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LEE MONOIDS ARE NONFINITELY BASED WHILE THE SETS OF THEIR ISOTERMS ARE FINITELY BASED

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Abstract

We establish a new sufficient condition under which a monoid is nonfinitely based and apply this condition to Lee monoids L_ℓ^1, obtained by adjoining an identity element to the semigroup generated by two idempotents a and b with the relation $0 = abab\cdots$ (length ℓ). We show that every monoid M which generates a variety containing L_5^1 and is contained in the variety generated by L_ℓ^1 for some $\ell \geq 5$ is nonfinitely based. We establish this result by analysing τ-terms for M, where τ is a certain nontrivial congruence on the free semigroup. We also show that if τ is the trivial congruence on the free semigroup and $\ell \leq 5$, then the τ-terms (isoterms) for L_ℓ^1 carry no information about the nonfinite basis property of L_ℓ^1.

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1. Introduction

An algebra is said to be *finitely based* (FB) if there is a finite subset of its identities from which all of its identities may be deduced. Otherwise, an algebra is said to be *nonfinitely based* (NFB). Throughout this article, elements of a countably infinite alphabet $\mathfrak{A}$ are called *variables* and elements of the free monoid $\mathfrak{A}^*$ and free semigroup $\mathfrak{A}^+$ are called *words*.

In 1968, Perkins [9] found the first sufficient condition under which a monoid (semigroup with an identity element) is NFB. By using this condition, he constructed the first two examples of finite NFB semigroups. The first was the six-element Brandt monoid and the second was the 25-element monoid obtained from the set of words $W = \{abtba, atbab, abab, aat\}$, using the following construction attributed to Dilworth.

Let W be a set of words in the free monoid $\mathfrak{A}^*$. Let $S^1(W)$ denote the Rees quotient over the ideal of $\mathfrak{A}^*$ consisting of all words that are not subwords of words in W. For each set of words W, the semigroup $S^1(W)$ is a monoid with zero whose nonzero elements are the subwords of words in W. We say that W is *(non)finitely based* if the monoid $S^1(W)$ is (non)finitely based.

We regard monoids here as semigroups, that is, algebras with one operation. The finite basis property of a monoid does not depend on whether it is considered as an algebra with one operation or two operations [13]. Indeed, if an identity can be derived from a finite set of identities Σ by using the substitutions $\Theta : \mathfrak{A} \to \mathfrak{A}^*$, then it can also be derived from a finite set of identities Σ' by using only the substitutions $\Theta : \mathfrak{A} \to \mathfrak{A}^+$. (Just take Σ' to be the set of all identities obtained by deleting variables in the identities in Σ.)

If τ is an equivalence relation on the free semigroup $\mathfrak{A}^+$, then we say that a word $\mathbf{u}$ is a *τ-term* for a semigroup S if $\mathbf{u} \tau \mathbf{v}$ whenever S satisfies $\mathbf{u} \approx \mathbf{v}$. Recall from [9] that $\mathbf{u}$ is an *isoterm* for S if $\mathbf{u} = \mathbf{v}$ whenever S satisfies $\mathbf{u} \approx \mathbf{v}$. If $\mathbf{u}$ is an isoterm for S then, evidently, $\mathbf{u}$ is a τ-term for S for every equivalence relation τ on $\mathfrak{A}^+$. We use $\operatorname{var} S$ to refer to the variety of semigroups generated by S. The following result of Jackson gives the fundamental connection between monoids of the form $S^1(W)$ and isoterms for monoids.

Fact 1.1 [3, Lemma 3.3]. *Let W be a set of words and M be a monoid. Then* $\operatorname{var} M$ *contains $S^1(W)$ if and only if every word in W is an isoterm for M.*

Given a monoid M, we use $\operatorname{Isot}(M)$ to denote the set of all words in $\mathfrak{A}^*$ that are isoterms for M. Using Fact 1.1, it is easy to show that $W = \operatorname{Isot}(M)$ is the largest subset of $\mathfrak{A}^*$ such that $S^1(W)$ is contained in $\operatorname{var} M$ (see [12, Fact 8.1]).

A locally finite algebra is said to be *inherently not finitely based* (INFB) if any locally finite variety containing it is NFB. A finite semigroup S is INFB if and only if every Zimin word ($\mathbf{Z}_1 = x_1, \dots, \mathbf{Z}_{k+1} = \mathbf{Z}_k x_{k+1} \mathbf{Z}_k, \dots$) is an isoterm for S [10, Proposition 7]. Together with [10, Proposition 3], this implies that if M is a finite INFB monoid, then the set $\operatorname{Isot}(M)$ is NFB. It also implies that the Brandt monoid is INFB and, consequently, the set of its isoterms is nonfinitely based.

For the majority of the aperiodic monoids which are known to be NFB but not INFB, their nonfinite basis property can be established by exhibiting a certain finite set of words W and a certain set of identities Σ (without any bound on the number of variables involved) and proving the following statement.

- *If a monoid M satisfies all identities in Σ and all the words in W are isoterms for M, then M is NFB.*

If the nonfinite basis property of a monoid M is established by a sufficient condition of this form, then, evidently, the set $\operatorname{Isot}(M)$ is also NFB.

We say that a word $\mathbf{u}$ has *the same type* as $\mathbf{v}$ if $\mathbf{u}$ can be obtained from $\mathbf{v}$ by changing the individual exponents of variables. For example, the words $x^2yxzx^5y^2xzx^3$ and $xy^2x^3zxyx^2zx$ are of the same type. We will present a new sufficient condition (see Theorem 2.1 below) under which a monoid is nonfinitely based. Theorem 2.1 exhibits a certain finite set of words W, a certain set of identities Σ (without any bound on the number of variables involved) and states the following:

- *if a monoid M satisfies all identities in Σ and every word in W can form an identity of M only with a word of the same type, then M is NFB.*

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Recently, Lee suggested investigating the finite basis property of semigroups

$$L_\ell = \langle a, b \mid aa = a, bb = b, \underbrace{ababab\cdots}_{\text{length } \ell} = 0\rangle, \quad \ell \geq 2$$

and the monoids L_ℓ^1 obtained by adjoining an identity element to L_ℓ. The four-element semigroup $L_2 = A_0$ is long known to be finitely based [2]. Zhang and Luo [15] proved that the six-element semigroup L_3 is NFB and Lee generalised this to a sufficient condition [7] which implies that for all $\ell \geq 3$, the semigroup L_ℓ is NFB [6]. The five-element monoid L_2^1 was also proved to be FB by Edmunds [1], while the seven-element monoid L_3^1 was recently shown to be NFB by Zhang [14]. Lee conjectured that the monoids L_ℓ^1 are NFB for all $\ell \geq 3$. Theorem 2.1 implies that for each $\ell \geq 5$, the monoid L_ℓ^1 is NFB. This leaves the nine-element monoid L_4^1 as the only unsolved case in the finite basis problem for the monoids L_ℓ^1.

