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Spaces of functions and sections with paracompact domain

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We study spaces of continuous functions and sections with domain a paracompact Hausdorff k-space X and range a nilpotent CW complex Y, with emphasis on localization at a set of primes. For $\mathrm{map}_\phi(X, Y)$, the space of maps with prescribed restriction ϕ on a suitable subspace $A \subset X$, we construct a natural spectral sequence of groups that converges to $\pi_*(\mathrm{map}_\phi(X, Y))$ and allows for detection of localization on the level of E^2. Our applications extend and unify the previously known results.

Keywords: Function space; Federer spectral sequence; localization

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1. Introduction

Let $\mathrm{map}(X, Y)$ denote the space of continuous maps $X \to Y$ where Y has the homotopy type of a nilpotent CW complex, and let $l\colon Y \to Y_{(P)}$ denote localization at the set of primes P. The induced map

$$l_*\colon\ \mathrm{map}(X,Y) \to \mathrm{map}(X,Y_{(P)}) \tag{1.1}$$

is known to be localization at P on each path component under various assumptions on X. By the classical results of Hilton, Mislin, Roitberg and Steiner [13, Theorems A and B], X can be a homologically finite CW complex. The more recent results of Klein, Schochet, and Smith [16] (and [26]) show that this also holds whenever X is a compact metric space and one knows, *a priori*, that the function spaces involved are nilpotent. Similar results hold for spaces of sections (see Møller [24] for the case of a CW domain, and [16] for the case of a compact metric domain). Here we address the question of nilpotency and localization for function and section spaces with more general paracompact domains by a unified approach; the case of a locally compact subspace of a Euclidean space has served as a particular motivation.

To achieve our goal, we need a good grip on the homotopy groups of $\mathrm{map}(X, Y)$. Federer [10] constructed a spectral sequence of groups, arising from an exact couple, converging to $\pi_*(\mathrm{map}(X, Y))$ for a finite-dimensional CW complex X and a simple CW complex Y (see also [27] for a based rational version). Dyer [8] generalized Federer's spectral sequence to the case of a paracompact Hausdorff space of finite covering dimension X. The method was to represent X as an inverse limit of polyhedra and take the direct limit of Federer's exact couples. Analogous approaches were those of Dror Farjoun and Schochet [7] and Klein, Schochet and Smith [16].

Here, we take the dual approach (as in [13] and [24]), and express the nilpotent CW complex Y as a weak limit of the refined Postnikov tower of principal fibrations $Y_q \to Y_{q-1}$. We note that, similarly as in the case of a CW complex X, the associated tower of maps $\mathrm{map}(X, Y_q) \to \mathrm{map}(X, Y_{q-1})$ is also one of principal fibrations under reasonable conditions. Of course, there is the spectral sequence of Bousfield and Kan [5] associated to any tower of fibrations, but that involves sets and groups acting on sets, and is not ready-made for localization. We observe that Federer's method admits an abstract generalization to any tower of principal fibrations and obtain a spectral sequence of groups (arising from an exact couple) and a slightly tweaked version thereof that contains sufficient information to detect localization of the limiting homotopy groups at the E^2-level. It seems that these observations are novel. To state our main result, we first let $\mathrm{map}_\phi(X, Y)$ denote the subspace of $\mathrm{map}(X, Y)$ consisting of maps whose restriction to the closed subspace $A \subset X$ is a prescribed map $\phi\colon A \to Y$. Further let X be a paracompact Hausdorff k-space; we will be using the shorthand $X \in \mathrm{PHK}$. (The obvious examples are metric spaces, compact Hausdorff spaces, and CW complexes.) Assume that Y is a CW approximation to the inverse limit of successive principal fibrations with fibres $K(\pi^q, n_q)$ where $1 = n_1 \leqslant n_2 \leqslant \dots$. Let $\check{H}^*(X, A; \pi)$ denote Čech cohomology with coefficients in the abelian group π. Additionally, assume one of the following.

- **(i)** $\phi = \mathrm{const}_{y_0}$ where y_0 is nondegenerate; we use $\mathrm{map}_{A\to y_0}(X, Y)$ in this case.
- **(ii)** $A \hookrightarrow X$ is a closed Hurewicz cofibration (for example, A may be empty).
- **(iii)** (X, A) is a proper pair (see definition 1.5) and Y is an ANR, i.e., an absolute neighbourhood retract for metric spaces.

Theorem 1.1. *Take $f \in \mathrm{map}_\phi(X, Y)$. There exists an upper half-plane homology type spectral sequence of groups where $E^2_{-p,q}$ is isomorphic with $\check{H}^{n_q-q+p}(X, A; \pi^q)$ for $1 \leqslant q \geqslant p \geqslant 0$ and trivial otherwise. The n-th differential consists of morphisms $d^n_{-p,q}\colon E^n_{-p,q} \to E^n_{-p-n,q+n-1}$.*

Assume also that Y has only finitely many nontrivial homotopy groups or that $\dim X$ (respectively $\dim(X, A)$ under (iii) above) is finite. Then $\mathrm{map}_\phi(X, Y)$ is nilpotent and the spectral sequence converges to $\pi_k(\mathrm{map}_\phi(X, Y), f)$ for $k \geqslant 1$.

The spectral sequence is natural with respect to maps of pairs $(X, A) \to (X', A')$ over Y and with respect to maps $Y \to Y'$ induced by maps of classifying spaces $K(\pi^q_Y, n_q + 1) \to K(\pi^q_{Y'}, n_q + 1)$. Let $E^n_{-p,q} \to E'^n_{-p,q}$ be the associated map in any of the two cases and let $\pi_k \to \pi'_k$ be the associated morphism of homotopy groups of mapping spaces. Assume that both E^n and E'^n converge and that

- *$E^2_{-p,q} \to E'^2_{-p,q}$ P-localizes for $q > p \geqslant 0$, and*
- *the torsion in $E'^2_{-q,q}$ is P-local and $E^2_{-q,q} \to E'^2_{-q,q}$ is P-injective for $q > 0$.*

Then $\pi_k \to \pi'_k$ are P-localizations for all $k \geqslant 1$.

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If Y is connected and simple in all dimensions, we obtain a second-quadrant sequence with non-zero E^2-terms given by $E^2_{-p,q} \cong \check{H}^p(X,\, A; \pi_q(Y))$ for $q \geqslant p \geqslant 0$.

The theorem is proved in § 5 as a consequence of theorem 3.1 and the results in § 4; theorem 3.1 also contains a precise description of the first differential in terms of the k-invariants of Y. We emphasize that in the absolute case (i.e., $A = \emptyset$), the result always depends only on the homotopy type of Y, while if A is nonempty and not cofibred in X, then the topology of Y will play a role (§ 7 contains details for the 'twisted' relative case). For example, Y is an ANR if it is a smooth manifold or a locally finite CW complex.

The nilpotency of $\mathrm{map}_\phi(X,\, Y)$ and convergence of the spectral sequence are automatic if Y is a Postnikov section; this is a feature of our approach. The applications of [**16**] include cases where $Y = K_{(0)}$ is the rationalization of a finite CW complex K; if K is rationally elliptic, then Y will be a Postnikov section, guaranteeing convergence for any $X \in \mathrm{PHK}$. (Compare also with [**7**, Theorem A].)

The following is a formal consequence of the naturality in theorem 1.1.

Corollary 1.2. *Assume the conditions for convergence in theorem 1.1. If the canonical morphisms $\check{H}^i(X,\, A; \pi^q) \to \check{H}^i(X,\, A; \pi^q \otimes \mathbb{Z}_{(P)})$ are localizations at P for all q and all $i \leqslant n_q$, then $\mathrm{map}_\phi(X,\, Y) \to \mathrm{map}_\phi(X,\, Y_{(P)})$ localizes homotopy groups at all basepoints. (Here, $Y_{(P)}$ is required to be an ANR if Y is.)*

Universal coefficients for Čech cohomology of compact spaces immediately imply

Corollary 1.3. *Suppose that X is compact Hausdorff and Y has the homotopy type of a nilpotent CW complex. If $\dim X < \infty$ or Y is a Postnikov section, the induced map (1.1) is localization on each path component.*

Example 1.4. In [**15**], D. S. Kahn constructed a compact metric space X with trivial integral Čech cohomology groups $\check{H}^q(X; \mathbb{Z})$ for $q > 1$ but with essential maps $X \to \mathbb{S}^3$. Let $Y = B\mathbb{S}^3$ be the classifying space of the sphere $\mathbb{S}^3$ with basepoint y_0. Then $\pi_1(\mathrm{map}(X,\, Y),\, \mathrm{const}_{y_0}) \cong \pi_1(\mathrm{map}_*(X,\, Y),\, \mathrm{const}_{y_0}) \cong [X,\, \Omega B\mathbb{S}^3]$ is nontrivial. Therefore, the spectral sequence associated to $\mathrm{map}(X,\, Y)$ cannot converge to $\pi_*(\mathrm{map}(X,\, Y))$. In particular, X is infinite-dimensional (as noted by Kahn).

We need to impose restrictions on a noncompact X for $\mathrm{map}(X,\, Y)$ to be amenable to our methods. Presumably, it would be sufficient to have a regular complete lattice of zero sets on X that determines the topology and is generated by compact zero sets (see [**28**, pp. 1 and 2] for the definition of a regular lattice). To avoid technicalities, we content ourselves with the 'countable case' as follows.

Definition 1.5. *A topological pair $(X,\, A)$ will be called proper if $A \in \mathrm{PHK}$ and X is the union of an ascending chain of subspaces $A = X_0 \subset X_1 \subset X_2 \subset \ldots$ (called an admissible chain) such that for all $i \geqslant 1$ there is a pushout*

$$\begin{array}{ccc} S_i & \xrightarrow{\gamma_i|_{S_i}} & X_{i-1} \\ \big\downarrow & & \big\downarrow \\ C_i & \xrightarrow{\gamma_i} & X_i \end{array} \tag{1.2}$$

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*where S_i is a zero set in the compact Hausdorff space C_i and X has the weak (colimit) topology with respect to $\{X_i\}$. Each X_i is paracompact and it follows that $X \in \mathrm{PHK}$ by Michael [**21**, Theorem 8.2]. Also, the X_i are zero sets in X.*

We call X a proper space if $(X, \emptyset)$ is a proper pair. In this case, the X_i are compact Hausdorff. Examples include countable CW complexes and proper metric spaces (i.e. closed balls are compact). Also, locally compact subspaces of Euclidean spaces $\mathbb{R}^d$ are proper. More generally, if X is a Hausdorff k-space that admits a proper function $\phi\colon X \to [0, \infty)$ (the preimages of compact sets are compact), then one sees that X is proper by setting $X_i = \phi^{-1}([0, i])$.

