ is a Gaussian $\mathcal{N}(\mu_T, c_T(\theta))$ random variable. It implies that for all $c > 0$ such that $c > |\gamma|/|\theta|$,

$$\limsup_{T\to+\infty} \frac{1}{T} \log \mathbb{P}\{|\bar{X}_T| \ge c\} \le -\frac{\theta^2}{2}\left(c + \left|\frac{\gamma}{\theta}\right|\right)^2. \tag{4.4}$$

On the other hand, we immediately deduce from Lemma 3.2 together with Gärtner–Ellis's theorem that the sequence (Σ_T) satisfies an LDP with speed T and good rate function

$$I(c) = \begin{cases} \dfrac{(2\theta c + 1)^2}{8c} & \text{if } c > 0, \\ +\infty & \text{if } c \le 0. \end{cases}$$

Therefore, for all $c > 0$ such that $-2\theta c < 1$,

$$\lim_{T\to\infty} \frac{1}{T} \log \mathbb{P}\{\Sigma_T \le c\} = -\frac{(2\theta c + 1)^2}{8c}. \tag{4.5}$$

Finally, it follows from the conjunction of (4.2)–(4.5) that, for all $\varepsilon > 0$ and for $\xi > 1$ large enough,

$$\limsup_{T\to+\infty} \frac{1}{T} \log \mathbb{P}\{\|\widehat{V}_T - \widetilde{V}_T\| > \varepsilon\} \le -M_{\theta,\gamma}(\varepsilon, \xi),$$

where

$$M_{\theta,\gamma}(\varepsilon, \xi) = \min\left(\frac{\theta^2}{2}\left(\xi + \left|\frac{\gamma}{\theta}\right|\right)^2, \frac{\xi}{8}\left(\frac{2\theta}{\xi} + 1\right)^2\right).$$

We observe that if ξ goes to ∞, $M_{\theta,\gamma}(\varepsilon, \xi)$ tend to ∞, which is exactly what we wanted to prove.

We are now in the position to prove our LDP results. Our strategy is to establish an LDP for the triplet

$$\left(\frac{X_T}{\sqrt{T}}, \frac{1}{T}\int_0^T X_t^2\,\mathrm{d}t, \frac{1}{T}\int_0^T X_t\,\mathrm{d}t\right)$$

and then to make use of the contraction principle [4] in order to prove Theorem 2.1 via Lemma 4.1. The limiting cumulant generating function Λ of the above triplet was calculated in Lemma 3.1. It is not difficult to check that the function Λ is steep on its effective domain $\mathcal{D}_\Lambda$. Consequently, we deduce from Gärtner–Ellis's theorem that the above triplet satisfies an LDP with good rate function I given by the Fenchel–Legendre transform of Λ,

$$I(\lambda, \mu, \delta) = \sup_{(a,b,c)\in\mathcal{D}_\Lambda} \{\lambda a + \mu b + \delta c - \Lambda(a, b, c)\}.$$

After some straightforward calculations, we shall prove that

$$I(\lambda, \mu, \delta) = \begin{cases} \dfrac{\theta^2\mu - \theta\lambda^2}{2} + \dfrac{\theta + \gamma^2 + 2\theta\gamma\delta}{2} + \dfrac{(1+\lambda^2)^2}{8(\mu - \delta^2)} & \text{if } \delta^2 < \mu, \\ +\infty & \text{if } \delta^2 \geq \mu. \end{cases}$$

Hereafter, it follows from the well-known Itô formula that

$$\int_0^T X_t\,\mathrm{d}X_t = \frac{1}{2}(X_T^2 - T), \tag{4.6}$$

which implies that

$$\widetilde{V}_T = f\left(\frac{X_T}{\sqrt{T}}, \frac{1}{T}\int_0^T X_t^2\,\mathrm{d}t, \frac{1}{T}\int_0^T X_t\,\mathrm{d}t\right),$$

where f is the continuous function given, for all $(\lambda, \mu, \delta) \in \mathbb{R}^3$ such that $\mu > \delta^2$, by

$$f(\lambda, \mu, \delta) = \begin{pmatrix} \dfrac{\lambda^2 - 1}{2(\mu - \delta^2)} \\ -\dfrac{\delta(\lambda^2 - 1)}{2(\mu - \delta^2)} \end{pmatrix}.$$

Therefore, we infer from the contraction principle, see, e.g. [4, Theorem 4.2.1], together with Lemma 4.1, that the sequences of random vectors $(\widehat{V}_T)$ and $(\widetilde{V}_T)$ share the same LDP with good rate function

$$I_{(\theta,\gamma)}(c, d) = \inf\left\{ I(\lambda, \mu, \delta) \;\middle|\; (\lambda, \mu, \delta) \in \mathbb{R}^3, \mu > \delta^2, f(\lambda, \mu, \delta) = \begin{pmatrix} c \\ d \end{pmatrix} \right\},$$

where the infimum over the empty set is equal to $+\infty$. Finally, we obtain the rate function (2.1) thanks to elementary calculations, which completes the proof of Theorem 2.1.

5. Proofs of the sharp large deviations results

5.1. Proof of Theorem 2.2 (first part)

Firstly, it follows from straightforward calculations that the effective domain $\mathcal{D}_{\mathcal{L}}$ given in Lemma 3.2 can be rewritten as

$$\mathcal{D}_{\mathcal{L}} = \begin{cases} \left(-\infty, \dfrac{-\theta^2}{2c}\right) & \text{if } \theta < c \leq \dfrac{\theta}{2}, \\ (-\infty, 2(c-\theta)) & \text{if } \dfrac{\theta}{2} < c \leq 0, \\ \left(\dfrac{-\theta^2}{2c}, 2(c-\theta)\right) & \text{if } c > 0. \end{cases}$$

In addition, for all $a \in \mathcal{D}_{\mathcal{L}}$, let $\varphi(a) = \sqrt{\theta^2 + 2ac}$ and denote

$$L(a) = \mathcal{L}(a, -ac) = -\tfrac{1}{2}(a + \theta + \sqrt{\theta^2 + 2ac}), \tag{5.1}$$

$$H(a) = \mathcal{H}(a, -ac) = -\frac{1}{2}\log\left(\frac{(\varphi(a) - a - \theta)\theta^2}{2\varphi^3(a)}\right) - \frac{\gamma^2(a + \theta + \varphi(a))}{2\theta^2}. \tag{5.2}$$

