# Generalised Solutions of Laplace's Equation

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The present paper contains solutions of the tensor generalisation of Laplace's Equation. The results obtained are summarised in the two theorems enunciated in § 1. They apply only to the case when the Riemannian space forming the background of the theory is flat. In the concluding paragraph a special case is considered, and it is shown that the present theory is closely connected with Whittaker's well known general solution of the ordinary Laplace's Equation.[1]

## § 1. Introduction.

Let

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu \tag{1.1}$$

define the metric of an $n$-dimensional Euclidean space; that is, one for which the Riemann-Christoffel tensor is everywhere zero. Let $\Omega$ be one half of the square of the geodesic distance between the two points $(x^i)$ and $(\bar{x}^i)$ of this space,[2] so that $\Omega$ is a scalar function of the two sets of variables $(x^i)$, $(\bar{x}^i)$. Let further the coordinates $(\bar{x}^i)$ be each functions of a variable $\tau$. Then $\Omega$ is a function of $x^1, x^2, \ldots, x^n$, and $\tau$, and we shall later define $\tau$ as a function of the $x^i$ by means of the equation

$$\Omega = 0. \tag{1.2}$$

Greek suffixes will be used to denote covariant differentiations with respect to the $x^i$, with $\tau$ kept constant, the only exception to this rule being that the suffix $\tau$ will denote ordinary partial differentiations with respect to $\tau$. Thus, for example,

$$\left.\begin{aligned} \Omega_\mu &= \partial\,\Omega/\partial x^\mu, & \Omega_\tau &= \partial\,\Omega/\partial\tau, \\ \Omega_{\mu\nu} &= \frac{\partial^2\,\Omega}{\partial x^\mu\,\partial x^\nu} - \{\mu\nu,\, a\}\,\frac{\partial\,\Omega}{\partial x^a}\,, & \Omega_{\tau\tau} &= \frac{\partial^2\,\Omega}{\partial\tau^2}, \\ \Omega_{\tau\mu\nu} &= \frac{\partial^2\,\Omega_\tau}{\partial x^\mu\,\partial x^\nu} - \{\mu\nu,\, a\}\,\frac{\partial\,\Omega_\tau}{\partial x^a}\,, & \Omega_{\tau\mu} &= \frac{\partial^2\,\Omega}{\partial\tau\partial x^\mu}\,, \end{aligned}\right\} \tag{1.3}$$

[1] Whittaker and Watson, "Modern Analysis" (1920), § 18.3.

[2] Some properties of this function have been investigated in earlier papers, particularly (i) *Proc. London Math. Soc.*, **31** (1930), 225; (ii) *ibid.*, **32** (1931), 87. These will be referred to as papers 1 and 2 respectively.

and so on. It must be emphasised that in these definitions the partial differentiations with respect to the $x^i$ are strict, that is, they treat $\tau$ as a constant as well as the other $x$'s. The summation convention does not of course hold for the suffix $\tau$.

For convenience the Christoffel symbol $\{\lambda\mu, \nu\}$ will be denoted by $\Gamma^{\nu}_{\lambda\mu}$. Further, the evaluation at $(\bar{x}^i)$ of any function of the $x^i$ will be indicated by the superposing of a bar on the functional symbol.

The partial differential equation of which solutions are sought is

$$V^{\lambda}_{\lambda} \equiv g^{\lambda\mu} \left\{ \frac{\partial^2 V}{\partial x^{\lambda}\, \partial x^{\mu}} - \Gamma^{\alpha}_{\lambda\mu} \frac{\partial V}{\partial x^{\alpha}} \right\} = 0. \tag{1.4}$$

The following are the theorems proved.

THEOREM I. *If the functions $\bar{x}^i(\tau)$ are chosen to satisfy the differential equations*

$$\bar{g}_{\mu\nu} \frac{d\bar{x}^{\mu}}{d\tau} \frac{d\bar{x}^{\nu}}{d\tau} = 0, \tag{1.5}$$

$$\frac{d^2 \bar{x}^{\mu}}{d\tau^2} + \bar{\Gamma}^{\mu}_{\alpha\beta} \frac{d\bar{x}^{\alpha}}{d\tau} \frac{d\bar{x}^{\beta}}{d\tau} = 0, \tag{1.6}$$