Observe that if (X, A) is a proper pair with admissible chain $\{X_i\}$ and B is a zero set in X, then $(X, A \cup B)$ is a proper pair with admissible chain $\{X_i \cup B\}$. In particular, if X is a proper space, then (X, B) is a proper pair for any zero set B.

A proper pair (X, A) will be called locally finite-dimensional if the C_i have finite covering dimension. Every countable relative CW complex is then a locally finite-dimensional proper pair. We write $\dim(X, A) \leqslant d$ if $\dim C_i \leqslant d$ for all i and call such a pair finite-dimensional.

Any admissible chain of a compact proper space (or pair) is essentially finite. In particular, a locally finite-dimensional compact proper space is finite-dimensional.

Hilton, Mislin, Roitberg, and Steiner pointed out in [**13**] that the natural class of CW complexes X for which (1.1) always localizes is that of globally homologically finite ones; X is such when $\oplus_{n=0}^{\infty} H_n(X)$ is finitely generated. From our point of view, that is precisely because a CW complex X is globally homologically finite if and only if $\mathrm{map}(X, Y)$ has the homotopy type of a CW complex for every nilpotent CW complex Y. (For sufficiency, use [**28**, Proposition 2.6.4] and [**30**, Corollary 1.2]. For necessity, use the proof of [**28**, **c.** of Theorem 4.5.3].) Thus in this case, (1.1) is actually localization in the category of nilpotent spaces of CW homotopy type.

The following is a generalization of Theorems A and B of [**13**].

THEOREM 1.6. *Let (X, A) be a proper pair and let Y be a nilpotent CW complex. Fix a map $\phi\colon A \to Y$. If $\mathrm{map}_\phi(X, Y)$ and $\mathrm{map}_{l\phi}(X, Y_{(P)})$ have CW homotopy type, then they are nilpotent and (1.1) is CW localization at P on path components.*

It turns out that the question of whether or not $\mathrm{map}(X, Y)$ has CW homotopy type is intimately related to the behaviour of the spectral sequence of theorem 1.1.

DEFINITION 1.7. *Let A be a closed subspace of $X \in \mathrm{PHK}$. The pair (X, A) is quasi-finite, $(X, A) \in \mathcal{QF}$, if for each abelian group G and all $n \geqslant 1$, the space $\mathrm{map}_{A\to *}(X, K(G, n))$ has the homotopy type of a CW complex. (We assume $K(G, n)$ to be well-pointed.) We call X quasi-finite if $(X, \emptyset)$ is a quasi-finite pair.*

By [**28**, Theorem 4.5.3, **a.**], a CW complex belongs to $\mathcal{QF}$ if and only if all its homology groups are finitely generated.

PROPOSITION 1.8. *Let (X, A) be a quasi-finite and locally finite-dimensional proper pair, and let Y have the homotopy type of a nilpotent CW complex; if A is nonempty and not cofibred in X, we assume that Y is an ANR. Let $\phi\colon A \to Y$ be any map.*

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If $\operatorname{map}_\phi(X, Y)$ *has the homotopy type of a CW complex, then the spectral sequence of theorem* 1.1 *converges to* $\pi_*(\operatorname{map}_\phi(X, Y), f)$ *for any* $f \in \operatorname{map}_\phi(X, Y)$*.*

We note that if (X, A) is a skeleton-finite relative CW pair, then it is a quasi-finite and locally finite-dimensional proper pair.

For proper pairs, quasi-finiteness is actually a cohomological property as follows. Recall that a tower of groups $\cdots \to G_2 \to G_1$ is *Mittag-Leffler* if for each i, the images of $G_j \to G_i$ stabilize for all big enough $j \geqslant i$.

Proposition 1.9. *Let* (X, A) *be a proper pair with an admissible chain* $\{X_i\}$*. Then* $(X, A) \in \mathcal{QF}$ *if and only if for each abelian group* G*, the towers of abelian groups*

$$\cdots \to \check{H}^k(X_3, A; G) \to \check{H}^k(X_2, A; G) \to \check{H}^k(X_1, A; G)$$

are Mittag-Leffler for $0 \leqslant k < \infty$ *and, in addition, for each* k *there exists* i *such that restriction* $\check{H}^k(X, A; G) \to \check{H}^k(X_i, A; G)$ *is injective.*

We turn to spaces of sections.

Let X be connected and let $Q \to X$ be a Hurewicz fibration with space of sections $\Gamma(X, Q) \subset \operatorname{map}(X, Q)$. Assume that $Q \to X$ is a pullback of a fibration $p\colon E \to B$ of spaces of CW homotopy type where B is connected and the typical fibre of p, call it Y, is nilpotent. Let P be a set of primes and let $\ell\colon E \to E_{(P)}$ be the fibrewise localization at P over B. By pulling back, we obtain a fibrewise localization $\ell\colon Q \to Q_{(P)}$ over X. Note that ℓ induces a map

$$\ell_*\colon \Gamma(X, Q) \to \Gamma(X, Q_{(P)}). \tag{1.3}$$

In [**26**], the authors studied the effect of ℓ_* for compact X provided $\Gamma(X, Q)$ is known, *a priori*, to be nilpotent. We give conditions for nilpotency and enhance their localization result [**26**, Theorem 3] as follows.

Theorem 1.10. *Suppose* X *is a connected proper space and assume that, in addition,* X *is locally finite-dimensional or* Y *is a Postnikov section.*

- (**a**) *If* X *is compact, then* $\Gamma(X, Q)$ *and* $\Gamma(X, Q_{(P)})$ *are nilpotent spaces of CW type and* (1.3) *is CW localization at* P *on each path component.*
- (**b**) *If* $\Gamma(X, Q)$ *has CW type, then* $\Gamma(X, Q)$ *and* $\Gamma(X, Q_{(P)})$ *are nilpotent and* (1.3) *localizes at* P *on path components. If, in addition,* Y *is a Postnikov section, also* $\Gamma(X, Q_{(P)})$ *has CW type and* (1.3) *is CW localization at* P*.*

A relative version for sections prescribed on $A \subset X$, which also generalizes the main results of [**24**], is stated in § 7.

Underlying categories and techniques. Our results are valid in the category $\mathcal{K}$ of k-spaces and in the category $\mathcal{Top}$ of all topological spaces (see [**20**, Sections 1.1 and 4.1]). There are the underlying Strø m closed model category structures on $\mathcal{K}$ and $\mathcal{Top}$ with Hurewicz fibrations, closed Hurewicz cofibrations, and homotopy equivalences (called h-equivalences in [**20**]). While for some of our results, Serre fibrations would be sufficient, we also lean on the results of Stasheff [**33**] and

Brown and Heath [4] that require Hurewicz fibrations, so we stick to the latter. Our mapping spaces, which have the (k-ified) compact open topology, are homotopy metrizable in case of a proper domain, and remarks of [29, Section 2.3] apply. Some of our key technical results have been obtained by replacing the target space with a homotopy equivalent absolute neighbourhood retract; Milnor [22] proved that ANRs and CW complexes contain the same homotopy types. The fact that ANRs together with homotopy equivalences and Hurewicz fibrations form a fibration category in the sense of Baues [2] (see Miyata [23] as well as remark 7.1) is of importance.

2. A generalized Federer's spectral sequence

Assume a tower of Hurewicz fibrations

$$\cdots \to Z_n \to Z_{n-1} \to \cdots \to Z_1 \to Z_0 \to Z_{-1} = \{\text{point}\}\,. \tag{2.1}$$

We consider a spectral sequence converging to the homotopy groups of the inverse limit, taking on the approach of Federer [10]. He considered only the special case where $Z_i = \operatorname{map}(X^{(i)}, Y)$ is the space of maps from the i-skeleton of a CW complex X to a simple CW complex Y, and the maps $Z_i \to Z_{i-1}$ are restriction fibrations. However, his treatment applies to a fairly general setting as we proceed to explain.

Let the space Z_∞, together with projections $P^i\colon Z_\infty \to Z_i$, be the (topological) inverse limit of (2.1). Let $R\colon Z \to Z_\infty$ be a map of another space Z into Z_∞ and denote $R^i = P^i \circ R\colon Z \to Z_i$. In our applications, $Z \to Z_\infty$ typically would not be a homotopy equivalence (i.e., it would not have a homotopy inverse), but it will induce isomorphisms on homotopy groups in favourable circumstances.

Finally, pick $\zeta \in Z$ and set $\zeta_i = R^i(\zeta)$ for all i. Let Z'_i denote the path component of ζ_i in Z_i, and let F'_i denote the fibre of $Z'_i \to Z'_{i-1}$ over ζ_{i-1}. Note that F'_i may be disconnected. In such a setting, Federer defines a generalized exact couple

$$\begin{array}{ccc} A & \xrightarrow{\;r\;} & A \\ {\scriptstyle i}\nwarrow & & \swarrow{\scriptstyle \partial} \\ & C & \end{array} \tag{2.2}$$

where $A_{p,q} = \pi_p(Z'_q,\, \zeta_q)$ are groups (generally noncommutative for $p = 1$), and $C_{p,q} = \pi_p(F'_q,\, \zeta_q)$ are abelian groups for all $p \geqslant 0$. When $p < 0$ or $q < 0$, everything is trivial. Morphisms r, ∂, and i arise from the long exact sequences of fibrations

$$\cdots \to \pi_p(F'_q, \zeta_q) \xrightarrow{i} \pi_p(Z'_q, \zeta_q) \xrightarrow{r} \pi_p(Z'_{q-1}, \zeta_{q-1}) \xrightarrow{\partial} \pi_{p-1}(F'_q, \zeta_q) \to \cdots \tag{2.3}$$

and are morphisms of groups throughout. Federer notes that the theory of 'abelian' exact couples carries over to this setting. We recall that the differential is $d = \partial \circ i$; we set $(C^{(0)},\, d^{(0)}) = (C,\, d)$ and let $C^{(n)}$ be the homology of $(C^{(n-1)},\, d^{(n-1)})$.

In the following lemma we provide a sufficient condition for the existence of a Federer's exact couple in the setting of a tower of fibrations (2.1), and give an abstract identification of the groups $C_{p,q}$ and the first differential.

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Lemma 2.1. *Assume that for each $q \geqslant 0$, the fibration $Z_q \to Z_{q-1}$ is principal, obtained as the pullback of a fibration $EL_q \to BL_q$ along a classifying map $l_q \colon Z_{q-1} \to BL_q$ where EL_q is contractible and BL_q is a not necessarily connected space with abelian fundamental group (in each component). (For example, BL_q may be an H-group.) By taking $A_{p,q} = \pi_p(Z'_q, \zeta_q)$ and $C_{p,q} = \pi_p(F'_q, \zeta_q)$ with morphisms (2.3), one obtains an exact couple (2.2) in the sense of Federer.*

In the associated spectral sequence we have that $C_{p,q} \cong \pi_{p+1}(BL_q, l_q(\zeta_{q-1}))$ if $p, q \geqslant 1$, and that $C_{0,q}$ is isomorphic with a subgroup of $\pi_1(BL_q, l_q(\zeta_{q-1}))$ for $q > 0$. If $q \leqslant 0$ or $p < 0$, then $C_{p,q} = 0$. Under those isomorphisms, the first differential $d \colon C_{p,q} \to C_{p-1,q+1}$ for $p \geqslant 1$ corresponds to the composite

$$\pi_{p+1}(BL_q, l_q(\zeta_{q-1})) \xrightarrow{\delta_q} \pi_p(F_q, \zeta_q) \xrightarrow{i} \pi_p(Z_q, \zeta_q) \xrightarrow{(l_{q+1})_*} \pi_p(BL_{q+1}, l_{q+1}(\zeta_q)) \tag{2.4}$$

where δ_q can be identified with the connecting morphism in the homotopy exact sequence of the fibration $EL_q \to BL_q$.