The function L is not steep as the derivative of L is finite at the boundary of $\mathcal{D}_{\mathcal{L}}$. Moreover, $L'(a) = 0$ if and only if $a = a_c$ with a_c given by (2.5). Finally, we observe that $a_c \in \mathcal{D}_{\mathcal{L}}$ only if $c < \theta/3$. We shall focus our attention on the SLDP in the easy case $\theta < c < \theta/3$. Denote by L_T the normalized cumulant generating function of the random variable $Z_T(a) = \mathcal{Z}_T(a, -ca)$. We can split $\mathbb{P}\{\widehat{\theta}_T \geq c\}$ into two terms, $\mathbb{P}\{\widehat{\theta}_T \geq c\} = A_T B_T$ with

$$A_T = \exp(T L_T(a_c)), \qquad B_T = \mathbb{E}_T\{\exp(-Z_T(a_c))\, \mathbf{1}_{\{Z_T(a_c) \geq 0\}}\},$$

where $\mathbf{1}$ is the indicator function and $\mathbb{E}_T$ denotes the expectation after the usual change of probability

$$\frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\mathbb{P}} = \exp(Z_T(a_c) - T L_T(a_c)). \tag{5.3}$$

On the one hand, from Lemma 3.2, it follows that

$$A_T = \exp(T L(a_c) + H(a_c))(1 + o(1)) = \exp(-T I_\theta(c) + J(c))(1 + o(1)). \tag{5.4}$$

It remains to establish an asymptotic expansion for B_T, which can be rewritten as

$$B_T = \mathbb{E}_T\{\exp(-a_c\sigma_c\sqrt{T}\boldsymbol{U}_T)\, \mathbf{1}_{\{\boldsymbol{U}_T \geq 0\}}\}, \tag{5.5}$$

where $\boldsymbol{U}_T = Z_T(1)/\sigma_c\sqrt{T}$.

Lemma 5.1. *For all $\theta < c < \theta/3$, we have*

$$B_T = \frac{1}{a_c\sigma_c\sqrt{2\pi T}}(1 + o(1)). \tag{5.6}$$

Proof. Denote by Φ_T the characteristic function of $\boldsymbol{U}_T$ under $\mathbb{P}_T$. As $\theta < c < \theta/3$, from (2.5) it follows that $a_c > 0$ and $\sigma_c > 0$. Moreover, (5.3) immediately implies that

$$\Phi_T(u) = \exp\left(T L_T\left(a_c + \frac{\mathrm{i}u}{\sigma_c\sqrt{T}}\right) - T L_T(a_c)\right). \tag{5.7}$$

Firstly, from Lemma 3.3 we deduce that for large enough T, Φ_T belongs to $L^2(\mathbb{R})$. As soon as $1 - 2a_c\alpha_k^T > 0$ for all $k \geq 1$ and large enough T, we obtain, from Lemma 3.3 and (5.7),

$$\begin{aligned} |\Phi_T(u)|^2 &= \prod_{k=1}^{\infty}\left(1 + \frac{4u^2(\alpha_k^T)^2}{\sigma_c^2 T(1-2a_c\alpha_k^T)^2}\right)^{-1/2} \exp\left(-\frac{(a_c\beta_k^T)^2}{1-2a_c\alpha_k^T}\right) \\ &\leq \prod_{k=1}^{\infty}\left(1 + \frac{4u^2(\alpha_k^T)^2}{\sigma_c^2 T(1-2a_c\alpha_k^T)^2}\right)^{-1/2}. \end{aligned} \tag{5.8}$$

For all $\varepsilon > 0$ small enough such that $1 - 2a_c\varepsilon > 0$, we denote

$$q_T(\varepsilon) = \sum_{k=1}^{\infty} \mathbf{1}_{\{|\alpha_k^T| > \varepsilon\}}.$$

From Lemma 3.3, it follows that there exist some constants $\varepsilon > 0$ and $\eta > 0$, depending only on ε, such that

$$\liminf_{T\to\infty} \frac{q_T(\varepsilon)}{T} \geq 2\eta. \tag{5.9}$$

Hence, we infer from (5.8) and (5.9) that, for large enough T,

$$|\Phi_T(u)|^2 \leq \left(1 + \frac{\xi^2 u^2}{T}\right)^{-\eta T},$$

where $\xi = 2\varepsilon/\sigma_c(1 + 2a_c\varepsilon)$, which clearly ensures, whenever $\eta T \geq 1$, that

$$|\Phi_T(u)|^2 \leq \frac{1}{1 + \eta\xi^2 u^2}. \tag{5.10}$$

Consequently, from (5.10) it follows that, for large enough T, Φ_T belongs to $L^2(\mathbb{R})$. Therefore, from Parseval's formula, it follows that B_T, given by (5.5), can be rewritten as

$$B_T = \frac{1}{2\pi a_c\sigma_c\sqrt{T}} \int_{\mathbb{R}} \left(1 + \frac{\mathrm{i}u}{a_c\sigma_c\sqrt{T}}\right)^{-1} \Phi_T(u)\,\mathrm{d}u. \tag{5.11}$$

However, from Lemma 3.2 we obtain, for all $u \in \mathbb{R}$,

$$\lim_{T\to\infty} \Phi_T(u) = \lim_{T\to\infty} \exp\left(TL_T\left(a_c + \frac{\mathrm{i}u}{\sigma_c\sqrt{T}}\right) - TL_T(a_c)\right) = \exp\left(-\frac{u^2}{2}\right) \tag{5.12}$$

as $L''(a_c) = \sigma_c^2$, which means that the distribution of $\boldsymbol{U}_T$ under $\mathbb{P}_T$ converges to the standard $\mathcal{N}(0, 1)$ distribution. Finally, (5.6) follows from (5.11), (5.12), and the Lebesgue dominated convergence theorem.

Finally, (2.3) immediately follows from the conjunction of (5.4) and (5.6). The proof of (2.4) follows exactly the same lines, the only notable point to mention being a change of sign in Parseval's formula.

5.2. Proof of Theorem 2.2 (second part)

We shall now proceed to the proof of the SLDP in the more complex case $c > \theta/3$ with $c \neq 0$. We can easily check that the function L, given by (5.1), is decreasing and reaches its minimum at the right boundary point $a_c = 2(c-\theta)$ of the domain $\mathcal{D}_{\mathcal{L}}$. Therefore, as in [1] and [3], it is necessary to make use of a slight modification of the usual strategy of changing the probability given in (5.3). There exists a unique a_T, which belongs to the interior of $\mathcal{D}_{\mathcal{L}}$ and converges to its border a_c, solution of the implicit equation

$$L'(a_T) + \frac{1}{T} H'(a_T) = 0. \tag{5.13}$$

It leads to the decomposition $\mathbb{P}(\widehat{\theta}_T \geq c) = A_T B_T$ with

$$A_T = \exp(T L_T(a_T)), \qquad B_T = \mathbb{E}_T\{\exp(-Z_T(a_T))\, \mathbf{1}_{\{Z_T(a_T)\geq 0\}}\},$$

where $\mathbb{E}_T$ stands for the expectation after the time-varying change of probability

$$\frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\mathbb{P}} = \exp(Z_T(a_T) - T L_T(a_T)). \tag{5.14}$$