*then, for all values of $n$, a solution of the partial differential equation $V^{\lambda}_{\lambda} = 0$ is given by*

$$V = f(\Omega_{\tau}), \tag{1.7}$$

*where, after differentiating,*[1] *$\tau$ is expressed as a function of $x^1, x^2, \ldots, x^n$ by means of the equation $\Omega = 0$, and where $f(\Omega_{\tau})$ is an arbitrary function of $\Omega_{\tau}$.*

That the equations (1.5) and (1.6) are compatible is well known.[2]

THEOREM II. *A solution of the equation $V^{\lambda}_{\lambda} = 0$ is given by*

$$V = \phi(\tau) / \Omega_{\tau}^{\frac{1}{2}(n-2)}, \tag{1.8}$$

*where $\phi(\tau)$ is an arbitrary function of $\tau$, and $\tau$ is expressed as a function of the $x$'s by means of the equation $\Omega = 0$. If the number $n$ of the*

[1] $\partial\Omega/\partial\tau$ is in general a function of $\tau$ as well as of the $x$'s, so $\tau$ must be eliminated in order that the solution should be expressed as a function of the $x$'s only.

[2] See, for example, Veblen, "Invariants of Quadratic Differential Forms" *Camb.*). *Math. Tract.* No. 24) (1927), 95.

*variables is equal to* 2 *or* 4 *there is no limitation on the choice of the functions* $\bar{x}^i(\tau)$; *but if n has any other value, these functions must satisfy the conditions* (1.5) *and* (1.6).

## § 2. PRELIMINARY FORMULAE.

It is well known that $\Omega$ satisfies[1] the partial differential equation

$$\Omega^\lambda \Omega_\lambda = 2\Omega, \tag{2.1}$$

where $\Omega^\lambda = g^{\lambda a}\Omega_a$.

Moreover, it is shown elsewhere[2] that, the space being flat,

$$\Omega_{\mu\nu} = g_{\mu\nu}. \tag{2.2}$$

Furthermore, since we have put

$$\Omega = 0 \tag{2.3}$$

it follows, by differentiating partially with respect to $x^\lambda$, that

$$\Omega_\lambda + \Omega_\tau \tau_\lambda = 0,$$

where $\tau_\lambda = \partial\tau/\partial x^\lambda$. Hence

$$\tau_\lambda = -\Omega_\lambda/\Omega_\tau. \tag{2.4}$$

Differentiating (2.1) twice in succession with respect to $\tau$, we get

$$\Omega_{\tau\lambda}\Omega^\lambda = \Omega_\tau, \tag{2.5}$$

$$\Omega_{\tau\tau\lambda}\Omega^\lambda + \Omega_{\tau\lambda}\Omega^\lambda_\tau = \Omega_{\tau\tau}. \tag{2.6}$$

From (2.2), raising the suffix $\nu$ and contracting,

$$\Omega^\mu_\mu = n, \tag{2.7}$$

and the differentiation of this equation with respect to $\tau$ gives

$$\Omega^{\ \mu}_{\tau\mu} = 0. \tag{2.8}$$

By (2.1), (2.3), (2.4), it follows that

$$\tau^\lambda \tau_\lambda = 0. \tag{2.9}$$

By (2.4),

$$\tau^\lambda_\lambda = g^{\lambda\alpha}_{\ }\tau_{\lambda a} = -g^{\lambda a}\left[\left(\frac{\Omega_\lambda}{\Omega_\tau}\right)_a + \left(\frac{\Omega_\lambda}{\Omega_\tau}\right)_\tau \tau_a\right]$$

$$= \Omega_\tau^{-2}(\Omega^\lambda_\tau\Omega_\lambda - \Omega_\tau\Omega^\lambda_\lambda - \Omega_\tau\Omega_{\tau\lambda}\tau^\lambda + \Omega_{\tau\tau}\Omega_\lambda\tau^\lambda),$$

whence, by (2.4), (2.5), (2.7) and (2.9),

$$\tau^\lambda_\lambda = -(n-2)\Omega_\tau^{-1}. \tag{2.10}$$

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[1] This in fact follows at once from equations (1) and (10) of paper 2.

[2] Paper 1, § 2, where $\Omega$ denotes twice the function here represented by $\Omega$.