We prove Theorem 2.1 by using the general method in [11]. This general method can be used to establish the majority of existing sufficient conditions under which a semigroup is NFB. In particular, it can also be used to reprove the sufficient condition of Lee [7], which implies that for all $\ell \geq 3$, the semigroup L_ℓ is NFB. (The proof is the same as that of [11, Theorem 5.2] but uses [7, Lemma 14] instead of [5, Lemma 13].) Thus, this method can be used to establish the nonfinite basis properties of both Lee semigroups and Lee monoids.

In Section 7 we introduce monoids of the form $S_\tau^1(W)$ and show that both Lee monoids and the monoids of the form $S^1(W)$ can be viewed as special cases of this general construction. We also generalise Fact 1.1 into Lemma 7.1, which gives us the connection between monoids of the form $S_\tau^1(W)$ and τ-terms when τ is not necessarily the equality relation on $\mathfrak{A}^+$.

2. A sufficient condition under which a monoid is nonfinitely based

If $\mathbf{u}$ is a word and $x \in \mathrm{Cont}(\mathbf{u})$, the set of all variables contained in the word $\mathbf{u}$, then an *island* formed by x in $\mathbf{u}$ is a maximal subword of $\mathbf{u}$ which is a power of x. For example, the word $xyyx^5yx^3$ has three islands formed by x and two islands formed by y. We use x^+ to denote x^n when n is a positive integer and its exact value is unimportant. If $\mathbf{u}$ is a word over a two-letter alphabet, then the *height* of $\mathbf{u}$ is the number of islands in $\mathbf{u}$. For example, the word x^+ has height 1, x^+y^+ has height 2, $x^+y^+x^+$ has height 3 and so on. For each $\ell \geq 2$, consider the following property of a semigroup S.

($\mathbf{C}_\ell$) *If the height of* $\mathbf{u} \in \{x, y\}^+$ *is at most* ℓ*, then* $\mathbf{u}$ *can form an identity of* S *only with a word of the same type.*

The following words were used by Jackson to prove [3, Lemma 5.4]:

$$\mathbf{J}_n = (x_1x_{1+n}\cdots x_{1+n^2-n})(x_2x_{2+n}\cdots x_{2+n^2-n})\cdots(x_nx_{2n}\cdots x_{n^2}), \quad n > 3.$$

For example, $\mathbf{J}_4 = (x_1x_5x_9x_{13})(x_2x_6x_{10}x_{14})(x_3x_7x_{11}x_{15})(x_4x_8x_{12}x_{16})$. We generalise Jackson's words slightly as follows:

$$\mathbf{J}_{n,k} = (x_1^kx_{1+n}^k\cdots x_{1+n^2-n}^k)(x_2^kx_{2+n}^k\cdots x_{2+n^2-n}^k)\cdots(x_n^kx_{2n}^k\cdots x_{n^2}^k), \quad n > 3,\ k > 0.$$

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Notice that the words $\mathbf{J}_n$ and $\mathbf{J}_{n,k}$ are of the same type for all $n > 3$ and $k > 0$. We use $\overline{\mathbf{u}}$ to denote the reverse of a word $\mathbf{u}$. The following theorem gives a sufficient condition under which a monoid is NFB and will be proved in Section 5.

THEOREM 2.1. *Let M be a monoid that satisfies Property* (C_5). *If, for each $n > 3$, M satisfies the identity*

$$\begin{aligned} \mathbf{U}_n &= (x_1 x_2 \cdots x_{n^2-1} x_{n^2})\mathbf{J}_{n,k}(x_{n^2} x_{n^2-1} \cdots x_2 x_1) \\ &\approx (x_1 x_2 \cdots x_{n^2-1} x_{n^2})\overline{\mathbf{J}_{n,k}}(x_{n^2} x_{n^2-1} \cdots x_2 x_1) = \mathbf{V}_n \end{aligned} \tag{2.1}$$

for some $k \geq 1$, then M is NFB.

An identity $\mathbf{u} \approx \mathbf{v}$ is called *regular* if $\mathrm{Cont}(\mathbf{u}) = \mathrm{Cont}(\mathbf{v})$. If a semigroup S satisfies an identity $\mathbf{u} \approx \mathbf{v}$, we write $S \models \mathbf{u} \approx \mathbf{v}$. The following lemma will be generalised and reversed in Corollary 7.2.

LEMMA 2.2. *For each $\ell \geq 2$, the monoid L_ℓ^1 satisfies Property* (C_ℓ). *In other words, if $L_\ell^1 \models \mathbf{u} \approx \mathbf{v}$, where $\mathrm{Cont}(\mathbf{u}) = \{x, y\}$ and the height of $\mathbf{u}$ is at most ℓ, then $\mathbf{v}$ is of the same type as $\mathbf{u}$.*

PROOF. Since the word x is an isoterm for L_ℓ^1, the monoid L_ℓ^1 satisfies only regular identities. In particular, $\mathrm{Cont}(\mathbf{v}) = \{x, y\}$. If $\mathbf{v}$ is not of the same type as $\mathbf{u}$, then consider the substitution $\Theta : \mathfrak{A} \to L_\ell^1$ such that $\Theta(x) = b$ and $\Theta(y) = a$. Then $\Theta(\mathbf{u})$ is a subword of $\underbrace{bababa\cdots}_{\text{length } \ell} \neq 0$ and $\Theta(\mathbf{u}) \neq \Theta(\mathbf{v})$. Therefore, $\mathbf{v}$ must be of the same type as $\mathbf{u}$. □

Theorem 2.1 and Lemma 2.2 immediately imply the following result.

COROLLARY 2.3. *Let M be a monoid such that* $\mathrm{var}(M)$ *contains L_5^1. If, for each $n > 3$, M satisfies the identity (2.1) for some $k \geq 1$, then M is NFB.*

The next lemma shows that the identities (2.1) belong to a wider class of identities satisfied by L_ℓ^1 for each $\ell \geq 1$.

LEMMA 2.4. *Let $k \geq 2$ and let $\mathbf{X}$ be a word such that $\mathrm{Cont}(\mathbf{X}) = \{x_1, \ldots, x_n\}$ and $occ_{\mathbf{X}}(x_i) \geq k - 1$ for $1 \leq i \leq n$. Then, for each $n > 0$, the monoids*

$$L_{2k}^1 = \langle a, b, 1 \mid aa = a, bb = b, (ab)^k = 0\rangle, \quad L_{2k+1}^1 = \langle a, b, 1 \mid aa = a, bb = b, (ab)^k a = 0\rangle$$

satisfy the identity

$$\mathbf{U}_n = x_1 x_2 \cdots x_{n-1} x_n \mathbf{X} x_n x_{n-1} \cdots x_2 x_1 \approx x_1 x_2 \cdots x_{n-1} x_n \overline{\mathbf{X}} x_n x_{n-1} \cdots x_2 x_1 = \mathbf{V}_n.$$

PROOF. First, notice that each variable appears at least $k + 1$ times in $\mathbf{U}_n$ and $\mathbf{V}_n$. Fix some substitution $\Theta : \mathfrak{A} \to L_{2k}^1$ (L_{2k+1}^1). If the set $\mathrm{Cont}(\Theta(x_i))$ contains both a and b for some i with $1 \leq i \leq n$, then both $\Theta(\mathbf{U}_n)$ and $\Theta(\mathbf{V}_n)$ contain $(ab)^{k+1}$ or $(ba)^{k+1}$ as a subword and, consequently, both are equal to zero. Therefore, we may assume that $\Theta(x_i) \in \{a, b, 1\}$ for $1 \leq i \leq n$. To avoid some trivial cases, we may also assume that $\Theta(x_1 x_2 \cdots x_{n-1} x_n)$ contains both letters a and b. Consider three cases.