From (5.1), (5.2), and (5.13), we obtain

$$(\theta^2 T + \gamma^2)\tau(a_T) = \frac{\theta^2(c(2\varphi(a_T) - a_T) + \theta(\theta - 3c))}{\varphi(a_T)(\varphi(a_T) + c)},$$

where $\varphi(a) = \sqrt{\theta^2 + 2ac}$ and $\tau(a) = \varphi(a) - a - \theta$. Consequently, from straightforward calculations, it follows that

$$\lim_{T\to\infty} T(\varphi(a_T) + 2c - \theta) = -\frac{c}{3c-\theta}, \tag{5.15}$$

$$\lim_{T\to\infty} T(a_T - a_c) = -\frac{(2c-\theta)}{3c-\theta}, \tag{5.16}$$

$$\lim_{T\to\infty} T\tau(a_T) = \frac{c-\theta}{3c-\theta}. \tag{5.17}$$

Moreover, we can show that $R_T(a_T) = \mathcal{R}_T(a_T, -ca_T)$ remains bounded when T goes to ∞. Hence, Lemma 3.2 together with (5.15), (5.16), and (5.17) imply that

$$\begin{aligned} A_T &= \exp(TL(a_T) + H(a_T))(1 + o(1)), \\ &= \exp\Big(-TI_\theta(c) - \frac{\gamma^2}{\theta^2}(2c-\theta)\Big)\left(\frac{2eT(2c-\theta)^3(3c-\theta)}{\theta^2(c-\theta)}\right)^{1/2}(1+o(1)). \end{aligned} \tag{5.18}$$

Moreover, the second term B_T can be rewritten as

$$B_T = \mathbb{E}_T\{\exp(-a_T T \boldsymbol{U}_T)\, \mathbf{1}_{\{\boldsymbol{U}_T \geq 0\}}\}, \tag{5.19}$$

where $\boldsymbol{U}_T = Z_T(1)/T$.

Lemma 5.2. *For $c > \theta/3$ with $c \neq 0$, we have*

$$B_T = \frac{1}{a_c b_c T\sqrt{2\pi \mathrm{e}}}(1 + o(1)), \tag{5.20}$$

where

$$b_c = -L'(a_c) = \frac{3c-\theta}{2(2c-\theta)}. \tag{5.21}$$

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Proof. Denote by Φ_T the characteristic function of $\boldsymbol{U}_T$ under $\mathbb{P}_T$. We infer from (5.14) that for all $u \in \mathbb{R}$,

$$\Phi_T(u) = \exp\biggl(TL_T\Bigl(a_T + \frac{\mathrm{i}u}{T}\Bigr) - TL_T(a_T)\biggr).$$

Moreover, from (5.15) and (5.16) we obtain for large enough T and for all $u \in \mathbb{R}$ such that $|u| = o(T)$,

$$\exp\biggl(TL\Bigl(a_T + \frac{\mathrm{i}u}{T}\Bigr) - TL(a_T)\biggr) = \exp\biggl(-\mathrm{i}b_c u - \frac{\sigma_c^2 u^2}{2T}\biggr)\biggl(1 + O\Bigl(\frac{|u|^3}{T^2}\Bigr)\biggr),$$

where σ_c^2 and b_c are given by (2.7) and (5.21). Consequently, as soon as $|u| = o(T^{2/3})$,

$$\exp\biggl(TL\Bigl(a_T + \frac{\mathrm{i}u}{T}\Bigr) - TL(a_T)\biggr) = \exp\biggl(-\mathrm{i}b_c u - \frac{\sigma_c^2 u^2}{2T}\biggr)(1 + o(1)) \tag{5.22}$$

and the remainder $o(1)$ is uniform. By the same token,

$$\lim_{T\to\infty} \exp\biggl(H\Bigl(a_T + \frac{\mathrm{i}u}{T}\Bigr) - H(a_T)\biggr) = \frac{1}{\sqrt{1 - 2\mathrm{i}b_c u}}. \tag{5.23}$$

Therefore, from Lemma 3.2 together with (5.22), (5.23), and the boundedness of $R_T(a_T)$, we obtain for all $u \in \mathbb{R}$ such that $|u| = o(T^{2/3})$,

$$\Phi_T(u) = \Phi(u) \exp\biggl(-\frac{\sigma_c^2 u^2}{2T}\biggr)(1 + o(1)), \tag{5.24}$$

where $\Phi(u) = 1/\sqrt{1 - 2\mathrm{i}b_c u}\,\exp(-\mathrm{i}b_c u)$. It means that the distribution of $\boldsymbol{U}_T$ under $\mathbb{P}_T$ converges to $b_c(\xi^2 - 1)$, where ξ stands for an $\mathcal{N}(0, 1)$ random variable. It also implies that, for large enough T, Φ_T belongs to $L^2(\mathbb{R})$. Hereafter, we deduce from Parseval's formula that B_T, given by (5.19), can be rewritten as

$$B_T = \frac{1}{2\pi T a_T} \int_{\mathbb{R}} \Bigl(1 + \frac{\mathrm{i}u}{a_T T}\Bigr)^{-1} \Phi_T(u)\,\mathrm{d}u.$$

We split B_T into two terms, $B_T = C_T + D_T$, where

$$C_T = \frac{1}{2\pi T a_T} \int_{|u|\le s_T} \Bigl(1 + \frac{\mathrm{i}u}{T a_T}\Bigr)^{-1} \Phi_T(u)\,\mathrm{d}u,$$

$$D_T = \frac{1}{2\pi T a_T} \int_{|u|> s_T} \Bigl(1 + \frac{\mathrm{i}u}{T a_T}\Bigr)^{-1} \Phi_T(u)\,\mathrm{d}u \tag{5.25}$$

with $s_T = T^{2/3}$. On the one hand, from (5.25), it follows that D_T is negligible, since

$$D_T = o\biggl(\exp\Bigl(-\frac{\sigma_c^2 s_T^2}{2T}\Bigr)\biggr). \tag{5.26}$$

On the other hand, from (5.24), it follows that for large enough T,

$$2\pi T a_T C_T = \int_{|u|\le s_T} \Phi(u) \exp\biggl(-\frac{\sigma_c^2 u^2}{2T}\biggr)(1 + o(1))\,\mathrm{d}u,$$

which leads, thanks to [1, Lemma 7.3], to

$$\lim_{T\to\infty} 2\pi T a_T C_T = \frac{\sqrt{2\pi}}{b_c\sqrt{\mathrm{e}}}. \tag{5.27}$$

Hence, (5.26) together with (5.27) clearly imply (5.20).

Finally, we immediately deduce (2.6) from (5.18) and (5.20).