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Lastly, it follows from (2.2), by interchanging the $x$'s and the $\bar{x}$'s, that

$$\frac{\partial^2 \Omega}{\partial \bar{x}^\mu \partial \bar{x}^\nu} - \bar{\Gamma}^\alpha_{\mu\nu} \frac{\partial \Omega}{\partial \bar{x}^\alpha} = \bar{g}_{\mu\nu}. \tag{2.11}$$

But
$$\frac{\partial \Omega}{\partial \tau} = \frac{\partial \Omega}{\partial \bar{x}^\alpha} \frac{\partial \bar{x}^\alpha}{\partial \tau}$$

and
$$\frac{\partial^2 \Omega}{\partial \tau^2} = \frac{\partial \Omega}{\partial \bar{x}^\alpha} \frac{d^2 \bar{x}^\alpha}{d\tau^2} + \frac{\partial^2 \Omega}{\partial \bar{x}^\alpha \partial \bar{x}^\beta} \frac{d\bar{x}^\alpha}{d\tau} \frac{d\bar{x}^\beta}{d\tau}$$

and therefore, by (2.11),

$$\Omega_{\tau\tau} = \bar{g}_{\mu\nu} \frac{d\bar{x}^\mu}{d\tau} \frac{d\bar{x}^\nu}{d\tau} + \frac{\partial \Omega}{\partial \bar{x}^\alpha} \left( \frac{d^2 \bar{x}^\alpha}{d\tau^2} + \bar{\Gamma}^\alpha_{\mu\nu} \frac{d\bar{x}^\mu}{d\tau} \frac{d\bar{x}^\nu}{d\tau} \right). \tag{2.12}$$

## § 3. PROOF OF THEOREM I.

We now show, by direct substitution, that any function of $\Omega_\tau$, say

$$U = f(\Omega_\tau), \tag{3.1}$$

is a solution of the equation

$$V^\lambda_\lambda = 0 \tag{3.2}$$

provided that the conditions (1.5) and (1.6) are satisfied.

For

$$U_\lambda = f'(\Omega_\tau)(\Omega_{\tau\lambda} + \Omega_{\tau\tau}\tau_\lambda), \tag{3.3}$$

and

$$\begin{aligned} U^\lambda_\lambda &= f''(\Omega_\tau)(\Omega_{\tau\lambda} + \Omega_{\tau\tau}\tau_\lambda)(\Omega^\lambda_\tau + \Omega_{\tau\tau}\tau^\lambda) \\ &\quad + f'(\Omega_\tau)(\Omega^{\ \lambda}_{\tau\lambda} + 2\,\Omega_{\tau\tau\lambda}\tau^\lambda + \Omega_{\tau\tau\tau}\tau_\lambda\tau^\lambda + \Omega_{\tau\tau}\tau^\lambda_\lambda) \\ &= f''(\Omega_\tau)(\Omega_{\tau\lambda}\Omega^\lambda_\tau + 2\,\Omega_{\tau\tau}\Omega^\lambda_\tau\tau_\lambda) + f'(\Omega_\tau)(\Omega^{\ \lambda}_{\tau\lambda} + 2\,\Omega_{\tau\tau\lambda}\tau^\lambda + \Omega_{\tau\tau}\tau^\lambda_\lambda), \end{aligned}$$

using equation (2.9).

By (2.4), (2.5), (2.6), (2.8) and (2.10), it quickly follows that

$$U^\lambda_\lambda = -f''(\Omega_\tau)(\Omega_{\tau\tau\lambda}\Omega^\lambda + \Omega_{\tau\tau}) - \Omega_\tau^{-1} f'(\Omega_\tau)\{2\,\Omega_{\tau\tau\lambda}\Omega^\lambda + (n-2)\,\Omega_{\tau\tau}\}. \tag{3.4}$$

By (2.12), if the functions $\bar{x}^i(\tau)$ satisfy the relations

$$\bar{g}_{\mu\nu} \frac{d\bar{x}^\mu}{d\tau} \frac{d\bar{x}^\nu}{d\tau} = 0, \tag{3.5}$$

$$\frac{d^2 \bar{x}^\mu}{d\tau^2} + \bar{\Gamma}^\mu_{\alpha\beta} \frac{d\bar{x}^\alpha}{d\tau} \frac{d\bar{x}^\beta}{d\tau} = 0, \tag{3.6}$$

then
$$\Omega_{\tau\tau} = 0. \tag{3.7}$$

Differentiating with respect to $x^\lambda$,

$$\Omega_{\tau\tau\lambda} + \Omega_{\tau\tau\tau}\,\tau_\lambda = 0,$$

and therefore

$$\Omega_{\tau\tau\lambda}\,\Omega^\lambda = 0, \tag{3.8}$$

since $\Omega^\lambda\,\tau_\lambda = 0$ by (2.4) and (2.9).