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Case 1. $\Theta(\mathbf{X})$ starts and ends with the same letter a or b. In this case, $\Theta(\overline{\mathbf{X}}) = \Theta(\mathbf{X})$ and, consequently, $\Theta(\mathbf{U}_n) = \Theta(\mathbf{V}_n)$.

Case 2. $\Theta(\mathbf{X}) = (ab)^m$ for some $m > 0$. In this case, $\Theta(\overline{\mathbf{X}}) = (ba)^m$ and, consequently,

$$\Theta(\mathbf{U}_n) = (ab)(ab)^m(ba) = (ab)(ba)^m(ba) = \Theta(\mathbf{V}_n).$$

Case 3. $\Theta(\mathbf{X}) = (ba)^m$ for some $m > 0$. This case is dual to Case 2. □

Corollary 2.3 and Lemma 2.4 immediately imply the following result.

COROLLARY 2.5. *Let M be a monoid such that M is contained in* $\mathrm{var}(L_\ell^1)$ *for some $\ell \geq 5$ and* $\mathrm{var}(M)$ *contains* $L_5^1 = \langle a, b, 1 \mid aa = a, bb = b, ababa = 0\rangle$. *Then M is NFB.*

3. Identities of monoids that satisfy Property (C_ℓ)

FACT 3.1. *If, for some $k \geq 1$, a monoid M satisfies Property* (C_{2k}), *then the word x^k is an isoterm for M.*

PROOF. If $M \models x^k \approx x^r$ for some $r \neq k$, then $M \models (xy)^k \approx (xy)^r$. To avoid a contradiction to Property (C_{2k}), we conclude that x^k is an isoterm for M. □

FACT 3.2. *Let M be a monoid that satisfies Property* (C_2). *Then:*

(i) *xy is an isoterm for M;*
(ii) *if $M \models x^+t \approx \mathbf{v}$, then $\mathbf{v} = x^+t$;*
(iii) *if $M \models tx^+ \approx \mathbf{v}$, then $\mathbf{v} = tx^+$;*
(iv) *if $M \models x^+tx^+ \approx \mathbf{v}$, then $\mathbf{v} = x^+tx^+$.*

PROOF. Part (i). Since the word x is an isoterm for M by Fact 3.1 and xy can form an identity of M only with a word of the same type, the word xy is also an isoterm for M.

Parts (ii) and (iii) follow immediately from the fact that t is an isoterm for M and Property (C_2). Part (iv) follows immediately from the fact that t is an isoterm for M and Parts (ii)–(iii). □

FACT 3.3. *Let M be a monoid that satisfies Property* (C_3). *If $M \models x^+t_1x^+t_2x^+ \approx \mathbf{v}$, then $\mathbf{v} = x^+t_1x^+t_2x^+$.*

PROOF. If $\mathbf{v} \neq x^+t_1x^+t_2x^+$, then $\mathbf{v} = x^+t_1t_2x^+$ by Fact 3.2(i) and (iv). Substituting y for t_1 and t_2 gives $M \models x^+yx^+yx^+ \approx x^+yyx^+$, contradicting the fact that M satisfies Property (C_3). □

If a variable t occurs exactly once in a word $\mathbf{u}$, then we say that t is *linear* in $\mathbf{u}$. If a variable x occurs more than once in $\mathbf{u}$, then we say that x is *nonlinear* in $\mathbf{u}$. Evidently, $\mathrm{Cont}(\mathbf{u}) = \mathrm{Lin}(\mathbf{u}) \cup \mathrm{Non}(\mathbf{u})$, where $\mathrm{Lin}(\mathbf{u})$ is the set of all linear variables in $\mathbf{u}$ and $\mathrm{Non}(\mathbf{u})$ is the set of all nonlinear variables in $\mathbf{u}$. A *block* of $\mathbf{u}$ is a maximal subword of $\mathbf{u}$ that does not contain any linear variables of $\mathbf{u}$.

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Lemma 3.4. *Let M be a monoid that satisfies Property* (C_3). *If $M \models \mathbf{u} \approx \mathbf{v}$, then:*

(i) $\mathrm{Lin}(\mathbf{u}) = \mathrm{Lin}(\mathbf{v})$, $\mathrm{Non}(\mathbf{u}) = \mathrm{Non}(\mathbf{v})$ *and the order of occurrences of linear variables in* $\mathbf{v}$ *is the same as in* $\mathbf{u}$*;*

(ii) *the corresponding blocks of* $\mathbf{u}$ *and* $\mathbf{v}$ *have the same content.*

That is, if $\mathbf{u} = \mathbf{a}_0 t_1 \mathbf{a}_1 t_2 \cdots t_{m-1} \mathbf{a}_{m-1} t_m \mathbf{a}_m$, *where* $\mathrm{Non}(\mathbf{u}) = \mathrm{Cont}(\mathbf{a}_0 \mathbf{a}_1 \cdots \mathbf{a}_{m-1} \mathbf{a}_m)$ *and* $\mathrm{Lin}(\mathbf{u}) = \{t_1, \ldots, t_m\}$, *then* $\mathbf{v} = \mathbf{b}_0 t_1 \mathbf{b}_1 t_2 \cdots t_{m-1} \mathbf{b}_{m-1} t_m \mathbf{b}_m$ *and* $\mathrm{Cont}(\mathbf{a}_q) = \mathrm{Cont}(\mathbf{b}_q)$ *for* $0 \le q \le m$.

Proof. Part (i) is an immediate consequence of Fact 3.2(i).

In order to verify Part (ii), we can assume that $\mathbf{u}$ contains exactly one nonlinear variable x. In this case, in view of Fact 3.2(ii)–(iv), $\mathbf{u}$ and $\mathbf{v}$ begin and end with the same variables. If some corresponding blocks of $\mathbf{u}$ and $\mathbf{v}$ did not have the same content, then $\mathbf{u}$ would contain $t_q t_{q+1}$ as a subword for some q with $1 \le q \le m$ while $\mathbf{v}$ contained $t_q x^+ t_{q+1}$ as a subword or *vice versa*. Since this contradicts Fact 3.3, the corresponding blocks of $\mathbf{u}$ and $\mathbf{v}$ must have the same content. □

If $\mathrm{Cont}(\mathbf{u}) \supseteq \{x_1, \ldots, x_n\}$, we write $\mathbf{u}(x_1, \ldots, x_n)$ to refer to the word obtained from $\mathbf{u}$ by deleting all occurrences of all variables that are not in $\{x_1, \ldots, x_n\}$.