5.3. Proof of Theorem 2.2 (third part)

Now assume that $c = \theta/3$, which means that $a_c = a_\theta$ with $a_\theta = -4\theta/3$. There exists a unique a_T, which belongs to the interior of $\mathcal{D}_{\mathcal{L}}$ and converges to its border a_θ, solution of the implicit equation

$$L'(a_T) + \frac{1}{T}H'(a_T) = 0. \tag{5.28}$$

From (5.1), (5.2), and (5.28), we obtain

$$(\theta^2 T + \gamma^2)\tau(a_T) = \frac{\theta^2 c(2\varphi(a_T) - a_T)}{\varphi(a_T)(\varphi(a_T)+c)},$$

where $\varphi(a) = \sqrt{\theta^2 + 2ac}$ and $\tau(a) = \varphi(a) - a - \theta$. We obviously have

$$\tau(a) = -\frac{(\varphi(a)+c)(\varphi(a)-\theta)}{2c},$$

which leads to

$$(\theta^2 T + \gamma^2)(\varphi(a_T)+c)^2 = \frac{2\theta^2 c^2(a_T - 2\varphi(a_T))}{\varphi(a_T)(\varphi(a_T)-\theta)}.$$

After some elementary calculations, it implies that

$$\lim_{T\to\infty} T(\varphi(a_T)+c)^2 = -\frac{\theta}{3}, \tag{5.29}$$

$$\lim_{T\to\infty} T(a_T - a_\theta)^2 = -\frac{\theta}{3}, \tag{5.30}$$

$$\lim_{T\to\infty} \sqrt{T}\tau(a_T) = 2\sqrt{-\frac{\theta}{3}}. \tag{5.31}$$

Hereafter, we shall make use of the decomposition $\mathbb{P}\{\widehat{\theta}_T \geq c\} = A_T B_T$ given by

$$A_T = \exp(T L_T(a_T)), \qquad B_T = \mathbb{E}_T\{\exp(-Z_T(a_T))\,\mathbf{1}_{\{Z_T(a_T)\geq 0\}}\},$$

$$\frac{\mathrm{d}\mathbb{P}_T}{\mathrm{d}\mathbb{P}} = \exp(Z_T(a_T) - T L_T(a_T)).$$

We shall show that $R_T(a_T) = \mathcal{R}_T(a_T, -ca_T)$ remains bounded when T goes to ∞. Hence, from Lemma 3.2 together with (5.29), (5.30), and (5.31), it follows that

$$A_T = \frac{\sqrt{2}}{3}\exp\left(-T I_\theta(c) + \frac{\gamma^2}{3\theta}\right)\left(\frac{-\theta \mathrm{e} T}{3}\right)^{1/4}(1+o(1)). \tag{5.32}$$

On the other hand, B_T can be rewritten as

$$B_T = \mathbb{E}_T\{\exp(-a_T\sqrt{T}\boldsymbol{U}_T)\,\mathbf{1}_{\{\boldsymbol{U}_T\geq 0\}}\},$$

where $\boldsymbol{U}_T = Z_T(1)/\sqrt{T}$.

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Lemma 5.3. *For $c = \theta/3$, we have*

$$B_T = \frac{1}{4\pi\sqrt{a_\theta T}} \exp\Big(-\frac{1}{4}\Big)\Gamma\Big(\frac{1}{4}\Big)(1+o(1)). \tag{5.33}$$

Proof. Via the same lines as in the proof of Lemma 5.1, we find that the characteristic function Φ_T of $\boldsymbol{U}_T$, under $\mathbb{P}_T$, belongs to $L^2(\mathbb{R})$. Hence, from Parseval's formula, it follows that

$$B_T = \frac{1}{2\pi a_T\sqrt{T}} \int_{\mathbb{R}} \Big(1 + \frac{\mathrm{i}u}{a_T\sqrt{T}}\Big)^{-1} \Phi_T(u)\,\mathrm{d}u. \tag{5.34}$$

However, from (5.29) and (5.30), we obtain

$$\lim_{T\to\infty} T\Big(L\Big(a_T + \frac{\mathrm{i}u}{\sqrt{T}}\Big) - L(a_T)\Big) = -\mathrm{i}d_\theta u - \frac{\sigma_\theta^2 u^2}{2}, \tag{5.35}$$

where σ_θ^2 is given by (2.9) and $d_\theta = \sigma_\theta/\sqrt{2}$. By the same token,

$$\lim_{T\to\infty} \exp\Big(H\Big(a_T + \frac{\mathrm{i}u}{\sqrt{T}}\Big) - H(a_T)\Big) = \frac{1}{\sqrt{1-2\mathrm{i}d_\theta u}}. \tag{5.36}$$

Therefore, from Lemma 3.2 together with (5.35), (5.36), and the boundedness of $R_T(a_T)$, we obtain the pointwise convergence

$$\lim_{T\to\infty} \Phi_T(u) = \Phi(u) = \frac{1}{\sqrt{1-2\mathrm{i}d_\theta u}} \exp\Big(-\mathrm{i}d_\theta u - \frac{\sigma_\theta^2 u^2}{2}\Big). \tag{5.37}$$

We see that the distribution of $\boldsymbol{U}_T$ under $\mathbb{P}_T$ converges to $\sigma_\theta \zeta + d_\theta(\xi^2 - 1)$, where ζ and ξ are two independent random variables sharing the same $\mathcal{N}(0,1)$ distribution. Finally, from (5.34) together with (5.37) and the Lebesgue dominated convergence theorem, we obtain

$$B_T = \frac{1}{2\pi a_T\sqrt{T}} \int_{\mathbb{R}} \Phi(u)\,\mathrm{d}u(1+o(1)) = \frac{1}{4\pi\sqrt{a_\theta T}} \exp\Big(-\frac{1}{4}\Big)\Gamma\Big(\frac{1}{4}\Big)(1+o(1)),$$

which completes the proof of Lemma 5.3.

The proof of (2.8) immediately follows from the conjunction of (5.32) and (5.33).

5.4. Proof of Theorem 2.2 (fourth part)

Assume now that $c = 0$. We shall obtain the leading asymptotic behavior of $\mathbb{P}\{\widehat{\theta}_T \geq 0\} = \mathbb{P}\{X_T^2 - 2X_T\overline{X}_T \geq T\}$. For all $\alpha > 0$, we have the decomposition $\mathbb{P}\{X_T^2 - 2X_T\overline{X}_T \geq T\} = A_T + B_T$, where

$$A_T = \mathbb{P}\{X_T^2 - 2X_T\overline{X}_T \geq T, |\overline{X}_T| \leq \alpha\}, \qquad B_T = \mathbb{P}\{X_T^2 - 2X_T\overline{X}_T \geq T, |\overline{X}_T| > \alpha\}.$$

Firstly, if

$$\alpha = \Big|\frac{\gamma}{\theta}\Big| + \frac{2}{\sqrt{-\theta}},$$

it is not hard to see that B_T is negligible. We deduce from the simple upper bound $B_T \leq \mathbb{P}\{|\overline{X}_T| > \alpha\}$ together with (4.4), we obtain

$$\limsup_{T\to+\infty} \frac{1}{T}\log B_T \leq 2\theta, \qquad B_T = o\Big(\exp\Big(\frac{3\theta T}{2}\Big)\Big).$$