Remark 2.2. As $Z_{-1} = \{\text{point}\}$, Z_0 is assumed to be a loop space of BL_0. Also, when $p = 1$, the codomain of (2.4) has to be understood as the image of $(l_{q+1})_*$.

Remark 2.3. Under $\pi_{p+1}(BL_q, l_q(\zeta_{q-1})) \cong \pi_p(\Omega(BL_q, l_q(\zeta_{q-1})), \mathrm{const}_{l_q(\zeta_{q-1})})$, the morphism δ_q from (2.4) is induced by a continuous map $\Omega(BL_q, l_q(\zeta_{q-1})) \to F_q$ (taking $\mathrm{const}_{l_q(\zeta_{q-1})}$ to ζ_q) if $EL_q \to BL_q$ is a regular fibration (i.e. stationary homotopies can be lifted to stationary homotopies).

Remark 2.4. The spectral sequence is natural with respect to maps of towers induced by maps of the classifying spaces BL_q. This holds also for the identification of the first differential (2.4).

Addendum 2.5. *Starting with $C_{0,q} \leqslant \widetilde{C}_{0,q} = \pi_0(F_q, \zeta_q) \cong \pi_1(BL_q, l_q(\zeta_{q-1}))$ with incoming differential $\tilde{d} \colon C_{1,q-1} \xrightarrow{d} C_{0,q} \leqslant \widetilde{C}_{0,q}$, each derived group $C^{(n)}_{0,q}$ is a subgroup of the group $\widetilde{C}^{(n)}_{0,q} = \widetilde{C}^{(n-1)}_{0,q} / \operatorname{im} \tilde{d}^{(n-1)}$ with incoming differential $\tilde{d}^{(n)} \colon C^{(n)}_{1,q-n-1} \xrightarrow{d^{(n)}} C^{(n)}_{0,q} \leqslant \widetilde{C}^{(n)}_{0,q}$. Moreover, $\widetilde{C}^{(n)}_{0,q}$ and $\tilde{d}^{(n)}$ are natural with respect to morphisms of towers induced by maps of classifying spaces BL_q. The exact couple (A, C) together with natural subgroup inclusions $C^{(n)}_{0,q} \leqslant \widetilde{C}^{(n)}_{0,q}$ will be called an augmented Federer's exact couple and denoted $(A, C, \widetilde{C})$.*

Proof. As $Z_q \to Z_{q-1} \xrightarrow{l_q} BL_q$ is a homotopy fibration, $\partial \colon \pi_{p+1}(Z_{q-1}, \zeta_{q-1}) \to \pi_p(F'_q, \zeta_q) = C_{p,q}$ can be viewed as $(l_q)_* \colon \pi_{p+1}(Z_{q-1}, \zeta_{q-1}) \to \pi_{p+1}(BL_q, l_q(\zeta_{q-1}))$. Moreover, $C_{0,q}$ can be identified with the image of $(l_q)_*$ in $\pi_1(BL_q, l_q(\zeta_{q-1}))$ which is abelian by our assumption on BL_q. This provides the identification of $C_{p,q}$ and that of ∂ in the differential $\partial \circ i \colon C_{p,q} \to C_{p-1,q+1}$. More precisely, as the fibration $Z_q \to Z_{q-1}$ is principal, F_q is homotopy equivalent to the loop-space $\Omega(BL_q, l_q(\zeta_{q-1}))$, and the connecting morphism in the homotopy exact sequence of $EL_q \to BL_q$ induces the isomorphism $\delta_q \colon \pi_{p+1}(BL_q, l_q(\zeta_{q-1})) \to \pi_p(F_q, \zeta_q)$. Noting that $\partial = (\delta_q)^{-1} \circ (l_q)_* \colon \pi_{p+1}(Z_{q-1}, \zeta_{q-1}) \to \pi_p(F_q, \zeta_q)$ completes the proof of the main statement.

For remark 2.3, consider a general fibration $f\colon E \to B$ and let $f(e_0) = b_0$ be any coherent pair of basepoints. The connecting morphism $\pi_*(\Omega B,\, \mathrm{const}_{b_0}) \to \pi_*(F,\, e_0)$ is induced by a map $\Omega(B,\, b_0) \to F$ which is most easily constructed by means of a lifting function that continuously lifts loops at b_0 to paths in E beginning at e_0 and takes the end point (in F). A regular fibration admits a lifting function sending constant paths to constant paths, forcing const_{b_0} to map to e_0.

To prove the addendum by induction, note that if $\tilde{d}^{(n-1)}\colon C^{(n-1)}_{1,q-n} \to \widetilde{C}^{(n-1)}_{0,q}$ is just $d^{(n-1)}$ followed by the inclusion $C^{(n-1)}_{0,q} \leqslant \widetilde{C}^{(n-1)}_{0,q}$, the images of $\tilde{d}^{(n-1)}$ and $d^{(n-1)}$ in $\widetilde{C}^{(n-1)}_{0,q}$ coincide, yielding subgroup inclusion $C^{(n)}_{(0,q)} = C^{(n-1)}_{0,q}/\operatorname{im} d^{(n-1)} \leqslant \widetilde{C}^{(n-1)}_{0,q}/\operatorname{im} \tilde{d}^{(n-1)} = \widetilde{C}^{(n)}_{(0,q)}$. □

Let $G_{p,q}$ denote the kernel of the induced morphism $R^q_*\colon \pi_p(Z,\, \zeta) \to A_{p,q} = \pi_p(Z'_q,\, \zeta_q)$ and consider the normal chain for $\pi_p(Z,\, \zeta)$:

$$\pi_p(Z,\zeta) = G_{p,-1} \geqslant G_{p,0} \geqslant G_{p,1} \geqslant \dots \tag{2.5}$$

The question of convergence is covered by the following proposition which is a straightforward generalization of the argument of Federer (see [**10**, pp. 351–352]).

Proposition 2.6. *Suppose that for each pair $(p,\, q)$, where $p \geqslant 1$, there exists an integer $j(p,\, q) \geqslant q$ for which the images of*

$$\pi_p(Z_{j(p,q)}, \zeta_{j(p,q)}) \to \pi_p(Z_q, \zeta_q) \text{ and } R^q_*\colon \pi_p(Z, \zeta) \to \pi_p(Z_q, \zeta_q)$$

coincide. Then for $n \geqslant \max\{j(p,\, q) + 1 - q,\, q\}$, the derived terms $C^{(n)}_{p,q}$ are stable:

$$G_{p,q-1}/G_{p,q} \cong C^{(n)}_{p,q}. \quad \square \tag{2.6}$$

Corollary 2.7. *Fix $p \geqslant 1$. If, in addition to the assumption of proposition 2.6, $G_{p,q}$ is trivial for $q \geqslant Q = Q(p)$, then the normal chain (2.5) terminates and its successive quotients are given by (2.6) for all $n \geqslant \max\{j(p,\, Q),\, Q\}$. In this sense the spectral sequence converges classically to $\pi_p(Z,\, \zeta)$.*

In practice, it is difficult to verify the assumptions of proposition 2.6 and corollary 2.7 (when they are not ‘automatic’). The entire § 4 is devoted to verifying those assumptions in a particular case, while proposition 1.8 shows they are implied by strong assumptions on homotopy type.

Routine diagram chasing coupled with exactness properties of localization of nilpotent groups yield the following theorem and its corollary.

Theorem 2.8. *Assume an abstract morphism of augmented Federer's exact couples $(A,\, C,\, \widetilde{C}) \to (A',\, C',\, \widetilde{C}')$. Fix a set of primes P (allowed to be the set of all primes). Assume that the following two conditions hold for some $n \geqslant 1$.*

(i) *The morphisms $C^{(n-1)}_{p,q} \to C'^{(n-1)}_{p,q}$ are localizations at P for $p,\, q \geqslant 1$.*

(ii) *The torsion in $\widetilde{C}'^{(n-1)}_{0,q}$ is P-local and the morphisms $\widetilde{C}^{(n-1)}_{0,q} \to \widetilde{C}'^{(n-1)}_{0,q}$ are P-injective for $q > 0$.*

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Then the same holds on level n. □

COROLLARY 2.9. *Assume that $C^{(n)}_{p,q}$ and $C'^{(n)}_{p,q}$ converge to successive quotients of finite normal chains $G_{p,q}$ and $G'_{p,q}$ as in (2.5). If the assumptions of theorem 2.8 hold, then the morphisms $G_{p,-1} \to G'_{p,-1}$ are P-localizations for $p \geqslant 1$.*

3. The homotopy spectral sequence of a function space

In this section, we associate to the function space $\mathrm{map}_\phi(X, Y)$ a tower of principal fibrations as studied above. We carry out the construction of the ensuing spectral sequence and the identification of the first differential. The questions of convergence and nilpotency will be addressed in the following sections.

Let $\check{H}^i(X, A; G)$ denote the i-th Čech cohomology group of the pair (X, A) with coefficients in the discrete group G. We refer to Dowker [**6**] and Bredon [**3**] for properties of Čech cohomology.

Let Y have the homotopy type of a connected nilpotent CW complex. Assume that Y is approximated (up to weak homotopy type) with the inverse limit of successive principal fibrations with fibres $K(\pi^q, n_q)$ where $1 = n_1 \leqslant n_2 \leqslant \dots$. Precisely, we assume a tower of fibrations

$$\cdots \to Y_q \xrightarrow{\eta_q} Y_{q-1} \to \cdots \to Y_2 \to Y_1 \to Y_0 = \{*\} \tag{3.1}$$

where $Y_1 = K(\pi^1, 1) =: K_1$ for an abelian group π^1, and each map $\eta_q\colon Y_q \to Y_{q-1}$ is a principal fibration obtained from the k-invariant $k_q\colon Y_{q-1} \to K(\pi^q, n_q + 1) = BK_q$ where π^q is abelian. This is to say that $Y_q \to Y_{q-1}$ is obtained by pulling back the path fibration $\varepsilon\colon P(BK_q) \to BK_q$ along k_q. We also assume that the BK_q and consequently $P(BK_q)$ are ANRs; it follows that so are Y_q (see [**23**] and remark 7.1). There are compatible maps $\eta^q\colon Y \to Y_q$ whose connectivity tends to infinity with q. These induce a continuous map $\eta\colon Y \to Y_\infty$ where Y_∞ is the inverse limit of the tower. The map η is a weak homotopy equivalence.