Lemma 3.5. *Let $\ell > 2$ and let M be a monoid that satisfies Property* (C_ℓ). *Let* $\mathbf{u}$ *be a word with* $\mathrm{Non}(\mathbf{u}) = \{x, y\}$ *such that the height of* $\mathbf{u}(x, y)$ *is at most* ℓ. *If* $M \models \mathbf{u} \approx \mathbf{v}$, *then the corresponding blocks of* $\mathbf{u}$ *and* $\mathbf{v}$ *begin and end with the same variables.*

Proof. By Lemma 3.4,

$$\mathbf{u} = \mathbf{a}_0 t_1 \mathbf{a}_1 t_2 \cdots t_{m-1} \mathbf{a}_{m-1} t_m \mathbf{a}_m, \quad \mathbf{v} = \mathbf{b}_0 t_1 \mathbf{b}_1 t_2 \cdots t_{m-1} \mathbf{b}_{m-1} t_m \mathbf{b}_m,$$

where $\mathrm{Cont}(\mathbf{a}_q) = \mathrm{Cont}(\mathbf{b}_q) \subseteq \{x, y\}$ for each q with $0 \le q \le m$. Since the height of $\mathbf{u}(x, y)$ is at most ℓ, Property (C_ℓ) implies that $\mathbf{u}$ and $\mathbf{v}$ begin and end with the same variables. The rest of the lemma follows from the following claim. □

Claim 3.6. $\mathbf{u}$ *contains a subword* $c_1 t c_2$ *for some* $c_1, c_2 \in \{x, y\}$ *and* $t \in \{t_1, \ldots, t_m\}$ *if and only if* $\mathbf{v}$ *contains the identical three-letter subword.*

Proof. To obtain a contradiction, suppose that $\mathbf{u}(x, y, t)$ and $\mathbf{v}(x, y, t)$ have different three-letter subwords with t in the middle. Modulo renaming variables and duality, there are three cases.

Case 1. $\mathbf{u}$ contains ytx as a subword but $\mathbf{v}$ contains xtx as a subword. In this case, let $\Theta : \mathfrak{A} \to \mathfrak{A}^+$ be a substitution such that $\Theta(t) = yx$ and is identical on all other variables. Then $\Theta(\mathbf{u}(x, y, t))$ has the same type as $\mathbf{u}(x, y)$ but $\Theta(\mathbf{v}(x, y, t))$ has bigger height than $\mathbf{u}(x, y)$. This contradicts Property (C_ℓ).

Case 2. $\mathbf{u}$ contains ytx as a subword but $\mathbf{v}$ contains xty as a subword. In this case, let $\Theta : \mathfrak{A} \to \mathfrak{A}^+$ be a substitution such that $\Theta(t) = yx$ and is identical on all other variables. Then $\Theta(\mathbf{u}(x, y, t))$ has the same type as $\mathbf{u}(x, y)$ but $\Theta(\mathbf{v}(x, y, t))$ has bigger height than $\mathbf{u}(x, y)$. This contradicts Property (C_ℓ).

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Case 3. $\mathbf{u}$ contains yty as a subword but $\mathbf{v}$ contains xtx as a subword. In this case, let $\Theta : \mathfrak{A} \to \mathfrak{A}^+$ be a substitution such that $\Theta(t) = y$ and is identical on all other variables. Then $\Theta(\mathbf{u}(x, y, t))$ has the same type as $\mathbf{u}(x, y)$ but $\Theta(\mathbf{v}(x, y, t))$ has bigger height than $\mathbf{u}(x, y)$. This contradicts Property (C_ℓ). □

Lemma 3.7. *Let $\ell > 2$ and let M be a monoid that satisfies Property (C_ℓ). Let $\mathbf{u}$ be a word with $\mathrm{Non}(\mathbf{u}) = \{x, y\}$ such that:*

(i) *the height of $\mathbf{u}(x, y)$ is at most ℓ;*
(ii) *every block of $\mathbf{u}$ has height at most 3.*

Then $\mathbf{u}$ can form an identity of M only with a word of the same type.

Proof. Write $\mathbf{u} = \mathbf{a}_0 t_1 \mathbf{a}_1 t_2 \cdots t_{m-1} \mathbf{a}_{m-1} t_m \mathbf{a}_m$, where $\{x, y\} = \mathrm{Cont}(\mathbf{a}_0 \mathbf{a}_1 \cdots \mathbf{a}_{m-1} \mathbf{a}_m)$ and $\mathrm{Lin}(\mathbf{u}) = \{t_1, \ldots, t_m\}$. If $M \models \mathbf{u} \approx \mathbf{v}$, then, by Lemmas 3.4 and 3.5,

$$\mathbf{v} = \mathbf{b}_0 t_1 \mathbf{b}_1 t_2 \cdots t_{m-1} \mathbf{b}_{m-1} t_m \mathbf{b}_m,$$

where $\mathrm{Cont}(\mathbf{a}_q) = \mathrm{Cont}(\mathbf{b}_q) \subseteq \{x, y\}$ for each q with $0 \le q \le m$ and the corresponding blocks $\mathbf{a}_q$ and $\mathbf{b}_q$ begin and end with the same variable. Condition (ii) implies that either the block $\mathbf{a}_q$ is empty or $\mathbf{a}_q \in \{x^+, y^+, x^+y^+, y^+x^+, x^+y^+x^+, y^+x^+y^+\}$ for each q with $0 \le q \le m$. Thus, if the corresponding blocks $\mathbf{a}_q$ and $\mathbf{b}_q$ are not of the same type for some q with $0 \le q \le m$, then only the following two cases are possible modulo renaming variables.

Case 1. $\mathbf{a}_q = x^+y^+$ but $\mathbf{b}_q = (x^+y^+)^r$ for some $r > 1$.

Case 2. $\mathbf{a}_q = x^+y^+x^+$ but $\mathbf{b}_q = (x^+y^+x^+)^r$ for some $r > 1$.

If $\mathbf{u}$ and $\mathbf{v}$ are not of the same type, then some blocks of $\mathbf{v}$ have bigger height than the corresponding blocks in $\mathbf{u}$. Therefore, $\mathbf{v}(x, y)$ has bigger height than $\mathbf{u}(x, y)$. To avoid a contradiction to Property (C_ℓ), we conclude that $\mathbf{u}$ and $\mathbf{v}$ are of the same type. □

Lemma 3.8. *Let $\ell > 2$ and let M be a monoid that satisfies Property (C_ℓ). Let $\mathbf{u}$ be a word such that:*

(i) *for each $\{x, y\} \subseteq \mathrm{Cont}(\mathbf{u})$, the height of $\mathbf{u}(x, y)$ is at most ℓ;*
(ii) *for each $x \in \mathrm{Non}(\mathbf{u})$, there is a linear variable $t \in \mathrm{Lin}(\mathbf{u})$ between any two islands formed by x.*

Then $\mathbf{u}$ can form an identity of M only with a word of the same type.