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Next, we recall that the sequence $(\widehat{\theta}_T)$ satisfies an LDP with good rate function I_θ given by (2.2). Consequently,

$$\lim_{T\to+\infty}\frac{1}{T}\log\mathbb{P}\{\widehat{\theta}_T\geq 0\}=\lim_{T\to+\infty}\frac{1}{T}\log\mathbb{P}\{X_T^2-2X_T\overline{X}_T\geq T\}=\theta,$$

which clearly implies that $\mathbb{P}\{X_T^2-2X_T\overline{X}_T\geq T\}=A_T(1+o(1))$. From now on, it remains to establish only the leading asymptotic behavior of A_T. We already saw at the beginning of Section 3 that the random vector

$$\begin{pmatrix}X_T\\ \overline{X}_T\end{pmatrix}\sim\mathcal{N}\left(\begin{pmatrix}m_T\\ \mu_T\end{pmatrix},\Gamma_T(\theta)\right).$$

Therefore,

$$A_T=\int_{-\alpha}^{\alpha}h_T(x)f_T(x)\,\mathrm{d}x,$$

where $h_T(x)=\mathbb{P}\{X_T^2-2X_T\overline{X}_T\geq T\mid\overline{X}_T=x\}$ and f_T is the Gaussian probability density function of $\overline{X}_T$. Moreover, as $c_T>0$, the conditional distribution of X_T given $\overline{X}_T=x$ is $\mathcal{N}(\nu_T,s_T^2)$ with $\nu_T=m_T+b_T(x-\mu_T)/cT$ and $s_T^2=a_T-b_T^2/c_T$. Furthermore, for all $x\in\mathbb{R}$, $h_T(x)$ can be rewritten as

$$h_T(x)=\mathbb{P}\left\{\frac{X_T-\nu_T}{s_T}\leq -y_T\,\middle|\,\overline{X}_T=x\right\}+\mathbb{P}\left\{\frac{X_T-\nu_T}{s_T}\geq z_T\,\middle|\,\overline{X}_T=x\right\},$$

where $y_T=(-x+\sqrt{x^2+T}+\nu_T)/s_T$ and $z_T=(x+\sqrt{c^2+T}-\nu_T)/s_T$. We can easily check that

$$\liminf_{T\to+\infty}\{y_T,z_T\}\geq\liminf_{T\to+\infty}\frac{-\alpha+\sqrt{T}-m_T-b_Tc_T^{-1}(\alpha-\mu_T)}{s_T}=+\infty.$$

From standard asymptotic analysis of Gaussian distribution tails, it follows that

$$h_T(x)=\frac{1}{y_T\sqrt{2\pi}}\exp\left(-\frac{y_T^2}{2}\right)(1+o(1))+\frac{1}{z_T\sqrt{2\pi}}\exp\left(-\frac{z_T^2}{2}\right)(1+o(1)),$$

where $o(1)$ is uniform with respect to x. We split A_T into two terms, $A_T=C_T+D_T$,

$$C_T=\int_{-\alpha}^{\alpha}\frac{1}{y_T\sqrt{2\pi}}\exp\left(-\frac{y_T^2}{2}\right)\frac{1}{\sqrt{2\pi c_T}}\exp\left(-\frac{(x-\mu_T)^2}{2c_T}\right)\mathrm{d}x(1+o(1)),$$
$$D_T=\int_{-\alpha}^{\alpha}\frac{1}{z_T\sqrt{2\pi}}\exp\left(-\frac{z_T^2}{2}\right)\frac{1}{\sqrt{2\pi c_T}}\exp\left(-\frac{(x-\mu_T)^2}{2c_T}\right)\mathrm{d}x(1+o(1)).$$

From a careful asymptotic expansion inside the integral C_T together with the change of variables $y=-\theta(x+\gamma/\theta)\sqrt{T}$ and Lebesgue's dominated convergence theorem, we obtain

$$\lim_{T\to+\infty}\sqrt{2\pi}\sqrt{-2\theta T}\mathrm{e}^{-\theta T}C_T=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\exp\left(2y+\frac{\gamma^2}{\theta}-\frac{y^2}{2}\right)\mathrm{d}y=\exp\left(\frac{\gamma^2}{\theta}+2\right).$$

By the same token, we also obtain

$$\lim_{T\to+\infty}\sqrt{2\pi}\sqrt{-2\theta T}\mathrm{e}^{-\theta T}D_T=\exp\left(\frac{\gamma^2}{\theta}+2\right),$$

which completes the proof.

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Appendix A. Proof of Lemma 3.1.

For all $(a, b, c) \in \mathbb{R}^3$, let

$$\mathcal{Z}_T(a, b, c) = a\sqrt{T}X_T + b\int_0^T X_t^2 \,\mathrm{d}t + c\int_0^T X_t \,\mathrm{d}t.$$

We shall calculate the limit Λ of the normalized cumulant generating function Λ_T of the random variable $\mathcal{Z}_T(a, b, c)$. Firstly, as in [6], from Girsanov's formula associated with (1.1), it follows that

$$\begin{aligned}\Lambda_T(a, b, c) &= \frac{1}{T}\log \mathbb{E}\{\exp(\mathcal{Z}_T(a, b, c))\},\\ &= \frac{1}{T}\log \mathbb{E}_{\varphi,\delta}\biggl\{\exp\biggl((\theta-\varphi)\int_0^T X_t \,\mathrm{d}X_t + \frac{1}{2}(2b-\theta^2+\varphi^2)\int_0^T X_t^2\,\mathrm{d}t - \xi_T\biggr)\biggr\},\end{aligned}$$

where $\mathbb{E}_{\varphi,\delta}$ stands for the expectation after the change of probability

$$\frac{\mathrm{d}\mathbb{P}_{\varphi,\delta}}{\mathrm{d}\mathbb{P}_{\theta,\gamma}} = \exp\biggl((\varphi-\theta)\int_0^T X_t\,\mathrm{d}X_t - \frac{1}{2}(\varphi^2-\theta^2)\int_0^T X_t^2\,\mathrm{d}t + \zeta_T\biggr)$$

with $\xi_T = \zeta_T - a\sqrt{T}X_T - cT\overline{X}_T$ and

$$\zeta_T = (\theta\gamma - \varphi\delta)T\overline{X}_T - (\gamma-\delta)X_T + \tfrac{1}{2}(\gamma^2-\delta^2)T.$$