Let (X, A) be a pair where X is a Hausdorff k-space and let $\phi\colon A \to Y$ be a map. Set $\phi_q = \eta^q \circ \phi\colon A \to Y_q$; the maps ϕ_q induce a continuous map $\phi_\infty\colon A \to Y_\infty$. We may form an associated tower as follows.

$$\cdots \to \mathrm{map}_{\phi_q}(X, Y_q) \xrightarrow{(\eta_q)_*} \mathrm{map}_{\phi_{q-1}}(X, Y_{q-1}) \to \cdots \to \mathrm{map}_{\phi_1}(X, Y_1) \to \{*\} \tag{3.2}$$

Using the fact that the η_q are regular fibrations, the proof of [**29**, Lemma A.4] may be amended to render (3.2) a tower of fibrations (without conditions on A). The maps $\eta^q\colon Y \to Y_q$ induce compatible maps $R^q\colon \mathrm{map}_\phi(X, Y) \to \mathrm{map}_{\phi_q}(X, Y_q)$ and thus a limit map $R\colon \mathrm{map}_\phi(X, Y) \to \lim_q \mathrm{map}_{\phi_q}(X, Y_q) = \mathrm{map}_{\phi_\infty}(X, Y_\infty)$. Set $Z_q = \mathrm{map}_{\phi_q}(X, Y_q)$ and $Z = \mathrm{map}_\phi(X, Y)$. Pick $\zeta \in Z$ and let $\zeta_q = R^q(\zeta)$ for all q.

Finally assume that, in addition, $A \hookrightarrow X$ is a closed cofibration or (X, A) is a proper pair. One can use [**29**, Lemma A.2] in the first case and proposition 6.1 in the second to infer that restrictions $\mathrm{map}(X, T) \to \mathrm{map}(A, T)$ are fibrations for all T, respectively all T which are ANRs.

THEOREM 3.1. *The tower $\{Z_i\}$ satisfies the assumptions of lemma 2.1. In the associated spectral sequence we have that $C_{p,q} \cong \check{H}^{n_q-p}(X, A; \pi^q)$ if $p, q \geqslant 1$, and*

$\widetilde{C}_{0,q} \cong \check{H}^{n_q}(X,\, A; \pi^q)$ *for* $q > 0$*. If* $q \leqslant 0$ *or* $p < 0$*, then* $C_{p,q} = \widetilde{C}_{0,q} = 0$*. These identifications are natural in* $(X,\, A)$ *(i.e. with respect to maps over* Y*) as well as with respect to maps of towers (3.1) induced by maps* $K(\pi^q_{(1)}, n_q + 1) \to K(\pi^q_{(2)}, n_q + 1)$ *for* $q \geqslant 1$*.*

Proof. There are induced pullback diagrams as follows. (Amend [**29**, Lemma A.3].)

$$\begin{array}{ccc} \mathrm{map}_{\phi_q}(X, Y_q) & \longrightarrow & \mathrm{map}_{\psi_q}(X, P(BK_q)) \\ \Big\downarrow{\scriptstyle (\eta_q)_*} & & \Big\downarrow{\scriptstyle \varepsilon_*} \\ \mathrm{map}_{\phi_{q-1}}(X, Y_{q-1}) & \xrightarrow{(k_q)_*} & \mathrm{map}_{\varepsilon\psi_q}(X, BK_q) \end{array} \tag{3.3}$$

Here, ψ_q denotes the composite $A \xrightarrow{\phi_q} Y_q \to P(BK_q)$. Also, let Ψ_q denote the composite $X \xrightarrow{\zeta_q} Y_q \to P(BK_q)$. Note that $\varepsilon\psi_q = k_q\phi_{q-1}$ and $\varepsilon\Psi_q = k_q\zeta_{q-1}$. As mentioned above, the vertical arrows in (3.3) are fibrations. As $P(BK_q)$ is contractible, $\mathrm{map}(X,\, P(BK_q)) \to \mathrm{map}(A,\, P(BK_q))$ is a fibration and homotopy equivalence. Thus $EL_q = \mathrm{map}_{\psi_q}(X,\, P(BK_q))$, its fibre over ψ_q, is also contractible, rendering $(\eta_q)_*$ in diagram (3.3) a principal fibration. Set $BL_q = \mathrm{map}_{\varepsilon\psi_q}(X,\, BK_q)$. Assuming an H-group multiplication on BK_q with strict unit $\mathbb{1}$, we get induced H-group structures on $\mathrm{map}(X,\, BK_q)$ and $\mathrm{map}(A,\, BK_q)$ and, therefore, an induced map of fibrations as follows.

$$\begin{array}{ccc} \mathrm{map}(X, BK_q) & \xrightarrow{\xi \mapsto \xi\cdot(\varepsilon\Psi_q)} & \mathrm{map}(X, BK_q) \\ \Big\downarrow & & \Big\downarrow \\ \mathrm{map}(A, BK_q) & \xrightarrow{\alpha \mapsto \alpha\cdot(\varepsilon\psi_q)} & \mathrm{map}(A, BK_q) \end{array} \tag{3.4}$$

The horizontal arrows are homotopy equivalences and by coglueing homotopy equivalences (see [**4**, Corollary 1.5] or [**19**, Lemma 2.2.4]), so are the maps on the fibres. In particular,

$$\mathrm{map}_{A \to \mathbb{1}}(X, BK_q) \to \mathrm{map}_{\varepsilon\psi_q}(X, BK_q) \tag{3.5}$$

is a homotopy equivalence. This shows that, in fact, BL_q is an H-group and so lemma 2.1 can be applied. The identification of $C_{p,q}$ as (a subgroup of) $\check{H}^{n_q - p}(X,\, A; \pi^q)$ comes via (3.5) through the obvious homeomorphism of loop spaces

$$\Omega^k\big(\mathrm{map}_{A\to\mathbb{1}}(X, BK_q), \mathrm{const}_{\mathbb{1}}\big) \approx \mathrm{map}\,\big((X, A), (\Omega^k(BK_q, \mathbb{1}), \mathrm{const}_{\mathbb{1}}\big)$$

and representability of Čech cohomology over paracompact Hausdorff pairs (see [**14**] and [**11**]). The reader may check that this identification is natural with respect to maps of pairs $(X,\, A) \to (X',\, A')$ over Y.

For full naturality with respect to maps of towers (3.1), we assume that

$$K(\pi^q_{(1)}, n_q + 1) \to K(\pi^q_{(2)}, n_q + 1) \tag{3.6}$$

are strict H-maps of ANR H-groups with strict units. To achieve that, we deloop a given homotopy representative for (3.6) to a based map $K(\pi^q_{(1)},\, n_q + 2) \to K(\pi^q_{(2)},\, n_q + 2)$ between ANRs and take the induced map between the associated Moore loop spaces (which will also be ANRs). □

ADDENDUM 3.2. *The first differential* $C_{p,q} \to C_{p-1,q+1}$ *for* $p \geqslant 1$ *is obtained by applying* π_p *to the composite of (based) continuous maps*

$$\begin{aligned} &\Omega\big(\operatorname{map}_{A\to\mathbb{1}}(X, BK_q), \operatorname{const}_{\mathbb{1}}\big) \to \Omega\big(\operatorname{map}_{k_q\phi_{q-1}}(X, BK_q), k_q\zeta_{q-1}\big) \to \\ &\qquad \{f \mid \eta_q f = \zeta_{q-1}\} \subset \operatorname{map}_{\phi_q}(X, Y_q) \xrightarrow{(k_{q+1})_*} \operatorname{map}_{k_{q+1}\phi_q}(X, BK_{q+1}) \end{aligned} \tag{3.7}$$

The first map is induced by the self-homotopy equivalence $\operatorname{map}_{A\to\mathbb{1}}(X,\, BK_q) \to \operatorname{map}_{k_q\phi_{q-1}}(X,\, BK_q)$ *sending the map* $\operatorname{const}_{\mathbb{1}}$ *to* $k_q\zeta_{q-1}$ *that is in turn induced by an H-group multiplication on* BK_q *with strict unit* $\mathbb{1}$*. The following 3 maps correspond to the sequence* (2.4) *in conjunction with remark* 2.3*. This identification of the differential is natural in* $(X,\, A)$ *and* Y*.*

4. An auxiliary result for finite-dimensional domain

The results of this section will be used in § 5 to establish the convergence of the spectral sequence of theorem 1.1 in case of a finite dimensional domain.

In [**16**], the authors establish natural isomorphisms $\operatorname{colim}_j \pi_k(\operatorname{map}(X_j,\, Y),\, f_j) \to \pi_k(\operatorname{map}(X,\, Y),\, f)$ where X is a compactum and $\{X_j\}$ an inverse system of compact polyhedra with limit X. They use a method of Spanier [**32**, Theorem 13.4] and a clever trick to show validity for any map $f\colon X \to Y$ as basepoint.

We note that a generalization to any paracompact space X (pair, even) is possible, using methods of Barratt (see [**1**, Section 12]) and the same trick to allow for general basepoints in $\operatorname{map}(X,\, Y)$. To begin, let X be a paracompact Hausdorff space with closed subset A. For each locally finite open covering λ of X, let N_λ denote the nerve of λ and L_λ the subcomplex of N_λ obtained by the embedding of the nerve of $\lambda \cap A$. A simplicial map $p_{\mu\lambda}\colon N_\mu \to N_\lambda$ exists whenever μ refines λ, and partitions of unity provide for maps $h_\lambda\colon X \to N_\lambda$. Finally, we assume that $(Y,\, y_0)$ has the homotopy type of a pointed CW complex and allow for an additional compact Hausdorff parameter space T. Our auxiliary result is the following.

THEOREM 4.1. *Product maps* $h_\lambda \times \mathrm{id}_T$ *and* $p_{\lambda\mu} \times \mathrm{id}_T$ *yield a natural bijection*

$$\operatorname{colim}_\lambda[(N_\lambda \times T, L_\lambda \times T), (Y, y_0)] \to [(X \times T, A \times T), (Y, y_0)].$$

Essentially, this follows by an application of [**1**, Theorem 12.32] to the codomain $(\operatorname{map}(T,\, Y),\, \operatorname{const}_{y_0})$, which has the homotopy type of a pointed CW complex if $(Y,\, y_0)$ has, and the exponential law. Barratt's restriction to locally finite targets is not necessary (the straight-line homotopy used implicitly in his Lemma 12.31

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should be replaced with one in the sense of [25, Proposition 4.9.7]). Note that if A is empty, we do not need a basepoint in Y.