Proof. Write the word as $\mathbf{u} = \mathbf{a}_0 t_1 \mathbf{a}_1 t_2 \cdots t_{m-1} \mathbf{a}_{m-1} t_m \mathbf{a}_m$, where $\mathrm{Lin}(\mathbf{u}) = \{t_1, \ldots, t_m\}$ and $\mathrm{Cont}(\mathbf{a}_0 \mathbf{a}_1 \cdots \mathbf{a}_{m-1} \mathbf{a}_m) = \{x_1, \ldots, x_n\} = \mathrm{Non}(\mathbf{u})$.

Condition (ii) implies that for $1 \le i \le n$ and $0 \le q \le m$, each variable x_i forms at most one island in $\mathbf{a}_q$. Lemma 3.7 implies that $\mathbf{u}(x_i, x_j, t_1, \ldots, t_m)$ forms an identity of M only with a word of the same type for $1 \le i < j \le n$. Therefore, $\mathbf{u}$ can also form an identity of M only with a word of the same type. □

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4. Words and substitutions

LEMMA 4.1. *Let* $\mathbf{u}$ *and* $\mathbf{v}$ *be two words of the same type such that* $\mathrm{Lin}(\mathbf{u}) = \mathrm{Lin}(\mathbf{v})$ *and* $\mathrm{Non}(\mathbf{u}) = \mathrm{Non}(\mathbf{v})$. *Let* $\Theta : \mathfrak{A} \to \mathfrak{A}^+$ *be a substitution that has the following property.*

(*) *If* $\Theta(x)$ *contains more than one variable, then* x *is linear in* $\mathbf{u}$.

Then $\Theta(\mathbf{u})$ *and* $\Theta(\mathbf{v})$ *are also of the same type.*

PROOF. Since $\mathbf{u}$ and $\mathbf{v}$ are of the same type, $\mathbf{u} = c_1^{u_1} c_2^{u_2} \cdots c_r^{u_r}$ and $\mathbf{v} = c_1^{v_1} c_2^{v_2} \cdots c_r^{v_r}$ for some $r \geq 1$ and $u_1, \dots, u_r, v_1, \dots, v_r > 0$, where $c_1, \dots, c_r$ are not necessarily distinct variables.

We first prove that the words $\Theta(c_i^{u_i})$ and $\Theta(c_i^{v_i})$ are of the same type for $1 \leq i \leq r$. Indeed, if c_i is linear in $\mathbf{u}$ (and in $\mathbf{v}$), then $u_i = v_i = 1$ and $\Theta(c_i^{u_i}) = \Theta(c_i^{v_i})$. If c_i is nonlinear in $\mathbf{u}$ (and in $\mathbf{v}$), then $\Theta(c_i^{u_1}) = x^+$ for some variable x and $\Theta(c_i^{v_i})$ is a power of the same variable. Since $\Theta(\mathbf{u}) = \Theta(c_1^{u_1} c_2^{u_2} \cdots c_r^{u_r}) = \Theta(c_1^{u_1})\Theta(c_2^{u_2}) \cdots \Theta(c_r^{u_r})$ and $\Theta(\mathbf{v}) = \Theta(c_1^{v_1} c_2^{v_2} \cdots c_r^{v_r}) = \Theta(c_1^{v_1})\Theta(c_2^{v_2}) \cdots \Theta(c_r^{v_r})$, we conclude that $\Theta(\mathbf{u})$ and $\Theta(\mathbf{v})$ are of the same type. □

If x and y are two distinct variables, then $E_{x=y}$ denotes a substitution that renames y by x and is identical on all other variables.

FACT 4.2. *Given a word* $\mathbf{u}$ *and a substitution* $\Theta : \mathfrak{A} \to \mathfrak{A}^+$, *we can rename some variables in* $\mathbf{u}$ *so that the resulting word* $E(\mathbf{u})$ *has the following properties:*

(i) $\Theta(E(\mathbf{u}))$ *is of the same type as* $\Theta(\mathbf{u})$;

(ii) *for every* $x, y \in \mathrm{Cont}(E(\mathbf{u}))$, *if the words* $\Theta(x)$ *and* $\Theta(y)$ *are powers of the same variable, then* $x = y$.

PROOF. If $\mathbf{u}$ satisfies Property (ii), then take E to be the identity substitution $E_{x=x}$ and we are done. If $\mathbf{u}$ does not satisfy Property (ii), then for some $x \neq y \in \mathrm{Cont}(\mathbf{u})$ the words $\Theta(x)$ and $\Theta(y)$ are powers of the same variable. If $E_{x=y}(\mathbf{u})$ satisfies Property (ii), then take $E = E_{x=y}$. Notice that $\Theta(E_{x=y}(\mathbf{u}))$ is of the same type as $\Theta(\mathbf{u})$. If not, then for some $p \neq z \in \mathrm{Cont}(E_{x=y}(\mathbf{u}))$ the words $\Theta(p)$ and $\Theta(z)$ are powers of the same variable. If $E_{p=z}E_{x=y}(\mathbf{u})$ satisfies Property (ii), then take $E = E_{p=z}E_{x=y}$ and we are done. And so on. Since the number of variables in $E(\mathbf{u})$ decreases, eventually the word $E(\mathbf{u})$ will satisfy Property (ii). □

LEMMA 4.3. *Let* $\ell > 1$ *and let* $\mathbf{U}$ *be a word such that for each* $\{x, y\} \subseteq \mathrm{Cont}(\mathbf{U})$, *the height of* $\mathbf{U}(x, y)$ *is at most* ℓ. *Let* $\Theta : \mathfrak{A} \to \mathfrak{A}^+$ *be a substitution which satisfies Property* (ii) *in Fact 4.2. If* $\Theta(\mathbf{u}) = \mathbf{U}$, *then* $\mathbf{u}$ *satisfies Condition* (i) *in Lemma 3.8, that is, for each* $\{x, y\} \subseteq \mathrm{Cont}(\mathbf{u})$, *the height of* $\mathbf{u}(x, y)$ *is at most* ℓ.

PROOF. Suppose that for some $\{x, y\} \subseteq \mathrm{Cont}(\mathbf{u})$, the word $\mathbf{u}(x, y)$ has height bigger than ℓ. Since Θ satisfies Property (ii) in Fact 4.2, $\Theta(x)$ contains x' and $\Theta(y)$ contains y' for some $x' \neq y'$. Therefore, $\mathbf{U}(x', y')$ also has height bigger than ℓ, which is a contradiction. □

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5. Proof of Theorem 2.1

The following lemma implies [11, Corollary 2.2] and is a special case of [11, Fact 2.1].

