CHARACTERISATIONS OF PARTIALLY CONTINUOUS, STRICTLY COSINGULAR AND ϕ_- TYPE OPERATORS

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0. Introduction. We will denote the dimension of a subspace M of X by $\dim M$ and the codimension of M with respect to X by $\operatorname{cod}_X M$ or simply $\operatorname{cod} M$ if there is no danger of confusion. The classes of infinite dimensional and closed infinite codimensional subspaces of X will be denoted by $\mathscr{I}(X)$ and $\mathscr{I}_c(X)$ respectively with $\mathscr{F}(X)$ and $\mathscr{F}_c(X)$ denoting the classes of finite dimensional and of finite codimensional subspaces of X respectively. For a subspace M of X we denote the injection of M into X by J_M^X and the quotient map from X onto the quotient space X/M by Q_M^X. Where there is no danger of confusion we will write J_M and Q_M. The injection of X into its completion $\tilde{X}$ will be denoted by J_X. Letting X' denote the continuous dual of X we remark that since X' is isometric to $(\tilde{X})'$, these two spaces will be considered identical where convenient. The orthogonal complements of subsets $M \subset X$ in X' and $K \subset X'$ in X will be denoted by $M^\perp$ and ${}^\perp K$ respectively; $M^{\perp X}$ and ${}^{X\perp}K$ will be used if there is danger of confusion.

For an operator T we define the *adjoint* or *conjugate* T' of T to be the adjoint of $TJ_{D(T)}$ in the sense of [7]. The injective operator $\hat{T}$ induced by T is defined as in [7]. In general the dimension of the kernel $N(T)$ of T is denoted by $a(T)$ with $b(T)$ and $\bar{b}(T)$ denoting $\operatorname{cod}_Y R(T)$ and $\operatorname{cod}_Y \overline{R(T)}$ respectively.

An operator T is defined to be *strictly cosingular* [10] if there is an $M \in \mathscr{I}_c(Y)$ such that $(Q_M T)'$ has a continuous inverse, F_- [5] if there is no $M \in \mathscr{F}(Y)$ such that $(Q_M T)'$ has a continuous inverse and F_+ [4] if there is an $M \in \mathscr{F}_c(D(T))$ such that TJ_M has a continuous inverse. Furthermore T is said to be $\phi_-(\phi_+)$ if T is a normally solvable operator with $b(T) < \infty (a(T) < \infty)$. Whenever Y is complete, T is said to be *nuclear* if there exists sequences $\{x_n'\} \subset (D(T))'$ and $\{y_n\} \subset Y$ such that $\sum_{n=1}^{\infty} \|x_n'\| \, \|y_n\| < \infty$ and $Tx = \sum_{n=1}^{\infty} x_n'(x) y_n$ for each $x \in D(T)$. The classes of continuous, partially continuous, F_- and strictly cosingular operators in $L(X, Y)$ will be denoted by $B(X, Y)$, $PB(X, Y)$, $F_-(X, Y)$ and $SC(X, Y)$. Square brackets will be used to indicate that only everywhere defined operators are considered; for example $B[X, Y]$ and $PB[X, Y]$ denote the classes of everywhere defined continuous and partially continuous operators respectively. An operator T is said to be *bounded* if and only if $T \in B[X, Y]$. Observe that if $A \in B[Y, Z]$, then $(AT)' = T'A'$ and hence for any closed subspace M of Y we have $(Q_M T)' = T'Q_M' = T'J_{M^\perp}$ [7, I.6.4]. For $T \in B(X, Y)$, $\bar{T} \in B[D(T)^\sim, \tilde{Y}]$ will denote the unique bounded extension of $J_Y T J_{D(T)}$ to all of $D(T)^\sim$.

Note that our definition of strict cosingularity generalises that of [13] with the two definitions being equivalent in the classical case of bounded operators between Banach spaces [7, II.4.4]. We similarly conclude from [7, II.4.4 and IV.1.13] that F_- and ϕ_- operators coincide in the case of closed operators between Banach spaces.

The subspace $D(T) \subset X$, renormed with the norm $\|.\|_T = \|.\| + \|T.\|$, will be denoted by X_T, with G_T denoting the identity map from X_T into X with range $D(T)$. The operator T composed with G_T will be denoted by TG. Observe that both TG and G_T are bounded with norm not exceeding 1.

1. F_- and strictly cosingular operators.

1.1 THEOREM [5]. *The following are equivalent.*

(I) $T \notin F_-$.

(II) *There exists* $M \in \mathcal{I}_c(\tilde{Y})$ *such that* $Q_M J_Y T$ *is nuclear (compact).*

(III) *For any* $\varepsilon > 0$ *there exists* $M \in \mathcal{I}_c(\tilde{Y})$ *such that* $\|Q_M J_Y T\| \leq \varepsilon$.

1.2 PROPOSITION [5, 4.15]. *The following are equivalent.*

(I) $T \in SC(X, Y)$.

(II) *For each* $M \in \mathcal{I}_c(Y)$, $Q_M T \notin F_-$.