By (3.4), (3.7), (3.8) it follows that

$$U^\lambda_\lambda = 0$$

provided that the conditions (3.5) and (3.6) are satisfied. This concludes the proof of Theorem I.

## § 4. PROOF OF THEOREM II.

The next problem is to show that, if

$$V = \phi(\tau) / \Omega_\tau^{\frac{1}{2}(n-2)}, \tag{4.1}$$

where $\phi(\tau)$ is an arbitrary function of $\tau$, then $V$ satisfies the equation $V^\lambda_\lambda = 0$. It will be shown that the restrictions (3.5) and (3.6) must still be imposed except when $n = 2$ or $n = 4$.

Consider first the function $W$ defined by

$$W = \Omega_\tau^{-\frac{1}{2}(n-2)}. \tag{4.2}$$

Putting $f(\Omega_\tau) = \Omega_\tau^{-\frac{1}{2}(n-2)}$ in (3.3) and (3.4),

$$W_\lambda = -\tfrac{1}{2}(n-2)\,\Omega_\tau^{-\frac{1}{2}n}\,(\Omega_{\tau\lambda} + \Omega_{\tau\tau}\,\tau_\lambda), \tag{4.3}$$

$$W^\lambda_\lambda = \tfrac{1}{4}(n-2)(n-4)\,\Omega_\tau^{-\frac{1}{2}(n+2)}\,(\Omega_{\tau\tau} - \Omega_{\tau\tau\lambda}\,\Omega^\lambda_{,}). \tag{4.4}$$

Hence, if $n = 2$ or $n = 4$, we have

$$W^\lambda_\lambda = 0 \tag{4.5}$$

without any restrictions being placed on the choice of the functions $\bar{x}^i(\tau)$. But, if $n$ has neither of these values, it follows as in the previous paragraph that we shall still have

$$W^\lambda_\lambda = 0 \tag*{(4.5)*}$$

provided that the $\bar{x}^i(\tau)$ satisfy the relations

$$\bar{g}_{\mu\nu}\,\frac{d\bar{x}^\mu}{d\tau}\,\frac{d\bar{x}^\nu}{d\tau} = 0, \tag{4.6}$$

$$\frac{d^2\bar{x}^\mu}{d\tau^2} + \bar{\Gamma}^\mu_{\alpha\beta}\,\frac{d\bar{x}^\alpha}{d\tau}\,\frac{d\bar{x}^\beta}{d\tau} = 0. \tag{4.7}$$

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Now consider the function $V$ defined by

$$V = W\phi(\tau), \tag{4.8}$$

where $\phi(\tau)$ is any function of $\tau$. Covariant differentiation gives

$$V_\lambda = W_\lambda \phi(\tau) + W\tau_\lambda \phi'(\tau),$$

$$V^\lambda_\lambda = W^\lambda_\lambda \phi(\tau) + 2W_\lambda \tau^\lambda \phi'(\tau) + W\tau_\lambda \tau^\lambda \phi''(\tau) + W\tau^\lambda_\lambda \phi'(\tau)$$

$$= \phi'(\tau)(2W_\lambda \tau^\lambda + W\tau^\lambda_\lambda),$$

by (4.5), (4.5)* and (2.9).

Equation (4.8) therefore gives a solution of $V^\lambda_\lambda = 0$ provided that

$$2W_\lambda \tau^\lambda + W\tau^\lambda_\lambda = 0.$$

But by (4.3), (4.2) and (2.10), the left-hand side of this equation is equal to

$$-(n-2)\,\Omega_\tau^{-\frac{1}{2}n}(\Omega_{\tau\lambda}\tau^\lambda + \Omega_{\tau\tau}\tau_\lambda\tau^\lambda) - (n-2)\,\Omega_\tau^{-\frac{1}{2}n},$$

which is zero in virtue of equations (2.4), (2.5) and (2.9), for all values of $n$. We deduce finally, therefore, that

$$V = \phi(\tau)\,/\,\Omega_\tau^{\frac{1}{2}(\ \,-2)}$$

is a solution of $V^\lambda_\lambda = 0$ provided that, when $n$ has a value other than 2 or 4, the choice of the functions $x^i(\tau)$ is restricted by the equations (4.6) and (4.7).