LEMMA 5.1. *Let τ be an equivalence relation on the free semigroup $\mathfrak{A}^+$ and S be a semigroup. Suppose that, for infinitely many n, S satisfies an identity $\mathbf{U}_n \approx \mathbf{V}_n$ in at least n variables such that $\mathbf{U}_n$ and $\mathbf{V}_n$ are not τ-related. Suppose also that for every identity $\mathbf{u} \approx \mathbf{v}$ of S in fewer than n variables, every word $\mathbf{U}$ such that $\mathbf{U}\tau\mathbf{U}_n$ and every substitution $\Theta : \mathfrak{A} \to \mathfrak{A}^+$ such that $\Theta(\mathbf{u}) = \mathbf{U}$, we have $\mathbf{U}\tau\Theta(\mathbf{v})$. Then S is NFB.*

PROOF. Take an arbitrary $m > 0$ and let Σ be a set of identities of S in at most m variables. By our assumption, S satisfies an identity $\mathbf{U}_n \approx \mathbf{V}_n$ in at least n variables such that $n > m$ and the words $\mathbf{U}_n$ and $\mathbf{V}_n$ are not τ-related.

If $\mathbf{U}_n \approx \mathbf{V}_n$ was a consequence from Σ, then we could find a sequence of words $\mathbf{U}_n = \mathbf{W}_1 \approx \mathbf{W}_2 \approx \cdots \approx \mathbf{W}_l = \mathbf{V}_n$ and substitutions $\Theta_1, \dots, \Theta_{l-1}(\mathfrak{A} \to \mathfrak{A}^+)$ such that for each $i = 1, \dots, l-1$, we have $\mathbf{W}_i = \Theta_i(\mathbf{u}_i)$ and $\mathbf{W}_{i+1} = \Theta_i(\mathbf{v}_i)$ for some identity $\mathbf{u}_i \approx \mathbf{v}_i \in \Sigma$. Since every identity in Σ involves fewer than n variables, we have $\mathbf{U}_n = \mathbf{W}_1\tau\mathbf{W}_2\tau\cdots\tau\mathbf{W}_{l-1}\tau\mathbf{W}_l = \mathbf{V}_n$. Thus, $\mathbf{U}_n\tau\mathbf{V}_n$. But $\mathbf{U}_n$ and $\mathbf{V}_n$ are not τ-related, so $\mathbf{U}_n \approx \mathbf{V}_n$ is not a consequence from Σ. Since m and Σ were arbitrary, S is NFB. □

Let $\mathbf{U}$ be a word of the same type as $\mathbf{U}_n = x_1x_2\cdots x_{n^2}\mathbf{J}_n x_{n^2}\cdots x_2x_1$. Then the occurrences of x_{n^2} form two islands in $\mathbf{U}$. We refer to these two islands as ${}_1x^+_{n^2}$ and ${}_2x^+_{n^2}$ counting rightwards from the left. For each i with $1 \le i < n^2$, the occurrences of x_i form three islands in $\mathbf{U}$. We refer to these three islands as ${}_1x^+_i$, ${}_2x^+_i$ and ${}_3x^+_i$ counting rightwards from the left.

LEMMA 5.2. *Let $\mathbf{U}$ be a word of the same type as*

$$\mathbf{U}_n = x_1x_2\cdots x_{n^2}\mathbf{J}_n x_{n^2}\cdots x_2x_1.$$

Then $\mathbf{U}$ has the following properties:

(P1) *for $1 \le i \ne j \le n^2$, the word x_ix_j appears at most once in $\mathbf{U}$ as a subword;*

(P2) *for $1 \le i \le n^2$, there are occurrences of at least n pairwise distinct variables between any two islands formed by x_i in $\mathbf{U}$.*

PROOF. Property (P1) is evident. To verify Property (P2), notice that there are occurrences of $n^2 - 1$ pairwise distinct variables between ${}_1x^+_{n^2}$ and ${}_2x^+_{n^2}$. If $1 \le i < n^2$, consider two cases.

Case 1. $n^2 - i < n$. The $n-1$ islands $\{{}_2x^+_1, {}_2x^+_{1+n}, {}_2x^+_{1+2n}, \dots, {}_2x^+_{1+(n-2)n}\}$ are located between ${}_1x^+_{n^2}$ and ${}_2x^+_i$, so there are at least n pairwise distinct variables between ${}_1x^+_i$ and ${}_2x^+_i$. The $n-1$ islands $\{{}_2x^+_n, {}_2x^+_{2n}, {}_2x^+_{3n}, \dots, {}_2x^+_{n^2-n}\}$ are located between ${}_2x^+_i$ and ${}_2x^+_{n^2}$, so there are at least n pairwise distinct variables between ${}_2x^+_i$ and ${}_3x^+_i$.

Case 2. $n^2 - i \ge n$. The $n-1$ islands $\{{}_1x^+_{i+1}, {}_1x^+_{i+2}, {}_1x^+_{i+3}, \dots, {}_1x^+_{n^2-1}\}$ are located between ${}_1x^+_i$ and ${}_1x^+_{n^2}$, so there are at least n pairwise distinct variables between ${}_1x^+_i$ and ${}_2x^+_i$. The $n-1$ islands $\{{}_2x^+_{n^2-1}, {}_2x^+_{n^2-2}, {}_2x^+_{n^2-3}, \dots, {}_2x^+_{i+1}\}$ are located between ${}_2x^+_{n^2}$ and ${}_3x^+_i$, so there are at least n pairwise distinct variables between ${}_2x^+_i$ and ${}_3x^+_i$. □

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Proof of Theorem 2.1. Let τ be the equivalence relation on $\mathfrak{A}^+$ defined by $\mathbf{u}\tau\mathbf{v}$ if $\mathbf{u}$ and $\mathbf{v}$ are of the same type. First, notice that the words $\mathbf{U}_n$ and $\mathbf{V}_n$ are not of the same type. Indeed, $\mathbf{U}_n$ contains $x_{n^2}x_1$ as a subword but $\mathbf{V}_n$ does not have this subword.

Now let $\mathbf{U}$ be of the same type as $\mathbf{U}_n$. Let $\mathbf{u} \approx \mathbf{v}$ be an identity of M in fewer than n variables and let $\Theta : \mathfrak{A} \to \mathfrak{A}^+$ be a substitution such that $\Theta(\mathbf{u}) = \mathbf{U}$. The word $E(\mathbf{u})$ also involves fewer than n variables and $E(\mathbf{u}) \approx E(\mathbf{v})$ is also an identity of M.

Since $\mathbf{U}(x_i, x_j) = x_i^+ x_j^+ x_i^+ x_j^+ x_i^+$ for $1 \le i < j \le n^2$, the height of $\mathbf{U}(x_i, x_j)$ is 5. By Lemma 4.3, $E(\mathbf{u})$ satisfies Condition (i) in Lemma 3.8 for $\ell = 5$, that is, for each $\{x, y\} \subseteq \mathrm{Cont}(E(\mathbf{u}))$, the height of $E(\mathbf{u})(x, y)$ is at most 5.

If $y \in \mathrm{Non}(E(\mathbf{u}))$, then, in view of Property (P1) in Lemma 5.2, $\Theta(y) = x_i^*$ for some i with $1 \le i \le n^2$. Since the occurrences of x_i form at most three islands in $\mathbf{U}$ and Θ satisfies Property (ii) in Fact 4.2, the occurrences of y also form at most three islands in $\mathbf{u}$.