Consequently, if we assume that $\theta^2 - 2b > 0$ and if we choose $\varphi = \sqrt{\theta^2-2b}$ and $\delta = 0$, $\Lambda_T(a, b, c)$ reduces to

$$\Lambda_T(a, b, c) = \frac{\varphi-\theta-\gamma^2}{2} + \frac{1}{T}\log \mathbb{E}_{\varphi,0}\biggl\{\exp\biggl(-\frac{1}{2}\boldsymbol{V}_T^\top \boldsymbol{J}\boldsymbol{V}_T + \boldsymbol{U}_T^\top \boldsymbol{V}_T\biggr)\biggr\},$$

where the vectors $\boldsymbol{U}_T$ and $\boldsymbol{V}_T$ are given by

$$\boldsymbol{U}_T = \begin{pmatrix} a\sqrt{T}+\gamma \\ T(c-\theta\gamma)\end{pmatrix}, \qquad \boldsymbol{V}_T = \begin{pmatrix} X_T \\ \overline{X}_T\end{pmatrix},$$

and $\boldsymbol{J}$ is the diagonal matrix of order 2,

$$\boldsymbol{J} = \begin{pmatrix}\varphi-\theta & 0\\ 0 & 0\end{pmatrix}.$$

Under the new probability $\mathbb{P}_{\varphi,0}$, $\boldsymbol{V}_T$ is a Gaussian random vector with 0 mean and covariance matrix $\Gamma_T(\varphi)$ given by (3.1). Denote by $\boldsymbol{M}_T(a, b, c)$ the square matrix of order 2

$$\boldsymbol{M}_T(a, b, c) = \boldsymbol{I}_2 + \boldsymbol{J}\Gamma_T(\varphi) = \begin{pmatrix} 1+(\varphi-\theta)a_T(\varphi) & (\varphi-\theta)b_T(\varphi)\\ 0 & 1\end{pmatrix},$$

where $\boldsymbol{I}_2$ stands for the identity matrix of order 2. Clearly, we have

$$\det \boldsymbol{M}_T(a, b, c) = 1 + (\varphi-\theta)a_T(\varphi) = 1 + \frac{\varphi-\theta}{2\varphi}(\mathrm{e}^{2\varphi T}-1),$$

which leads to

$$\lim_{T\to\infty} \frac{\det \boldsymbol{M}_T(a,b,c)}{\mathrm{e}^{2\varphi T}} = \frac{\varphi-\theta}{2\varphi}. \tag{A.1}$$

Hence, as $\theta < 0 < \varphi$, from (A.1), it follows that for large enough T, the matrix $\boldsymbol{M}_T(a,b,c)$ is positive definite. It is also not hard to see that, from (3.1),

$$\lim_{T\to\infty} \frac{T \det \Gamma_T(\varphi)}{\mathrm{e}^{2\varphi T}} = \frac{1}{2\varphi^3}. \tag{A.2}$$

Therefore, from standard Gaussian calculations, we obtain

$$\Lambda_T(a,b,c) = \frac{\varphi-\theta-\gamma^2}{2} - \frac{1}{2T}\log(\det \boldsymbol{M}_T(a,b,c)) + \frac{1}{T}\boldsymbol{H}_T(a,b,c), \tag{A.3}$$

where $\boldsymbol{H}_T(a,b,c) = 2^{-1}\boldsymbol{U}_T^\top \Gamma_T(\varphi)\boldsymbol{M}_T^{-1}(a,b,c)\boldsymbol{U}_T$. On the one hand, from (A.1), we immediately obtain

$$\lim_{T\to\infty} \frac{1}{2T}\log(\det \boldsymbol{M}_T(a,b,c)) = \varphi. \tag{A.4}$$

On the other hand, we clearly have

$$\boldsymbol{H}_T(a,b,c) = \frac{1}{2\det \boldsymbol{M}_T(a,b,c)}((a\sqrt{T}+\gamma)^2 a_T(\varphi) + 2T d_T(\varphi) + T^2(c-\theta\gamma)^2 e_T(\varphi)),$$

where $d_T(\varphi) = (a\sqrt{T}+\gamma)(c-\theta\gamma)b_T(\varphi)$ and $e_T(\varphi) = c_T(\varphi) + (\varphi-\theta)\det\Gamma_T(\varphi)$. Consequently, from (A.1) and (A.2), we obtain

$$\lim_{T\to\infty} \frac{1}{T}\boldsymbol{H}_T(a,b,c) = \frac{1}{2}\bigg(\frac{a^2}{\varphi-\theta} + \frac{(c-\theta\gamma)^2}{\varphi^2}\bigg). \tag{A.5}$$

Finally, from (A.3) together with (A.4) and (A.5), it follows that

$$\lim_{T\to\infty} \Lambda_T(a,b,c) = -\frac{1}{2}(\theta+\varphi+\gamma^2) + \frac{1}{2}\bigg(\frac{a^2}{\varphi-\theta}\bigg) + \frac{1}{2}\bigg(\frac{c-\theta\gamma}{\varphi}\bigg)^2,$$

which completes the proof.

Appendix B. Proof of Lemma 3.2.

Our goal is to establish the full asymptotic expansion for the normalized cumulant generating function $\mathcal{L}_T$ of the random variable $\mathcal{Z}_T(a,b)$. Firstly, as in the proof of Lemma 3.1, from Girsanov's formula associated with (1.1), it follows that

$$\begin{aligned}\mathcal{L}_T(a,b) &= \frac{1}{T}\log\mathbb{E}\{\exp(\mathcal{Z}_T(a,b))\},\\ &= \frac{1}{T}\log\mathbb{E}_{\varphi,\delta}\bigg\{\exp\bigg((a+\theta-\varphi)\int_0^T X_t\,\mathrm{d}X_t + \frac{1}{2}(2b-\theta^2+\varphi^2)\int_0^T X_t^2\,\mathrm{d}t - \xi_T\bigg)\bigg\},\end{aligned}$$

where $\xi_T = aX_T\overline{X}_T + \zeta_T + bT(\overline{X}_T)^2$ and $\zeta_T = (\theta\gamma-\varphi\delta)T\overline{X}_T - (\gamma-\delta)X_T + (\gamma^2-\delta^2)T/2$. Consequently, if we assume that $\theta^2-2b>0$ and if we choose $\varphi = \sqrt{\theta^2-2b}$ and $\delta = 0$, $\mathcal{L}_T(a,b)$ reduces to

$$\mathcal{L}_T(a,b) = \frac{\tau-\gamma^2}{2} + \frac{1}{T}\log\mathbb{E}_{\varphi,0}\bigg\{\exp\bigg(-\frac{1}{2}\boldsymbol{V}_T^\top \boldsymbol{J}_T\boldsymbol{V}_T + \gamma\boldsymbol{U}_T^\top\boldsymbol{V}_T\bigg)\bigg\}$$

with $\tau = \varphi - (a+\theta)$, where the vectors $\boldsymbol{U}_T$ and $\boldsymbol{V}_T$ are given by

$$\boldsymbol{U}_T = \begin{pmatrix} 1 \\ -\theta T \end{pmatrix}, \qquad \boldsymbol{V}_T = \begin{pmatrix} X_T \\ \overline{X}_T \end{pmatrix},$$

and $\boldsymbol{J}_T$ is the diagonal matrix of order 2,

$$\boldsymbol{J}_T = \begin{pmatrix} \tau & a \\ a & 2bT \end{pmatrix}.$$

In Appendix A, we saw that under the new probability $\mathbb{P}_{\varphi,0}$, $\boldsymbol{V}_T$ is Gaussian random vector with 0 mean and covariance matrix $\Gamma_T(\varphi)$ given by (3.1). Let $\boldsymbol{M}_T(a,b)$ be the square matrix of order 2,

$$\boldsymbol{M}_T(a,b) = \boldsymbol{I}_2 + \boldsymbol{J}_T\Gamma_T(\varphi) = \begin{pmatrix} 1+\tau a_T(\varphi) + ab_T(\varphi) & \tau b_T(\varphi) + ac_T(\varphi) \\ 2bTb_T(\varphi) + aa_T(\varphi) & 1 + 2bTc_T(\varphi) + ab_T(\varphi) \end{pmatrix}.$$