When $n = 2$ this theorem is the tensor generalisation of the well-known fact that any function of $x \pm iy$ is a solution of the equation $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$. When $n = 4$ it gives a generalisation of a solution, due to Conway, of the classical wave-equation of mathematical physics.[1]

## § 5. CONNECTION WITH WHITTAKER'S SOLUTION OF LAPLACE'S EQUATION.

Apply Theorem I to the case when $n = 3$ and the metric is given by

$$ds^2 = dx^2 + dy^2 + dz^2; \qquad (x^1 = x,\ x^2 = y,\ x^3 = z).$$

[1] See Bateman, "Electrical and Optical Wave Motion" (1915), 115.

The equation to be solved is then the ordinary Laplace's Equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0. \tag{5.1}$$

Also, of course,

$$2\,\Omega = (x - \bar{x})^2 + (y - \bar{y})^2 + (z - \bar{z})^2. \tag{5.2}$$

The restrictions (4.6) and (4.7) placed on the choice of $\bar{x}$, $\bar{y}$, $\bar{z}$ as functions of $\tau$ reduce in this case to

$$\left(\frac{d\bar{x}}{d\tau}\right)^2 + \left(\frac{d\bar{y}}{d\tau}\right)^2 + \left(\frac{d\bar{z}}{d\tau}\right)^2 = 0, \tag{5.3}$$

and
$$\frac{d^2\bar{x}}{d\tau^2} = 0 = \frac{d^2\bar{y}}{d\tau^2} = \frac{d^2\bar{z}}{d\tau^2}\,. \tag{5.4}$$

The most general solutions of (5.3) and (5.4) are

$$\left.\begin{aligned} \bar{x} &= a + \lambda i \tau \cos u \\ \bar{y} &= b + \lambda i \tau \sin u \\ \bar{z} &= c + \lambda \tau \end{aligned}\right\} \tag{5.5}$$

where $i = \sqrt{-1}$ and $a$, $b$, $c$, $\lambda$, $u$ are arbitrary constants. Take $a = b = c = 0$ and $\lambda = -1$. Substituting from (5.5) in (5.2), we have

$$2\,\Omega = r^2 + 2\tau\,(ix \cos u + iy \sin u + z),$$

where
$$r^2 = x^2 + y^2 + z^2,$$

and hence
$$\Omega_\tau = ix \cos u + iy \sin u + z.$$

A solution of equation (5.1) is therefore, by Theorem I,

$$V = f\,(ix \cos u + iy \sin u + z,\ u), \tag{5.6}$$

where $f$ is an arbitrary function and $u$ an arbitrary constant.[1]
It follows therefore that

$$\int f\,(ix \cos u + iy \sin u + z,\ u)\,du \tag{5.7}$$

is also a solution of (5.1), provided that the limits of integration are such that differentiation under the integral sign is permissible.

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[1] Since $u$ is an arbitrary constant, the function $f$ of the two arguments $ix \cos u + iy \sin u + z$ and $u$, is (regarded as a function of $x$, $y$, $z$), an arbitrary function of the former argument only; that is, of $\partial\Omega/\partial\tau$ only.

Whittaker has shown[1] that the most general solution of Laplace's Equation is of this form.

An application of Theorem II to the same special case leads ultimately to the conclusion that the integral

$$\int \frac{1}{r} \psi\left(\frac{ix\cos u + iy\sin u + z}{r^2}, u\right) du \tag{5.8}$$

gives a solution of (5.1), $\psi$ being an arbitrary function of its arguments. It is however a well known fact that if a function $\chi(x, y, z)$ satisfies Laplace's Equation, so also does $\frac{1}{r}\chi\left(\frac{x}{r^2}, \frac{y}{r^2}, \frac{z}{r^2}\right)$. The solution (5.8) is therefore deducible from (5.7).

[1] Whittaker and Watson, *loc. cit.*

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