To apply theorem 4.1, let P^nY be the n-th Postnikov section (up to homotopy) together with the projection $p\colon Y \to P^nY$ (pointed if Y is pointed).

Proposition 4.2. *If* $\dim X \leqslant d < n$, *then* $p_*\colon \operatorname{map}(X, Y) \to \operatorname{map}(X, P^nY)$ *and* $p_*\colon \operatorname{map}_{A\to y_0}(X, Y) \to \operatorname{map}_{A\to y_0}(X, P^nY)$ *are* $(n-d+1)$*-equivalences.*

Proof. The dimension assumption guarantees a cofinal subfamily $\{\lambda\}$ with $\dim N_\lambda \leqslant d$. Let N_λ be such and let T be a compact k-dimensional polyhedron. Consider

$$[N_\lambda \times T, Y] \to [N_\lambda \times T, P^nY]. \tag{4.1}$$

By relative cellular approximation, (4.1) is a surjection if $k + d \leqslant n + 1$ and an injection if $k + d \leqslant n$. By theorem 4.1, $[X \times T, Y] \to [X \times T, P^nY]$ has the same properties. In particular, taking $T = \{*\}$ we infer that $[X, Y] \to [X, P^nY]$ is a bijection. Let $f\colon X \to Y$ be a basepoint in $\operatorname{map}(X, Y)$ and let $*$ denote a basepoint in the sphere S^k. Making use of the projection retraction $X \times S^k \to X \times \{*\} \equiv X$ as in the proof of [16, Theorem 6.4], we infer that

$$[X \times S^k, Y]_f \to [X \times S^k, Y] \to [X, Y] \tag{4.2}$$

is a split short exact sequence of pointed sets. Here, $[X \times S^k, Y]_f$ is the set of homotopy classes of maps $X \times S^k \to Y$ that restrict to $f\colon X \times \{*\} \to Y$. Applying naturality of (4.2) to $p\colon Y \to P^nY$, a diagram chase shows that $p_*\colon [X \times S^k, Y]_f \to [X \times S^k, P^nY]_{pf}$, which can clearly be identified with $p_*\colon \pi_k(\operatorname{map}(X, Y), f) \to \pi_k(\operatorname{map}(X, P^nY), pf)$, is bijective for $k < n - d + 1$ and surjective for $k = (n - d + 1)$.

For $p_*\colon \operatorname{map}_{A\to y_0}(X, Y) \to \operatorname{map}_{A\to p(y_0)}(X, P^nY)$, use pairs everywhere. □

We apply proposition 4.2 to the relative case as follows. Let A be closed in X and let $\phi\colon A \to Y$ be a map. Consider the induced map

$$\operatorname{map}_\phi(X, Y) \to \operatorname{map}_{p\phi}(X, P^nY). \tag{4.3}$$

Corollary 4.3. *Assume that n is big enough.*

(**i**) *If* $\dim X \leqslant d$ *and A is cofibred in X, then* (4.3) *is an* $(n-d)$*-equivalence.*

(**ii**) *If* (X, A) *is a proper pair with* $\dim(X, A) \leqslant d$, *then* (4.3) *is a weak* $(n-d-1)$*-equivalence if Y and* P^nY *are ANRs.*

We call a map a *weak* n-equivalence if it induces an injection on π_0, isomorphisms on π_k for $1 \leqslant k < n$, and epimorphisms on π_n for all choices of basepoint.

Proof. Case (**i**) follows from proposition 4.2 applied to the morphism of the homotopy exact sequences of fibrations $\operatorname{map}(X, Y) \to \operatorname{map}(A, Y)$ and

$\mathrm{map}(X, P^nY) \to \mathrm{map}(A, P^nY)$. For (**ii**), consider the pullback diagrams induced by (1.2):

$$\begin{array}{ccc} \mathrm{map}_\phi(X_i, Y) & \longrightarrow & \mathrm{map}(C_i, Y) \\ \big\downarrow & & \big\downarrow \\ \mathrm{map}_\phi(X_{i-1}, Y) & \longrightarrow & \mathrm{map}(S_i, Y) \end{array} \tag{4.4}$$

Using proposition 6.1, it follows that the vertical arrows in (4.4) are fibrations. Reasoning as in case (**i**), it follows that the mapping induced on the fibres of

$$\begin{array}{ccc} \mathrm{map}(C_i, Y) & \longrightarrow & \mathrm{map}(C_i, P^nY) \\ \big\downarrow & & \big\downarrow \\ \mathrm{map}(S_i, Y) & \longrightarrow & \mathrm{map}(S_i, P^nY) \end{array}$$

is an $(n-d)$-equivalence. By naturality of (4.4) in Y,

$$\mathrm{map}_\phi(X_i, Y) \to \mathrm{map}_{p\phi}(X_i, P^nY)$$

is a weak $(n-d)$-equivalence for all i. Now $\mathrm{map}_\phi(X, Y)$ is the inverse limit of $\{\mathrm{map}_\phi(X_i, Y)\}$ (see corollary 6.2) and there are the associated $\lim^1 - \lim$ short exact sequences (see [**5**, Chapter IX] and also [**17**, p. 178, Theorem 1]). By naturality in Y, we get morphisms of those exact sequences identifying the morphisms $\pi_k(\mathrm{map}_\phi(X, Y), f) \to \pi_k(\mathrm{map}_\phi(X, P^nY), pf)$. From those we infer (**ii**). □

5. Convergence, nilpotency and localization

DEFINITION 5.1. *A topological space Z (not necessarily of CW homotopy type) is nilpotent if, for any choice of basepoint z_0, the fundamental group $\pi_1(Z, z_0)$ is nilpotent and operates nilpotently on all higher homotopy groups $\pi_n(Z, z_0)$, $n \geqslant 2$.*

Proof of theorem 1.1. Assume the notation of theorem 3.1. Diagram (3.3) exhibits $\mathrm{map}_{\phi_q}(X, Y_q)$ as the homotopy fibre of $(k_q)_*\colon \mathrm{map}_{\phi_{q-1}}(X, Y_{q-1}) \to BL_q$ where BL_q is an H-group and hence nilpotent. As $\mathrm{map}_{\phi_0}(X, Y_0) = \{*\}$, it follows by induction that $\mathrm{map}_{\phi_q}(X, Y_q)$ are nilpotent for all q (see [**12**, Theorem II.2.2] together with the final remark of the proof, as well as the proof of [**13**, Theorem A]).

Case 1. Y has finitely many nontrivial homotopy groups. Then $Y \to Y_q$ is a homotopy equivalence for some q. This finishes the proof if $A = \emptyset$ or if ϕ maps the entire A to a nondegenerate $y_0 \in Y$, for then we have homotopy equivalences $\mathrm{map}(X, Y) \to \mathrm{map}(X, Y_q)$ and $\mathrm{map}_{A\to y_0}(X, Y) \to \mathrm{map}_{A\to\eta^q(y_0)}(X, Y_q)$. (See the proof of (1) in the discussion on page 7.)

Suppose that $A \hookrightarrow X$ is a cofibration or (X, A) is a proper pair and Y is an ANR. Then we have restriction fibrations $\mathrm{map}(X, Y) \to \mathrm{map}(A, Y)$ and $\mathrm{map}(X, Y_q) \to \mathrm{map}(A, Y_q)$ (note Y_q is an ANR by construction), and a morphism between them induced by the homotopy equivalence $Y \to Y_q$. By [**4**, Corollary 1.5], we get the

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homotopy equivalence $\mathrm{map}_\phi(X, Y) \to \mathrm{map}_{\phi_q}(X, Y)$ between the fibres. Thus in Case 1, the assumptions of proposition 2.6 and corollary 2.7 are met trivially.

Case 2. Y has infinitely many nontrivial homotopy groups. By proposition 4.2 and corollary 4.3, the connectivity of $\mathrm{map}_\phi(X, Y) \to \mathrm{map}_{\phi_q}(X, Y_q)$ tends to infinity with q, implying nilpotency of $\mathrm{map}_\phi(X, Y)$ as well as the assumptions of proposition 2.6 and corollary 2.7.

Finally, we set $E^2_{-k,q} = C_{q-k,q}$ for $k < q$ and $E^2_{-q,q} = \widetilde{C}_{0,q}$ to obtain the upper half-plane spectral sequence with homology-type differentials as claimed. Localization on the level of E^2 implies localization in the limit in case of convergence by theorem 2.8. □

Example 5.2. By way of example, Federer [**10**] studied the rationalized higher homotopy groups of $\mathrm{map}_*(X, S^n)$ for a finite-dimensional CW complex X. To illustrate the usefulness of addendum 3.2, we also treat $\mathrm{map}_*(X, S^n_{(0)})$ where $S^n_{(0)}$ is the rationalized sphere. Our treatment is valid for $X \in \mathrm{PHK}$; by corollary 1.3, $\mathrm{map}_*(X, S^n) \to \mathrm{map}_*(X, S^n_{(0)})$ is rationalization on path components when X is compact of finite covering dimension.

The interesting case is of an even $n > 2$. Then, $Y = S^n_{(0)}$ can be represented as the homotopy fibre of a single Postnikov invariant $k\colon K(\mathbb{Q}, n) \to K(\mathbb{Q}, 2n)$ whose associated cohomology class in $H^{2n}(K(\mathbb{Q}, n); \mathbb{Q})$ is the square of the fundamental class $\iota_n \in H^n(K(\mathbb{Q}, n); \mathbb{Q})$.

First we consider $[X, Y]_*$. To this end, we investigate the Puppe sequence

$$\ldots \xrightarrow{\Omega k_\#} [X, \Omega K(\mathbb{Q}, 2n)]_* \Rightarrow [X, Y]_* \to [X, K(\mathbb{Q}, n)]_* \xrightarrow{k_\#} [X, K(\mathbb{Q}, 2n)]_*.$$

Here, $\Rightarrow$ is employed to imply that the group $[X, K(\mathbb{Q}, 2n-1)]_*$ acts on the set $[X, Y]_*$, and that $[X, Y]_* \to [X, K(\mathbb{Q}, n)]_*$ collapses precisely the orbits.