(III) *For each* $M \in \mathcal{I}_c(Y)$ *there exists* $N \in \mathcal{I}_c(\tilde{Y})$, $N \supset M$, *such that* $Q_N J_Y T$ *is nuclear (compact).*

(IV) *For each* $M \in \mathcal{I}_c(Y)$ *and each* $\varepsilon > 0$ *there exists* $N \in \mathcal{I}_c(\tilde{Y})$, $N \supset M$, *such that* $\|Q_N J_Y T\| < \varepsilon$.

Proof. The equivalence of (I) and (II) is an easy consequence of the definition. The equivalence of (II), (III) and (IV) is immediate from Theorem 1.1. ■

The following two results illustrate the close link between the properties $T \in F_-$ and $T \in SC$.

1.3 PROPOSITION. *The following are equivalent.*

(I) $T \in F_-(X, Y)$.

(II) *For each* $M \in \mathcal{I}_c(\tilde{Y})$, *there exists* $N \in \mathcal{I}_c(\tilde{Y})$, $N \supset M$, *such that* $Q_N J_Y T \notin SC$.

(III) *For each* $M \in \mathcal{I}_c(\tilde{Y})$, *there exists* $N \in \mathcal{I}_c(\tilde{Y})$, $N \supset M$, *such that* $Q_N J_Y T \in F_-$.

Proof. (I)$\Rightarrow$(III) Suppose there exists $F \in \mathcal{F}(Y)$ such that $(Q_F T)'$ has a continuous inverse. Let $M \in \mathcal{I}_c(\tilde{Y})$ be arbitrary. Then $M + F \in \mathcal{I}_c(\tilde{Y})$ and $(Q_{M+F} J_Y T)' = (J_Y T)' J_{(M+F)^\perp} = T' J_{M^\perp \cap F^\perp}$ is just a restriction of $(Q_F T)' = T' J_{F^\perp}$. Hence $(Q_{M+F} J_Y T)'$ also has a continuous inverse and so (III) follows.

(III)$\Rightarrow$(II) This is immediate from the definitions of F_- and strictly cosingular operators.

(II)$\Rightarrow$(I) Suppose $T \notin F_-$. By Theorem 1.1 there exists $M \in \mathcal{I}_c(\tilde{Y})$ such that $Q_M J_Y T$ is compact. Hence for any $N \in \mathcal{I}_c(\tilde{Y})$ with $N \supset M$, $Q_N J_Y T = Q_{N/M}^{\tilde{Y}/M}(Q_M^{\tilde{Y}} J_Y T)$ is still compact and hence strictly cosingular [7, III.2.5]. ■

1.4 PROPOSITION. *Let* Y *be complete. Then the following are equivalent.*

(I) $T \in SC$.

(II) *For each* $M \in \mathcal{I}_c(Y)$, *there exists* $N \in \mathcal{I}_c(Y)$, $N \supset M$, *such that* $Q_N T \in SC$.

(III) *For each* $M \in \mathcal{I}_c(Y)$, *there exists* $N \in \mathcal{I}_c(Y)$, $N \supset M$, *such that* $Q_N T \notin F_-$.

Proof. (I)$\Rightarrow$(II) This follows from Proposition 1.2 and the fact that compact operators are strictly cosingular.

(II)$\Rightarrow$(III) This is a consequence of the definitions of F_- and strictly cosingular operators.

(III)$\Rightarrow$(I) Suppose $T \notin SC$. Then there is some $M \in \mathcal{I}_c(Y)$ such that $(Q_M T)'$ has a continuous inverse. Now for any $N \in \mathcal{I}_c(Y)$ with $N \supset M$, $(Q_N T)' = T' J_{N^\perp}$ is just a

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restriction of $(Q_M T)' = T'J_{M^\perp}$ as $M^\perp \supset N^\perp$ and hence $(Q_N T)'$ has a continuous inverse; that is $Q_N T \in F_-$. We conclude that (I) follows from (III). ∎

Since ϕ_- operators agree with F_- operators in the classical case of closed operators between Banach spaces we note that in this case Propositions 1.3 and 1.4 can be formulated in terms of ϕ_- operators, thereby providing a characterisation of classical ϕ_- and strictly cosingular operators.

1.5 REMARK. From Proposition 1.3 we deduce that $T \in F_-(X, Y)$ if and only if $J_Y T \in F_-(X, \tilde{Y})$. Now let Y be complete. We then conclude from Proposition 1.2 that $T + S \in SC(X, Y)$ if $T, S \in SC(X, Y)$ and hence from Propositions 1.3 and 1.4 that $T + S \in F_-(X, Y)$ whenever $T \in F_-(X, Y)$, $S \in SC[X, Y]$ and Y is complete. By making use of Proposition 1.2 it may also be verified that, as in the classical case of bounded operators from one Banach space into another, $SC(X, Y)$, satisfies certain ideal properties. For example if $T \in SC(X, Y)$, $B \in B(Z, X)$ and $A \in B[Y, W]$, then $TB \in SC$ with $AT \in SC$ whenever Y is complete and T partially continuous. For a proof of this and related results the reader is referred to [**10**].