It is not hard to see that

$$\det(\boldsymbol{M}_T(a,b)) = 1 + 2ab_T(\varphi) + 2bTc_T(\varphi) + \tau a_T(\varphi) + (2\tau Tb - a^2)\det \Gamma_T(\varphi).$$

Hence, from (A.2), it follows that

$$\lim_{T\to\infty} \frac{\det \boldsymbol{M}_T(a,b)}{\mathrm{e}^{2\varphi T}} = \frac{\tau\theta^2}{2\varphi^3}. \tag{B.1}$$

Consequently, as soon as $\tau > 0$, from (B.1), we find that for large enough T, the matrix $\boldsymbol{M}_T(a,b)$ is positive definite. Therefore, from standard Gaussian calculations, it follows that

$$\mathcal{L}_T(a,b) = \frac{\tau-\gamma^2}{2} - \frac{1}{2T}\log(\det \boldsymbol{M}_T(a,b)) + \frac{\gamma^2}{2T}\boldsymbol{U}_T^\top \Gamma_T(\varphi)\boldsymbol{M}_T^{-1}(a,b)\boldsymbol{U}_T. \tag{B.2}$$

It remains for us to establish asymptotic expansions for the last two terms in (B.2), which can be achieved by straightforward but tedious calculations; see [3] for a detailed proof.

Appendix C. Proof of Lemma 3.3.

From (3.2) together with Itô's formula (4.6), it follows that

$$\mathcal{Z}_T(a,b) = \frac{a}{2}X_T^2 - \frac{aT}{2} - a\overline{X}_T X_T + b\int_0^T X_t^2\,\mathrm{d}t - bT(\overline{X}_T)^2. \tag{C.1}$$

In addition, from the beginning of Section 3, we can split $X_T = Y_T + m_T$ and $\overline{X}_T = \overline{Y}_T + \mu_T$. Consequently, we have the decomposition

$$\mathcal{Z}_T(a,b) = \mathcal{Z}_T^0(a,b) + \mathcal{Z}_T^1(a,b) + \mathcal{Z}_T^2(a,b),$$

where $\mathcal{Z}_T^0(a,b) = \mathbb{E}\{\mathcal{Z}_T(a,b)\}$,

$$\begin{aligned}
\mathcal{Z}_T^1(a,b) &= a(m_T - \mu_T)Y_T - am_T\overline{Y}_T + 2b\int_0^T (m_t - \mu_T)Y_t\,\mathrm{d}t, \\
\mathcal{Z}_T^2(a,b) &= \frac{a}{2}(Y_T^2 - \mathbb{E}\{Y_T^2\}) - a(\overline{Y}_T Y_T - \mathbb{E}\{\overline{Y}_T Y_T\}) - bT((\overline{Y}_T)^2 - \mathbb{E}\{(\overline{Y}_T)^2\}) \\
&\quad + b\bigg(\int_0^T Y_t^2\,\mathrm{d}t - \mathbb{E}\bigg\{\int_0^T Y_t^2\,\mathrm{d}t\bigg\}\bigg).
\end{aligned}$$

By using the same notations as in [9, Chapters 2 and 6], we clearly have

$$\mathcal{Z}_T^0(a,b) \in H^{:0:}, \qquad \mathcal{Z}_T^1(a,b) \in H^{:1:}, \qquad \mathcal{Z}_T^2(a,b) \in H^{:2:},$$

where $H^{:n:}$ stands for the homogeneous chaos of order n. Hence, from [9, Theorem 6.2], we obtain

$$\mathcal{Z}_T(a,b) = \mathbb{E}\{\mathcal{Z}_T(a,b)\} + \sum_{k=1}^{\infty} \alpha_k^T(\varepsilon_k^2 - 1) + \beta_k^T \varepsilon_k,$$

where (ε_k) are independent standard $\mathcal{N}(0,1)$ random variables. We also obtain, from [9, Theorem 6.2],

$$\sum_{k=1}^{+\infty} (\beta_k^T)^2 = \mathbb{E}\{(\mathcal{Z}_T^1(a,b))^2\}. \tag{C.2}$$

In addition, some rough estimates tell us that the right-hand side of (C.2) is uniformly bounded by some constant $B > 0$, depending only on a and b. There exists some constant $\zeta(a,b) > 0$ such that

$$\mathbb{E}\{(\mathcal{Z}_T^1(a,b))^2\} \le \zeta(a,b)(\mathbb{E}\{Y_T^2\} + \mathbb{E}\{(\overline{Y}_T)^2\} + \mathbb{E}\{\Delta_T^2\}), \tag{C.3}$$

where

$$\mathbb{E}\{Y_T^2\} = \int_0^T \mathrm{e}^{2\theta(T-t)}\,\mathrm{d}t \le -\frac{1}{2\theta}, \qquad \mathbb{E}\{(\overline{Y}_T)^2\} \le \frac{1}{T}\int_0^T \mathbb{E}\{Y_t^2\}\,\mathrm{d}s \le -\frac{1}{2\theta},$$

and

$$\begin{aligned}
\mathbb{E}\{\Delta_T^2\} &= \mathbb{E}\bigg\{\bigg(\int_0^T (m_t - \mu_T)Y_t\,\mathrm{d}t\bigg)^2\bigg\} \\
&= \frac{\gamma^2}{\theta^2}\mathbb{E}\bigg\{\bigg(\int_0^T \bigg(\frac{1-\mathrm{e}^{\theta T}}{\theta T} + \mathrm{e}^{\theta t}\bigg)Y_t\,\mathrm{d}t\bigg)^2\bigg\}, \\
&\le \frac{2\gamma^2}{\theta^4}\frac{1}{T}\int_0^T \mathbb{E}\{Y_t^2\}\,\mathrm{d}t + \frac{2\gamma^2}{\theta^2}\int_0^T \mathrm{e}^{\theta t}\,\mathrm{d}t \int_0^T \mathrm{e}^{\theta t}\mathbb{E}\{Y_t^2\}\,\mathrm{d}t \\
&\le \frac{-2\gamma^2}{\theta^5}.
\end{aligned}$$