By definition, $k_\#[f] = [k \circ f]$ which we may view as $f^*(\iota_n^2) = (f^*(\iota_n))^2$. Therefore, $k_\#\colon [X, K(\mathbb{Q}, n)]_* \to [X, K(\mathbb{Q}, 2n)]_*$ translates into the squaring operation $\check{H}^n(X, *; \mathbb{Q}) \to \check{H}^{2n}(X, *; \mathbb{Q})$. This holds for any X, implying that $\Omega k_\#$ is trivial as it corresponds, by adjunction, to squaring in a suspension

$$[SX, K(\mathbb{Q}, n)]_* \xrightarrow{k_\#} [SX, K(\mathbb{Q}, 2n)]_*.$$

This gives the short exact sequence

$$0 \to \check{H}^{2n-1}(X) \Rightarrow [X, Y]_* \to \big(\ker \mathrm{Sq}\colon \check{H}^n(X, *) \to \check{H}^{2n}(X, *)\big) \to *.$$

We consider the path component of a map $\zeta\colon X \to Y$. Diagram (3.3) reads

$$\begin{array}{ccc} \mathrm{map}_*(X, Y) & \longrightarrow & \mathrm{map}_*\big(X, PK(\mathbb{Q}, 2n)\big) \\ \Big\downarrow {\scriptstyle p_*} & & \Big\downarrow \\ \mathrm{map}_*\big(X, K(\mathbb{Q}, n)\big) & \xrightarrow{k_*} & \mathrm{map}_*\big(X, K(\mathbb{Q}, 2n)\big) \end{array} \tag{5.1}$$

We assume that $K(\mathbb{Q}, n)$ and $K(\mathbb{Q}, 2n)$ are well-pointed and that k is a fibration between well-pointed spaces. Next we assume that $\mu\colon K(\mathbb{Q}, n) \times$

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$K(\mathbb{Q},\, n) \to K(\mathbb{Q},\, n)$ is an H-group multiplication with strict unit $\mathbb{1}$. By virtue of the homotopy lifting property, we may also assume that $kp\zeta$ is the constant map.

For our spectral sequence, we assume that $Y_i = *$ for $1 \leqslant i \leqslant n-1$, next $Y_n = \Omega(K(\mathbb{Q},\, n+1),\, *) = K(\mathbb{Q},\, n)$, then $Y_i = Y_n$ for $n+1 \leqslant i \leqslant 2n-2$, and finally $Y = Y_{2n-1}$ is the homotopy fibre of $k\colon Y_n \to K(\mathbb{Q},\, 2n)$. The only nontrivial differential is $d^{(n-1)}$. The fibre F_n is equal to $\mathrm{map}_*(X,\, \Omega(K(\mathbb{Q},\, n+1),\, *)) = \mathrm{map}_*(X,\, Y_n)$ (with basepoint $\zeta_n = p \circ \zeta$). In this sense, δ_n in (2.4) can be viewed as the identity morphism, and therefore $d^{(n-1)}$ is obtained by applying $\pi_p(\_,\, p \circ \zeta)$ to $k_*\colon\, \mathrm{map}_*(X,\, K(\mathbb{Q},\, n)) \to \mathrm{map}_*(X,\, K(\mathbb{Q},\, 2n))$ (see also (3.7)). For our identification $C_{p,q} \cong \check{H}^{q-p}(X,\, *; \pi_q)$, we need to precompose k_* with the natural homotopy equivalence $\mathrm{map}_*(X,\, K(\mathbb{Q},\, n)) \to \mathrm{map}_*(X,\, K(\mathbb{Q},\, n))$, induced by μ, that sends the path component of $\mathrm{const}_{\mathbb{1}}$ to that of $p \circ \zeta$. Thus, the nontrivial differential corresponds to the map induced on $\pi_p(\_,\, \mathrm{const}_{\mathbb{1}})$ by

$$\mathrm{map}_*(X, K(\mathbb{Q}, n)) \to \mathrm{map}_*(X, K(\mathbb{Q}, 2n)), \quad f \mapsto k \circ \mu \circ (f, p\zeta).$$

We view the latter as the morphism $[S^p \wedge X,\, K(\mathbb{Q},\, n)]_* \to [S^p \wedge X,\, K(\mathbb{Q},\, 2n)]_*$ between sets of pointed homotopy classes. In light of the isomorphism $\check{H}^*(S^p \wedge X,\, *; \mathbb{Q}) \cong \check{H}^*(S^p \times X,\, S^p \vee X; \mathbb{Q})$ and the split short exact sequence

$$0 \to \check{H}^*(S^p \times X, S^p \vee X; \mathbb{Q}) \to \check{H}^*(S^p \times X; \mathbb{Q}) \to \check{H}^*(S^p \vee X; \mathbb{Q}) \to 0$$

we consider the composite

$$[(S^p \times X, S^p \vee X), (K(\mathbb{Q}, n), *)] \to [S^p \times X, K(\mathbb{Q}, n)] \to [S^p \times X, K(\mathbb{Q}, 2n)]. \tag{5.2}$$

The cross product with a generator $E^p \in \check{H}^p(S^p,\, *; \mathbb{Q}) = [S^p,\, K(\mathbb{Q},\, p)]_*$ realizes the iterated suspension isomorphism $[X,\, K(\mathbb{Q},\, n-p)]_* \to [(S^p \times X,\, S^p \vee X),\, (K(\mathbb{Q},\, n),\, *)]$, hence we compute the effect of (5.2) on $E^p| \times x|$ where $E^p| \in \check{H}^p(S^p; \mathbb{Q})$ and $x| \in \check{H}^{n-p}(X; \mathbb{Q})$ are restrictions. For general $A\colon S^p \times X \to K(\mathbb{Q},\, n)$ with $A(S^p \vee X) = \{\mathbb{1}\}$, the class of A maps to $[k \circ \mu \circ (A,\, p\zeta \circ \mathrm{pr}_X)]$; the latter can be viewed as

$$\begin{aligned}(A, p\zeta \circ \mathrm{pr}_X)^*(\mu^*(\iota_n))^2 &= (A, p\zeta \circ \mathrm{pr}_X)^*(1 \times \iota_n^2 + 2(\iota_n \times 1) \cup (1 \times \iota_n) + \iota_n^2 \times 1)\\ &= pr_X^*(p\zeta)^*(\iota_n^2) + 2 \cdot A^*(\iota_n) \cup \mathrm{pr}_X^*(p\zeta)^*(\iota_n) + A^*(\iota_n^2).\end{aligned}$$

Here, $(p\zeta)^*(\iota_n^2)$ corresponds to $[k \circ p\zeta]$ which is trivial. Finally, if $A^*(\iota_n) = E^p| \times x|$, we have $A^*(\iota_n^2) = (E^p| \times x|)^2 = 0$. Setting $I_n = p^*(\iota_n) \in \check{H}^n(Y; \mathbb{Q})$, we express

$$A^*(\iota_n) \cup \mathrm{pr}_X^*(p\zeta)^*(\iota_n) = (E^p| \times x|) \cup (1 \times \zeta^*(I_n)) = E^p| \times \big(x| \cup \zeta^*(I_n)\big).$$

Thus, $d^{(n-1)}\colon \check{H}^{n-p}(X,\, *; \mathbb{Q}) \to \check{H}^{2n-p}(X,\, *; \mathbb{Q})$ reads $\xi \mapsto 2\xi \cup \zeta^*(I_n)$; termination after $d^{(n-1)}$ yields exact sequences

$$\check{H}^{n-p-1}(X) \xrightarrow{d^{(n-1)}} \check{H}^{2n-p-1}(X) \to \pi_p \to \check{H}^{n-p}(X) \xrightarrow{d^{(n-1)}} \check{H}^{2n-p}(X)$$

where all cohomology is based with rational coefficients and $\pi_p = \pi_p(\mathrm{map}_*(X,\, Y); \zeta)$.

6. Proper domain

A useful approach to studying mapping spaces map(X, Y) where X is a CW complex is to view map(X, Y) as the inverse limit of the system $\{\mathrm{map}(K, Y)\}$ where K ranges over the finite subcomplexes of X and restriction mappings

$$\mathrm{map}(L, Y) \to \mathrm{map}(K, Y) \tag{6.1}$$

are associated to subcomplex inclusions $K \leqslant L$. As the latter are closed cofibrations, the restriction mappings (6.1) are Hurewicz fibrations for any space Y.

We cannot expect a general paracompact domain space X to be expressible as colimit of a suitable system of cofibrations. However, since we are interested in the homotopy type of map(X, Y), it is enough to replace the target space Y with a homotopy equivalent space in a way that renders restrictions (6.1) Hurewicz fibrations for sufficiently general compact pairs (L, K). To this end, we replace Y with a homotopy equivalent ANR; for example, we may take a simplicial complex with the metric topology. This can be done by Milnor [**22**, Theorem 2].

We say that $A \subset X$ is P-embedded if continuous pseudo-metrics on A extend to X. If, in addition, A is a zero set, then it is called P_0-embedded. We refer to Stramaccia [**34**] for more details; we need P-embeddings in the context of Morita's homotopy extension theorem used in the proof of proposition 6.1.

Recall that an absolute retract, with shorthand AR, is a contractible ANR.

PROPOSITION 6.1. *Let (X, A) be a compact proper pair and let Y be an ANR.*

(**i**) *Restriction $R\colon \mathrm{map}(X, Y) \to \mathrm{map}(A, Y)$ is a Hurewicz fibration.*

(**ii**) *Further let B be an AR, $f\colon X \to B$ a given map, and $p\colon Y \to B$ a Hurewicz fibration. For $L \subset X$ denote $\Gamma(L) = \{s\colon L \to Y \mid p \circ s = f|_L\} \subset \mathrm{map}(L, Y)$. Then $R|_{\Gamma(X)}\colon \Gamma(X) \to \Gamma(A)$ is a Hurewicz fibration.*

Proof. As R and $R|_{\Gamma(X)}$ are maps between metrizable spaces, it suffices to prove the homotopy lifting property for metrizable spaces (see [**9**, XX.2.3]).

To prove (**i**), we take a metric space Z. The solid part of the lifting diagram

$$\begin{array}{ccc} Z \times \{0\} & \longrightarrow & \mathrm{map}(X, Y) \\ \big\downarrow & \nearrow & \big\downarrow \\ Z \times [0, 1] & \longrightarrow & \mathrm{map}(A, Y) \end{array} \tag{6.2}$$

induces a map $\eta\colon Z \times X \times \{0\} \cup Z \times A \times [0, 1] \to Y$. Being the product of a paracompact and a compact space, $Z \times X$ is paracompact (Barratt [**1**, Lemma 12.43]) and hence collection-wise normal. Therefore its P_0-embedded subsets are precisely its zero sets ([**34**, Examples 1.2]). Thus, $Z \times A$ is P_0-embedded in $Z \times X$, and Morita's extension theorem (see [**34**, Theorem 2.2]) yields a map $Z \times X \times [0, 1] \to Y$ extending η. Its adjoint is the lifting sought in (6.2).