Due to Property (P2) in Lemma 5.2, there are occurrences of at least n pairwise distinct variables between any two islands formed by x_i in $\mathbf{U}$. Since $E(\mathbf{u})$ involves fewer than n variables, there is a variable $t \in \mathrm{Cont}(E(\mathbf{u}))$ between any two islands formed by y in $E(\mathbf{u})$ such that $\Theta(t)$ contains $x_i x_j$ as a subword for some i, j with $1 \le i \ne j \le n^2$. Due to Property (P1) in Lemma 5.2, t is linear in $E(\mathbf{u})$. Thus, $E(\mathbf{u})$ satisfies Condition (ii) in Lemma 3.8. Therefore, $E(\mathbf{v})$ is of the same type as $E(\mathbf{u})$ by Lemma 3.8.

Due to Property (P1) in Lemma 5.2, Θ satisfies Condition (*) in Lemma 4.1. Consequently, the word $\Theta(E(\mathbf{v}))$ has the same type as $\Theta(E(\mathbf{u}))$ by Lemma 4.1. Thus,

$$\mathbf{U} = \Theta(\mathbf{u}) \overset{\text{Fact 4.2}}{\tau} \Theta(E(\mathbf{u})) \overset{\text{Lemma 4.1}}{\tau} \Theta(E(\mathbf{v})) \overset{\text{Fact 4.2}}{\tau} \Theta(\mathbf{v}).$$

Since $\Theta(\mathbf{v})$ is of the same type as $\mathbf{U}$, Lemma 5.1 implies that M is NFB. □

6. Sets of isoterms for L_ℓ^1 are FB when $\ell \le 5$

If $\mathrm{var}\, S(W) = \mathrm{var}\, S(W')$, we say that sets of words W and W' are equationally equivalent and write $W \sim W'$. A word $\mathbf{u}$ is called *k-limited* if each variable occurs in $\mathbf{u}$ at most k times.

Fact 6.1. *We have:*

(i) $\mathrm{Isot}(L_2^1) = \mathrm{Isot}(L_3^1) \sim \{ab\}$;
(ii) $\mathrm{Isot}(L_4^1) = \mathrm{Isot}(L_5^1) \sim \{abab, a^2b^2, ab^2a\}$.

Proof. First, notice that for each $k \ge 1$, $L_{2k}^1 = \langle a, b, 1 \mid aa = a, bb = b, (ab)^k = 0\rangle$ and $L_{2k+1}^1 = \langle a, b, 1 \mid aa = a, bb = b, (ab)^k a = 0\rangle$ satisfy $xt_1xt_2x \cdots xt_kx \approx x^2t_1xt_2x \cdots xt_kx$. Therefore, every isoterm for L_{2k}^1 and for L_{2k+1}^1 is k-limited.

Part (i). Since L_2^1 satisfies Property (C_2) by Lemma 2.2, the word xy is an isoterm for L_2^1 by Fact 3.2. Since $\{ab\}$ is equationally equivalent to the set of all 1-limited words, we have $\mathrm{Isot}(L_2^1) = \mathrm{Isot}(L_3^1) \sim \{ab\}$.

Part (ii). Since L_4^1 satisfies Property (C_4) by Lemma 2.2, the word x^2 is an isoterm for L_4^1 by Fact 3.1.

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Let us show that if $\mathbf{u} \in \{abab, a^2b^2, ab^2a\}$, then $\mathbf{u}$ is an isoterm for L_4^1. Indeed, assume that $L_4^1 \models \mathbf{u} \approx \mathbf{v}$. Since x^2 is an isoterm for L_4^1, the identity $\mathbf{u} \approx \mathbf{v}$ is *balanced*, that is, every variable occurs the same number of times in $\mathbf{u}$ and $\mathbf{v}$. Then $\mathbf{v}$ can only be one of the words $\{abab, a^2b^2, ab^2a\}$ modulo renaming a and b. To avoid a contradiction to Property (C_4), we conclude that $\mathbf{v} = \mathbf{u}$.

Since $\{abab, a^2b^2, ab^2a\}$ is equationally equivalent to the set of all 2-limited words, we have $\mathrm{Isot}(L_4^1) = \mathrm{Isot}(L_5^1) \sim \{abab, a^2b^2, ab^2a\}$. □

Notice that the word $xyyxyx$ is 3-limited but is not an isoterm for L_6^1 because $L_6^1 \models xyyxyx \approx xyxyyx$.

Since for each $k > 0$ the set of all k-limited words is FB [4], the result of Zhang [14] that L_3^1 is NFB, Corollary 2.5 and Fact 6.1 immediately imply the following result.

Corollary 6.2. *The monoids L_3^1 and L_5^1 are NFB while the sets of their isoterms are FB.*

Presently, L_3^1, L_4^1 [8] and L_5^1 are the only known examples of NFB finite aperiodic monoids whose sets of isoterms are FB.

Question 6.3. Is there a finite aperiodic FB monoid whose set of isoterms is NFB? Is there a finite aperiodic NFB monoid with central idempotents whose set of isoterms is FB?

7. Monoids of the form $S_\tau^1(W)$

Let τ be a congruence on the free semigroup $\mathfrak{A}^+$ and W be a nonempty set of words in $\mathfrak{A}^+$ such that:

- W is a union of τ-classes, that is, $\mathbf{v} \in W$ whenever $\mathbf{u} \in W$ and $\mathbf{u}\tau\mathbf{v}$;
- W is closed under taking subwords, that is, $\mathbf{v} \in W$ whenever $\mathbf{u} \in W$ and $\mathbf{v}$ is a subword of $\mathbf{u}$.

Since W is a union of τ-classes, the set $I(W) = \mathfrak{A}^+ \setminus W$ is also a union of τ-classes if it is not empty. Let T denote the factor semigroup of $\mathfrak{A}^+$ over τ and T^1 denote the monoid obtained by adjoining an identity element to T. Let H_τ denote the homomorphism corresponding to τ extended to $\mathfrak{A}^*$ by $H_\tau(\epsilon) = 1$, where ϵ denotes the empty word and 1 denotes the identity element of T^1. Since W is closed under taking subwords, $H_\tau(I(W))$ is an ideal of $T = H_\tau(\mathfrak{A}^+)$ and of $T^1 = H_\tau(\mathfrak{A}^*)$. We define $S_\tau(W)$ as the Rees quotient of T over $H(I(W))$ and $S_\tau^1(W)$ as the Rees quotient of T^1 over $H(I(W))$. Notice that $S_\tau(W) = \phi H_\tau(\mathfrak{A}^+)$ and $S_\tau^1(W) = \phi H_\tau(\mathfrak{A}^*)$, where $\phi H_\tau(\mathbf{u}) = H_\tau(\mathbf{u})$ if $\mathbf{u} \in W \cup \{\epsilon\}$ and $\phi H_\tau(\mathbf{u}) = 0$ if $\mathbf{u} \in I(W)$.