Therefore, from (C.2) and (C.3), we obtain (3.4). It now remains for us to show that there exists some constant $A > 0$ that does not depend on T, such that $|\alpha_k^T| \le A$ for all $k \ge 1$. Since $\mathcal{D}_{\mathcal{L}}$ is an open set and the origin belongs to the interior of $\mathcal{D}_{\mathcal{L}}$, there exists $\varepsilon > 0$ such that $\{(xa, xb) \in \mathcal{D}_{\mathcal{L}} / |x| < \varepsilon\} \subset \mathcal{D}_{\mathcal{L}}$. For all $(a,b) \in \mathcal{D}_{\mathcal{L}}$ and for large enough T, we deduce from Lemma 3.2 that $\exp(T\mathcal{L}_T(xa,xb)) = \mathbb{E}\{\exp(x\mathcal{Z}_T(a,b))\}$ is finite. It means that the Laplace transform of $\mathcal{Z}_T(a,b)$ is well defined on $[-\varepsilon,\varepsilon]$. Hence, [9, Theorem 6.2] ensures that the characteristic function of $\mathcal{Z}_T(a,b)$ is analytic in the strip

$$\bigg\{z \in \mathbb{C} \;\bigg|\; |\mathrm{Im} z| < \frac{1}{2}\Big(\max_{k\ge 1}|\alpha_k^T|\Big)^{-1}\bigg\}.$$

So for large enough T, we obtain $\max|\alpha_k^T| < A$ with $A = 1/2\varepsilon$. Hereafter, the decomposition of $\mathcal{L}_T(xa,xb)$, given in Lemma 3.3, directly follows from [9, Equation (6.7), Theorem 6.2].

Our goal is now to pass through the limit in $\mathcal{L}_T(xa, xb)$. If we choose $x \in \mathbb{R}$ such that $|x| \leq 1/4A$, we have

$$\lim_{T\to\infty} \frac{1}{2T} \sum_{k=1}^{\infty} \frac{(x\beta_k^T)^2}{1-2x\alpha_k^T} \leq \lim_{T\to\infty} \frac{B}{16A^2T} = 0. \tag{C.4}$$

Moreover, from Lemma 3.2, we obtain

$$\lim_{T\to\infty} \mathcal{L}_T(xa, xb) = \mathcal{L}(xa, xb) = -\tfrac{1}{2}(xa + \theta + \sqrt{\theta^2 - 2xb}). \tag{C.5}$$

Furthermore, from the properties of (X_T) and $(\overline{X}_T)$ given at the beginning of Section 3, it follows that

$$\lim_{T\to\infty} \mathbb{E}\{X_T^2\} = \frac{\gamma^2}{\theta^2} - \frac{1}{2\theta}, \qquad \lim_{T\to\infty} \mathbb{E}\{(\overline{X}_T)^2\} = \frac{\gamma^2}{\theta^2},$$

which clearly implies that

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\{X_T^2\} = 0, \qquad \lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbb{E}\{X_t^2\}\,\mathrm{d}t = \frac{\gamma^2}{\theta^2} - \frac{1}{2\theta}, \qquad \lim_{T\to\infty} \mathbb{E}\{X_T\overline{X}_T\} = 0.$$

Then, from (C.1), it follows that

$$\lim_{T\to\infty} \frac{1}{T}\mathbb{E}\{\mathcal{Z}_T(xa, xb)\} = -\frac{x}{2}\left(a + \frac{b}{\theta}\right). \tag{C.6}$$

Finally, from the decomposition of $\mathcal{L}_T(xa, xb)$, (C.4), (C.5), and (C.6), we obtain

$$\lim_{T\to+\infty} \frac{1}{2T} \sum_{k=1}^{\infty} (\log(1-2x\alpha_k) + 2x\alpha_k^T) = \frac{1}{2\pi}\int_{\mathbb{R}} f_x(bg(y))\,\mathrm{d}y, \tag{C.7}$$

where the spectral density g is given by (3.3) and for all $x \in \mathbb{R}$ such that $|x| \leq 1/4A$,

$$f_x(y) = \tfrac{1}{2}(\log(1-2xy) + 2xy).$$

Hence, from (C.7) together with the elementary Taylor expansion of the logarithm and classical complex analysis results, it follows that, for any integer $p \geq 2$,

$$\lim_{T\to+\infty} \frac{1}{T} \sum_{k=1}^{\infty} (\alpha_k^T)^p = \frac{1}{2\pi}\int_{\mathbb{R}} (bg(y))^p\,\mathrm{d}y.$$

Therefore, we obtain the weak convergence (3.5) on the class of functions $\mathcal{F}$ from the Stone–Weierstrass theorem, which completes the proof of Lemma 3.3.

References

[1] Bercu, B. and Rouault, A. (2002). Sharp large deviations for the Ornstein–Uhlenbeck process. *Theory Prob. Appl.* **46,** 1–19.

[2] Bercu, B. and Richou, A. (2014). Large deviations for the Ornstein–Uhlenbeck process with shift. Available at http://www.arxiv.org/abs/1311.7039.

[3] Bercu, B., Coutin, L. and Savy, N. (2012). Sharp large deviations for the non-stationary Ornstein–Uhlenbeck process. *Stoch. Process. Appl.* **122,** 3393–3424.

[4] Dembo, A. and Zeitouni, O. (1998). *Large Deviations Techniques and Applications* (Appl. Math. (New York) **38**), 2nd edn. Springer, New York.
[5] Demni, N. and Zani, M. (2009). Large deviations for statistics of the Jacobi process. *Stoch. Process. Appl.* **119,** 518–533.
[6] Donati-Martin, C. and Yor, M. (1993). On some examples of quadratic functionals of Brownian motion. *Adv. Appl. Prob.* **25,** 570–584.
[7] Florens-Landais, D. and Pham, H. (1999). Large deviations in estimation of an Ornstein–Uhlenbeck model. *J. Appl. Prob.* **36,** 60–77.
[8] Gao, F. and Jiang, H. (2009). Deviation inequalities and moderate deviations for estimators of parameters in an Ornstein–Uhlenbeck process with linear drift. *Electron. Commun. Prob.* **14,** 210–223.
[9] Janson, S. (1997). *Gaussian Hilbert Spaces* (Camb. Tracts Math. **129**). Cambridge University Press.
[10] Jeanblanc, M., Yor, M. and Chesney, M. (2009). *Mathematical Methods for Financial Markets*. Springer, London.
[11] Jiang, H. (2012). Berry–Esseen bounds and the law of the iterated logarithm for estimators of parameters in an Ornstein–Uhlenbeck process with linear drift. *J. Appl. Prob.* **49,** 978–989.
[12] Kutoyants, Y. A. (2004). *Statistical Inference for Ergodic Diffusion Processes*. Springer, London.
[13] Vasicek, O. (1977). An equilibrium characterization of the term structure. *J. Financial Econom.* **5,** 177–188.
[14] Zani, M. (2002). Large deviations for squared radial Ornstein–Uhlenbeck processes. *Stoch. Process. Appl.* **102,** 25–42.