We turn to (**ii**). The lifting problem analogous to (6.2) but with $\mathrm{map}(X,\,Y)$ and $\mathrm{map}(A,\,Y)$ replaced by, respectively, $\Gamma(X)$ and $\Gamma(A)$, is equivalent to

$$\begin{array}{ccc} Z\times X\times\{0\}\cup Z\times A\times[0,1] & \xrightarrow{\ \eta\ } & Y \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ Z\times X\times[0,1] & \xrightarrow{\ f\circ \mathrm{pr}_X\ } & B \end{array} \tag{6.3}$$

By the proof of (**i**), η extends to a map $\bar{\eta}\colon Z\times X\times[0,\,1]\to Y$. We remedy the possible difference between $p\circ\bar{\eta}$ and $f\circ \mathrm{pr}_X$ (where pr_X is projection onto X) as follows. Let a partial map into B be defined by $p\circ\bar{\eta}$ on $Z\times X\times[0,\,1]\times\{0\}$ and $f\circ \mathrm{pr}_X$ on $Z\times X\times[0,\,1]\times\{1\}\cup(Z\times X\times\{0\}\sqcup Z\times A\times[0,\,1])\times[0,\,1]$. This partial map is defined on a zero (hence P_0-embedded) set in $Z\times X\times[0,\,1]\times[0,\,1]$ and since B is an AR, there exists an extension over all $Z\times X\times[0,\,1]\times[0,\,1]$ (see [**34**, Theorem 2.1]). We consider the lifting problem

$$\begin{array}{ccc} Z\times X\times[0,1]\times\{0\} & \xrightarrow{\ \bar{\eta}\ } & Y \\ \big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ Z\times X\times[0,1]\times[0,1] & \longrightarrow & B \end{array}$$

By construction, the homotopy into B is stationary on $Z\times X\times\{0\}\cup Z\times A\times[0,\,1]$. By regularity of p (see [**9**, XX.2.4]), the lifting into Y can also be assumed to be stationary there. Level 1 of that lifting is a map $Z\times X\times[0,\,1]\to Y$ that extends η and projects to $f\circ \mathrm{pr}_X$ by p, and thus it constitutes the lifting sought in (6.3). □

Corollary 6.2. *Let $(X,\,A)$ be a proper pair with an admissible chain $A=X_0\subset X_1\subset X_2\subset\dots$, let Y be an ANR, and let $\phi\colon A\to Y$ be a map. The induced tower*

$$\cdots\to \mathrm{map}_\phi(X_2,Y)\to \mathrm{map}_\phi(X_1,Y)\to \mathrm{map}_\phi(A,Y)$$

is one of Hurewicz fibrations and the terms $\mathrm{map}_\phi(X_i,\,Y)$ have CW homotopy type. Restrictions $R^i\colon \mathrm{map}_\phi(X,\,Y)\to \mathrm{map}_\phi(X_i,\,Y)$, which are also Hurewicz fibrations, exhibit $\mathrm{map}_\phi(X,\,Y)$ as the inverse limit.

Proof. The canonical inverse limit is the subspace $Z'\subset\prod_{i=1}^{\infty}\mathrm{map}_\phi(X_i,\,Y)$ consisting of compatible sequences of maps $f_i\colon X_i\to Y$ with $f_i|_A=\phi$. Restrictions R^i furnish the obvious homeomorphism $R\colon \mathrm{map}_\phi(X,\,Y)\to Z'$; it is an embedding since the X_i dominate compact sets in X and onto because $X=\mathrm{colim}_i\,X_i$. In diagram (4.4), restriction $\mathrm{map}(C_i,\,Y)\to \mathrm{map}(S_i,\,Y)$ is a fibration by (**i**) of proposition 6.1, and hence so is the pullback restriction $\mathrm{map}_\phi(X_i,\,Y)\to \mathrm{map}_\phi(X_{i-1},\,Y)$. The spaces $\mathrm{map}(C_i,\,Y)$ and $\mathrm{map}(S_i,\,Y)$ have CW homotopy type by [**22**, Theorem 3]. Now $\mathrm{map}_\phi(X_0,\,Y)$ is a point and it follows from [**29**, Proposition 4.2] that all $\mathrm{map}_\phi(X_i,\,Y)$ have CW homotopy type.

By way of composition, $\mathrm{map}_\phi(X_j,\,Y)\to\mathrm{map}_\phi(X_i,\,Y)$ are fibrations for all $j\geqslant i$. The universal property of the limit implies that also the R^i are fibrations. □

Proof of theorem 1.6. Corollary 1.3 guarantees that the maps

$$\mathrm{map}_\phi(X_i,Y)\to\mathrm{map}_{l\phi}(X_i,Y_{(P)}) \tag{6.4}$$

localize homotopy groups. Corollary 6.2 identifies the inverse limit of maps (6.4) as $\mathrm{map}_\phi(X,\,Y)\to\mathrm{map}_{l\phi}(X,\,Y_{(P)})$. By [**31**, Theorem 6.1], the spaces $\mathrm{map}(X,\,Y)$ and $\mathrm{map}(X,\,Y_{(P)})$ are nilpotent and this latter map localizes homotopy groups. □

Proof of proposition 1.8. Assume the notation of § 3. Diagram (3.3) exhibits $Z_q=\mathrm{map}_{\phi_q}(X,\,Y_q)$ as the homotopy fibre of $(k_q)_*\colon Z_{q-1}\to\mathrm{map}_{A\to *}(X,\,BK_q)$ where BK_q is an Eilenberg–MacLane space. Now $\mathrm{map}_{A\to *}(X,\,BK_q)$ has CW type since $(X,\,A)$ is quasi-finite, and as $Z_0=\{*\}$, an inductive application of [**33**, Proposition 0] shows that all Z_q have CW homotopy type.

Set $Z^i=\mathrm{map}_\phi(X_i,\,Y)$, $\zeta^i=\zeta|_{X_i}$, $Z^i_q=\mathrm{map}_\phi(X_i,\,Y_q)$, and $\zeta^i_q=\zeta_q|_{X_i}$. By corollary 6.2, $Z=\mathrm{map}_\phi(X,\,Y)$ is the limit of Z^i and Z_q is the limit of Z^i_q for all q, and all Z^i and Z^i_q have CW homotopy type. To establish the (additional) assumption of corollary 2.7, we show that for every $p\geqslant 1$, restrictions $\pi_p(Z,\,\zeta)\to\pi_p(Z_q,\,\zeta_q)$ are injective for all big enough q. By [**29**, Corollary 3.4,(**i**)], applied to $Z=\lim_i Z^i$, there exists an i such that $\pi_p(Z,\,\zeta)\to\pi_p(Z^i,\,\zeta^i)$ are injective for all $p\geqslant 1$. Fix some $p\geqslant 1$. By corollary 4.3, $\pi_p(Z^i,\,\zeta^i)\to\pi_p(Z^i_q,\,\zeta^i_q)$ is an isomorphism for all big enough q. Thus, $\pi_p(Z,\,\zeta)\to\pi_p(Z_q,\,\zeta_q)$ must be injective for those q.

To establish the assumptions of proposition 2.6, fix some q. By another application of [**29**, Corollary 3.4,(**i**)], this time to $Z_q=\lim_i Z^i_q$, there exists an i such that $\pi_p(Z_q,\,\zeta_q)\to\pi_p(Z^i_q,\,\zeta^i_q)$ are injective for all $p\geqslant 1$. By [**29**, Corollary 3.4,(**ii**)], applied to $Z=\lim_i Z^i$, there exists a $j\geqslant i$ such that the image of $\pi_p(Z,\,\zeta)\to\pi_p(Z^i,\,\zeta^i)$ coincides with that of $\pi_p(Z^j,\,\zeta^j)\to\pi_p(Z^i,\,\zeta^i)$ for all p. Finally, we fix some $p\geqslant 1$. By corollary 4.3, there exists a t such that $\pi_p(Z^j,\,\zeta^j)\to\pi_p(Z^j_t,\,\zeta^j_t)$ is an isomorphism. A straightforward diagram chase shows that every element in the image of $\pi_p(Z_t,\,\zeta_t)\to\pi_p(Z_q,\,\zeta_q)$ is also in the image of $\pi_p(Z,\,\zeta)\to\pi_p(Z_q,\,\zeta_q)$. □

Proof of proposition 1.9. By homotopy invariance of $\mathrm{map}_{A\to *}(X,\,Y)$ for a well-pointed Y, we may assume that $Y=K(G,\,n)$ is an H-group with strict unit $\mathbb{1}$ and an ANR. If $\{X_i\}$ is any admissible chain for $(X,\,A)$, then by corollary 6.2, $Z^i=\mathrm{map}_{A\to\mathbb{1}}(X_i,\,Y)$ defines a tower of fibrations whose terms have CW homotopy type and whose limit is $Z=\mathrm{map}_{A\to\mathbb{1}}(X,\,Y)$. Homotopical representation of Čech cohomology directly implies that restriction $Z\to Z^i$ induces the cohomological restriction $\check{H}^n(X,\,A;G)\to\check{H}^n(X_i,\,A;G)$.

Let $\zeta\colon X\to Y$ be a map with restrictions $\zeta^i=\zeta|_{X_i}$. Pointwise multiplications by ζ, respectively ζ^i, comprise a homotopy auto-equivalence of the tower $\{Z^i\}$ and its limit that maps the components of $\mathrm{const}_{\mathbb{1}}$ to those of ζ, respectively ζ^i. Thus we can identify the morphisms $\pi_k(Z^j,\,\zeta^j)\to\pi_k(Z^i,\,\zeta^i)$ with $\check{H}^{n-k}(X_j,\,A;G)\to\check{H}^{n-k}(X_i,\,A;G)$. An application of [**31**, Theorem 4.1] concludes the proof. □

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Example 6.3. Assume the notation of definition 1.5 for a proper space X and set

$$c(i) = \sup \left\{ n \,\middle|\, \check{H}^k(C_{i+1}, S_i; \mathbb{Z}) = 0 \text{ for all } k \leqslant n \right\}.$$

We claim that if $c(i) \geqslant n+2$ for $i \geqslant i_0$, then restriction $\mathrm{map}(X,\, K(G,\, n)) \to \mathrm{map}(X_{i_0},\, K(G,\, n))$ is a homotopy equivalence for every abelian group G. To verify this claim, note first that if $i \geqslant i_0$, universal coefficients imply $\check{H}^k(C_{i+1},\, S_i; G) = 0$ for any G and all $k \leqslant n+1$. Next, Wallace's theorem implies cohomology isomorphisms $\check{H}^*(X_{i+1},\, X_i; G) \to \check{H}^*(C_{i+1},\, S_i; G)$. Arguing as in the above proof, we use the cohomology exact sequence of the pair $(X_{i+1},\, X_i)$ to infer that if $i \geqslant i_0$, then restriction $\mathrm{map}(X_{i+1},\, K(G,\, n)) \to \mathrm{map}(X_i,\, K(G,\, n))$ is a homotopy equivalence. It follows that so also is $\mathrm{map}(X,\, K(G,\, n)) \to \mathrm{map}(X_{i_0},\, K(G,\, n))$ (see [**29**, Corollary 3.7]), rendering $\mathrm{map}(X,\, K(G,\, n))$ homotopy equivalent to a CW complex. In particular, if $\liminf_{i\to\infty} c(i) = \infty$, then $\mathrm{map}(X,\, K(G,\, n))$ has CW homotopy type for every G and n. For a simple-minded concrete example of a proper quasi-finite space that does not have the homotopy type of a CW complex, one can take the infinite wedge $X = \vee_{n=1}^{\infty} \mathbb{H}_n$ where $\mathbb{H}_n$ is the n-dimensional Hawaiian earring.