If τ is the trivial congruence on $\mathfrak{A}^+$, then $S_\tau^1(W)$ coincides with the widely studied monoid $S^1(W)$ defined in the introduction. Also, recall from the introduction that a word $\mathbf{u}$ is called a *τ-term* for a semigroup S if $\mathbf{u}\tau\mathbf{v}$ whenever $S \models \mathbf{u} \approx \mathbf{v}$. The following lemma generalises Fact 1.1.

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Lemma 7.1. *Let τ be a congruence on the free semigroup $\mathfrak{A}^+$ such that for each $x \in \mathfrak{A}$, if $x\tau\mathbf{u}$, then $\mathbf{u} = x^m$ for some $m > 0$. Let W be a nonempty set of words in $\mathfrak{A}^+$ which is a union of τ-classes and is closed under taking subwords. Let S be a semigroup (respectively monoid). Then* $\operatorname{var} S$ *contains $S_\tau(W)$ (respectively $S^1_\tau(W)$) if and only if every word in W is a τ-term for S.*

Proof. $\Rightarrow$ Let S be a semigroup such that $\operatorname{var} S$ contains $S_\tau(W)$. Take $\mathbf{u} \in W$. Let us show that $\mathbf{u}$ is a τ-term for S. Indeed, suppose that $S \models \mathbf{u} \approx \mathbf{v}$. Since $\operatorname{var} S$ contains $S_\tau(W)$, we have $\phi H_\tau(\mathbf{u}) = \phi H_\tau(\mathbf{v})$. Since $\mathbf{u} \in W$, we have $\phi H_\tau(\mathbf{u}) = H_\tau(\mathbf{u}) \neq 0$. If $\mathbf{v} \notin W$, then $\phi H_\tau(\mathbf{v}) = 0$. Thus, $\mathbf{v} \in W$ and $H_\tau(\mathbf{u}) = \phi H_\tau(\mathbf{u}) = \phi H_\tau(\mathbf{v}) = H_\tau(\mathbf{v})$. Therefore, $\mathbf{u}\tau\mathbf{v}$ and $\mathbf{u}$ is a τ-term for S.

$\Leftarrow$ Let S be a semigroup (respectively monoid) such that every word in W is a τ-term for S. Let $\mathbf{u} \approx \mathbf{v}$ be an identity of S and $\Theta : \mathfrak{A} \to S_\tau(W)$ a substitution (respectively $\Theta : \mathfrak{A} \to S^1_\tau(W)$). If $\mathbf{u} = c_1 \cdots c_r$ and $\mathbf{v} = d_1 \cdots d_l$ for some not necessarily distinct letters $c_1, \ldots, c_r$ and $d_1, \ldots, d_l$, then

$$\Theta(c_1) = \phi H_\tau(\mathbf{u}_1), \ldots, \Theta(c_r) = \phi H_\tau(\mathbf{u}_r), \quad \Theta(d_1) = \phi H_\tau(\mathbf{v}_1), \ldots, \Theta(d_l) = \phi H_\tau(\mathbf{v}_l)$$

for some not necessarily distinct words $\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{v}_1, \ldots, \mathbf{v}_l$ from $\mathfrak{A}^+$ (respectively from $\mathfrak{A}^*$). Modulo duality, three cases are possible.

Case 1. Both $\mathbf{u}_1 \cdots \mathbf{u}_r$ and $\mathbf{v}_1 \cdots \mathbf{v}_l$ belong to $I(W)$. In this case,

$$\Theta(\mathbf{u}) = \phi H_\tau(\mathbf{u}_1 \cdots \mathbf{u}_r) = 0 = \phi H_\tau(\mathbf{v}_1 \cdots \mathbf{v}_l) = \Theta(\mathbf{v}).$$

Case 2. $\mathbf{u}_1 \cdots \mathbf{u}_r \in W$. Since $S \models \mathbf{u} \approx \mathbf{v}$, we have $S \models (\mathbf{u}_1 \cdots \mathbf{u}_r) \approx (\mathbf{v}_1 \cdots \mathbf{v}_l)$. Since $\mathbf{u}_1 \cdots \mathbf{u}_r$ is a τ-term for S, we have $(\mathbf{u}_1 \cdots \mathbf{u}_r)\tau(\mathbf{v}_1 \cdots \mathbf{v}_l)$. Since W is a union of τ-classes, $\mathbf{v}_1 \cdots \mathbf{v}_l \in W$. Therefore,

$$\Theta(\mathbf{u}) = \phi H_\tau(\mathbf{u}_1 \cdots \mathbf{u}_r) = H_\tau(\mathbf{u}_1 \cdots \mathbf{u}_r) = H_\tau(\mathbf{v}_1 \cdots \mathbf{v}_l) = \phi H_\tau(\mathbf{v}_1 \cdots \mathbf{v}_l) = \Theta(\mathbf{v}).$$

Case 3. $\mathbf{u}_1 = \cdots = \mathbf{u}_r = \epsilon$. This case is only possible if S is a monoid. In this case, $\mathbf{u} \approx \mathbf{v}$ is a regular identity. (Indeed, if for some $y \in \mathfrak{A}$ we have $y \in \operatorname{Cont}(\mathbf{u})$ but $y \notin \operatorname{Cont}(\mathbf{v})$, then $S \models x \approx xy^n$ for some $x \neq y$ and $n > 0$. Since W is closed under taking subwords, x is a τ-term for S and, consequently, $x\tau xy^c$, which is forbidden by our assumption about τ.) Since $\operatorname{Cont}(\mathbf{u}) = \operatorname{Cont}(\mathbf{v})$, we have $\mathbf{v}_1 = \cdots = \mathbf{v}_l = \epsilon$. Consequently, $\Theta(\mathbf{u}) = 1 = \Theta(\mathbf{v})$.

Thus, we have proved that every identity of S holds in $S_\tau(W)$ and if S is a monoid then every identity of S holds in $S^1_\tau(W)$. Therefore, $\operatorname{var} S$ contains $S_\tau(W)$ and if S is a monoid then $\operatorname{var} S$ contains $S^1_\tau(W)$. □

Let τ be the relation on the free semigroup $\mathfrak{A}^+$ defined by $\mathbf{u}\tau\mathbf{v}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are of the same type and let W_ℓ be the set of all subwords of $\underbrace{b^+a^+b^+a^+b^+\cdots}_{\text{height } \ell}$. Then, for each $\ell \geq 2$, the Lee semigroup L_ℓ is isomorphic to $S_\tau(W_\ell)$ and the Lee monoid L^1_ℓ is isomorphic to $S^1_\tau(W_\ell)$. Thus, Lemma 7.1 immediately implies the following result.

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COROLLARY 7.2. *Let $\ell \geq 2$ and let S be a semigroup (respectively monoid). Then* var S *contains L_ℓ (respectively L_ℓ^1) if and only if S satisfies Property* (C_ℓ)*, that is, every word in $\{x, y\}^+$ of height at most ℓ can form an identity of S only with a word of the same type.*

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