7. Spaces of sections

Let $X \in \mathrm{PHK}$ be connected and let $\pi\colon Q \to X$ be a Hurewicz fibration with typical fibre of the homotopy type of a nilpotent CW complex. Let $\Gamma(\pi)$ denote the space of sections of π. We assume that $\pi = f^*(p')$ is the pullback of a fibration $p'\colon E' \to B$ of spaces of CW homotopy type along the continuous map $f\colon X \to B$. We can change π up to fibre homotopy equivalence (over X) and keep the homotopy type of $\Gamma(\pi)$. Thus we assume that B is in fact a simplicial complex with the strong topology; B is then an ANR with a covering of contractible open subsets (which are ANRs), see Milnor [**22**]. The corresponding total space E'' will still have CW homotopy type by results of Stasheff [**33**]. Choosing a homotopy equivalent ANR E''' we split the resulting map $E''' \to B$ into the composite of a homotopy equivalence and a fibration $p\colon E \to B$ where E is an ANR. The 'usual' splitting in the sense of the Hurewicz model structure has this property; see also Miyata [**23**]. Consequently, each fibre of p is an ANR (being cofibred in E by [**35**, Theorem 12]) and we will use Y to denote the fibre over a point understood from the context. As π is fibre homotopy equivalent to $f^*(p)$, we identify $\Gamma(\pi)$ with the space of maps $X \to E$ whose composite with p equals f.

Remark 7.1. In the fibration category on ANRs studied by [**23**], the fibrations are what we call regular Hurewicz fibrations (and the author calls maps with the strong homotopy lifting property). As remarked above, every Hurewicz fibration $E \to B$ between ANRs is regular on the ground of metrizability of B; this remark makes certain parts of [**23**] superfluous.

More generally, assume that $(X,\, A)$ is a proper pair with an admissible chain $\{X_i\}$ and that $\phi\colon A \to E$ is a map over $f|_A$. For $A \subset L \subset X$, we set

$$\Gamma_\phi(L, E) = \{ s\colon L \to E \,|\, p \circ s = f|_L,\ s|_A = \phi \},$$

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viewed as a subspace of $\mathrm{map}(L, E)$. For $A \subset K \subset L$ we have a restriction map $\Gamma_\phi(L, E) \to \Gamma_\phi(K, E)$. As noted above, if $A = \emptyset$, then the homotopy type of $\Gamma(X, E)$ is an invariant of the homotopy type of $p\colon E \to B$. Invariance in the relative version will be discussed after the following result.

THEOREM 7.2.

(**i**) *Restriction maps* $\Gamma_\phi(X_i, E) \to \Gamma_\phi(X_{i-1}, E)$ *are fibrations between spaces of CW homotopy type.*

(**ii**) *Let* $l\colon E \to E_{(P)}$ *be a fibrewise localization at the set of primes* P *where we assume that* $E_{(P)}$ *is an ANR. If* (X, A) *is locally finite dimensional or* Y *is a Postnikov section, then* $\Gamma_\phi(X_i, E)$ *are nilpotent and* $\Gamma_\phi(X_i, E) \to \Gamma_\phi(X_i, E_{(P)})$ *are localizations at* P *on path components.*

Proof. (**i**) By the assumption on B and normality of the C_i in (1.2), we may refine each step $X_{i-1} \subset X_i$ into a finite chain of adjunctions to assume, after reindexing, that $f(\gamma_i(C_i)) \subset K_i$ for a contractible open set K_i in B. We get induced pullbacks

$$\begin{array}{ccc} \Gamma_\phi(X_i, E) & \longrightarrow & \Gamma(C_i, E|_{K_i}) \\ \downarrow & & \downarrow \\ \Gamma_\phi(X_{i-1}, E) & \longrightarrow & \Gamma(S_i, E|_{K_i}) \end{array} \tag{7.1}$$

By proposition 6.1, the vertical arrow on the right is a fibration, and hence so is the one on the left. Consider $p_*\colon \mathrm{map}(C_i, E|_{K_i}) \to \mathrm{map}(C_i, K_i)$. As C_i is compact, this is a fibration between spaces of CW homotopy type. In fact $\mathrm{map}(C_i, K_i)$ is contractible and hence fibre inclusion $\Gamma(C_i, E|_{K_i}) \hookrightarrow \mathrm{map}(C_i, E|_{K_i})$ is a homotopy equivalence. On the other hand, $Y \hookrightarrow E|_{K_i}$ is also a homotopy equivalence, hence $\mathrm{map}(C_i, Y) \simeq \Gamma(C_i, E|_{K_i})$ is nilpotent (under the additional hypothesis) by theorem 1.1. The same holds for S_i in place of C_i and since $\Gamma_\phi(X_0, E)$ is a one point-space, it follows by induction that all $\Gamma_\phi(X_i, E)$ are nilpotent (see May and Ponto [**19**, Proposition 4.4.3]) of CW type (see Stasheff [**33**, Propositions 0 and 12]).

(**ii**) We apply the same reasoning to $\Gamma_{l\phi}(X_i, E_{(P)})$ and note that $\Gamma(C_i, E|_{K_i}) \to \Gamma(C_i, E_{(P)}|_{K_i})$ and $\Gamma(S_i, E|_{K_i}) \to \Gamma(S_i, E_{(P)}|_{K_i})$ may be identified with, respectively, $\mathrm{map}(C_i, Y) \to \mathrm{map}(C_i, Y_{(P)})$ and $\mathrm{map}(S_i, Y) \to \mathrm{map}(S_i, Y_{(P)})$ that localize by corollary 1.3. Thus if $\Gamma_\phi(X_{i-1}, E) \to \Gamma_\phi(X_{i-1}, E_{(P)})$ localizes (on path components), then so does $\Gamma_\phi(X_i, E) \to \Gamma_\phi(X_i, E_{(P)})$ by [**19**, Proposition 6.2.5]. □

We turn to the question of homotopy invariance of $\Gamma_\phi(X, E)$ under changes of $p\colon E \to B$ for general $X \in \mathrm{PHK}$ with closed subspace A. To this end, assume that p is homotopy equivalent to $p'\colon E' \to B'$, i.e., there are homotopy equivalences

$\eta\colon E \to E'$ and $\beta\colon B \to B'$ for which $p'\eta = \beta p$. Think of

$$\Gamma_\phi(X, E) \to \Gamma_{\eta\phi}(X, E') \tag{7.2}$$

as the restriction to the fibre of the transformation $p_* \to p'_*$:

$$\begin{array}{ccc} \mathrm{map}_\phi(X, E) & \xrightarrow{\eta_*} & \mathrm{map}_{\eta\phi}(X, E') \\ {\scriptstyle p_*}\downarrow & & \downarrow{\scriptstyle p'_*} \\ \mathrm{map}_{p\phi}(X, B) & \xrightarrow{\beta_*} & \mathrm{map}_{p'\eta\phi}(X, B') \end{array} \tag{7.3}$$

Assume that p_* and p'_* are fibrations. If η_* and β_* are homotopy equivalences, so is (7.2) by cogluеing [**4**, Corollary 1.5]. This will be true in the following cases.

(i) $\phi = \mathrm{const}_{e_0}$ (forcing $f|_A = \mathrm{const}_{b_0}$ where $b_0 = p(e_0)$) and p, p', η, β are pointed maps of well-pointed spaces,

(ii) $\phi = \mathrm{const}_{e_0}$ and p and p' are regular fibrations (for example if B and B' are metrizable) and η and β are homotopy equivalences in the pointed category,

(iii) E, E', B, B' are ANRs and (X, A) is a proper pair,

(iv) A is cofibred (in particular, A may be empty).

For (1), the proof of [**29**, Lemma A.4] shows that p_* and p'_* are fibrations, while if $\eta'\colon E' \to E$ is a pointed inverse for η and $\eta'\eta \simeq \mathrm{id}_E$ and $\eta\eta' \simeq \mathrm{id}_{E'}$ are pointed homotopies, then the standard proof that $\eta_*\colon \mathrm{map}(X, E) \to \mathrm{map}(X, E')$ is a homotopy equivalence (as for example [**18**, Theorem 6.2.25]) will show its restriction $\mathrm{map}(X, E)_{A\to e_0} \to \mathrm{map}(X, E')_{A\to\eta(e_0)}$ is also a homotopy equivalence.

For (2), the proof of [**29**, Lemma A.4] may be amended using regular lifting functions (as in Definition 2.1 in [**9**, Chapter XX]) to show that p_* and p'_* are fibrations, while the proof that η_* and β_* are homotopy equivalences is as above.

For (3), we note that p_* and p'_* are fibrations because p and p' are regular on the ground of metrizability of B. We consider $\eta_*\colon \mathrm{map}_\phi(X, E) \to \mathrm{map}_{\eta\phi}(X, E')$ as the map between the fibres induced by the transformation of restrictions:

$$\begin{array}{ccc} \mathrm{map}(X, E) & \xrightarrow{\eta_*} & \mathrm{map}(X, E') \\ \downarrow & & \downarrow \\ \mathrm{map}(A, E) & \xrightarrow{\eta_*} & \mathrm{map}(A, E') \end{array} \tag{7.4}$$

The horizontal arrows in (7.4) are homotopy equivalences ([**18**, ibid.]) and since E and E' are ANRs, the restrictions (i.e., the vertical arrows) are fibrations by proposition 6.1. Thus, the cogluеing theorem will guarantee that the maps between the fibres will be homotopy equivalences as well. The same argument applies to β_*, using the fact that B and B' are ANRs.

For (4), the argument that p_* and p'_* are Hurewicz fibrations is standard. In $\mathcal{K}$, we simply use the exponential law in $\mathcal{K}$, the fact that inclusion $X \times \{0\} \cup A \times [0, 1] \hookrightarrow X \times [0, 1]$ is a closed cofibration and the lifting properties of p_* and p'_*. For $\mathcal{T}op$, we extend the proof of [**29**, Lemma A.4]. When A is cofibred in X, the vertical arrows in (7.4) are fibrations for all E and E'.

Proof of theorem 1.10. The theorem deals with absolute sections ($A = \emptyset$), and we may take a representative for p as specified prior to the statement of theorem 7.2. Thus, statement (**a**) follows directly from theorem 7.2 as then $X = X_i$ for some i.

For (**b**), note that $\Gamma(X, E) \to \Gamma(X, E_{(P)})$ is the limit of maps $\Gamma(X_i, E) \to \Gamma(X_i, E_{(P)})$ (which localize by theorem 7.2). Apply [**31**, Theorems 6.1 and 6.2]. □

We remark that theorem 7.2 also contains [**24**, Theorem 5.3] as a special case.

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