2. Partially continuous operators. We note from [**9**, 4.1, 5.2 and 6.2] that $S = Q_{^\perp D(T')}T$ is closable with $S'J_{D(S')} = T'J_{D(T')}$. Consequently considering [**7**, II.5.1] we see that there exists a normed space Z and a bijection $B \in B[Y/{}^\perp D(T'), Z]$ such that $Z' = Y'_{S'} = Y'_{T'}$ and $B' = G_{S'} \equiv G_{T'}$ with both B and BS continuous (the injectivity of B follows from the way it was defined in [**7**, II.5.1] and the fact that $D(S')$ is total [**7**, II.2.11]). Let $H_T = BQ_{^\perp D(T')}$. Then both H_T and $H_T T$ are continuous with $(H_T)' = J_{(^\perp D(T'))^\perp} \cdot G_{S'} = G_{T'}$ and $(H_T T)' = T'G_{T'}$.

We will make use of the operator H_T in order to characterise partial continuity of T in terms of closed infinite codimensional subspaces of Y. Consequently we first investigate the relationship between the property of partial continuity and the operator H_T.

2.1 PROPOSITION. *The following are equivalent.*

(I) *There exists $F \in \mathcal{F}(Y)$ such that $Q_F T$ is continuous.*

(II) *H_T is an open map with $a(H_T) < \infty$.*

Proof. (I)$\Rightarrow$(II) Suppose $Q_F T$ is continuous for some $F \in \mathcal{F}(Y)$. Then $(Q_F T)' = T'J_{F^\perp}$ is bounded [**7**, II.2.8] and $\operatorname{cod} F^\perp < \infty$ [**7**, I.6.4]. Therefore $D(T')$ contains a $\sigma(Y', Y)$-closed finite codimensional subspace of Y' and hence $D(T')$ is $\sigma(Y', Y)$-closed and finite codimensional in Y'. Hence $a(H_T) = \dim {}^\perp D(T') = \operatorname{cod}({}^\perp D(T'))^\perp = \operatorname{cod} D(T') < \infty$ with H_T an open map by [**8**, 9.6.4].

(II)$\Rightarrow$(I) If H_T is open, then so is $(\hat{H}_T)$ [**8**, 4.2.4]. Thus $(\hat{H}_T)^{-1}$ and hence $(\hat{H}_T)^{-1}(H_T T) = Q_{^\perp D(T')}T$ is continuous. Since $\dim {}^\perp D(T') = a(H_T) < \infty$, we are done. ∎

2.2 LEMMA [**6**]. *T is partially continuous if and only if there is some $F \in \mathcal{F}(\tilde{Y})$ such that $Q_F J_Y T$ is continuous.*

2.3 LEMMA [**4**, Theorem 38]. *$T \in F_+$ if and only if $T' \in \phi_-$.*

We note from the above and from [**7**, Theorems IV.1.2 and IV.2.3] that if T is closed and X and Y complete, then $T \in F_+$ if and only if $T \in \phi_+$.

2.4 LEMMA. *Let T be closed, X and Y complete, and* $\operatorname{cod} D(T) < \infty$. *Then $D(T)$ is closed and hence T is continuous.*

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Proof. Note that if T is closed and X and Y complete, then X_T is complete. If in addition $D(T)=R(G_T)$ is finite codimensional in X, it is closed by [**7**, IV.1.13] since G_T is a closed operator. The rest of the result now follows from the closed graph theorem. ∎

2.5 Theorem. *The following are equivalent.*
(I) *T is partially continuous.*
(II) *$D(T')$ is finite codimensional in Y'.*
(III) *$H_T \in F_+$.*

Proof. (I)⇒(II) Suppose T is partially continuous. By Lemma 2.2 there is some $F \in \mathscr{F}(\tilde{Y})$ such that $Q_F^{\tilde{Y}} J_Y T$ is continuous. Hence $(Q_F J_Y T)' = (J_Y T)' J_{F^\perp} = T' J_{F^\perp}$ is bounded [**7**, II.2.8]. Therefore $\operatorname{cod} D(T') < \infty$ since $F^\perp \subset D(T')$ and $\operatorname{cod} F^\perp = \dim F < \infty$.

(II)⇒(III) Supposing that $\operatorname{cod} D(T') < \infty$ we see from Lemma 2.4 that $D(T') = R(G_{T'}) = R((H_T)')$ is closed and hence $(H_T)' \in \phi_-$. Consequently $H_T \in F_+$ by Lemma 2.3.

(III)⇒(I) Suppose $H_T \in F_+$. Thus $(H_T)' \in \phi_-$ by Lemma 2.3 whence $\operatorname{cod} D(T') = \operatorname{cod} R((H_T)') < \infty$. As $(J_Y T)' = T'$ is therefore continuous by Lemma 2.4, it follows that $(H_{J_Y T})' = G_{T'} = (H_T)'$ is an isomorphism and hence that $H_{J_Y T}$ is an open map [**7**, Theorem II.4.3]. Observe that $R(H_{J_Y T})$ is complete as $H_{J_Y T}$ is both bounded and open whence $a(H_{J_Y T}) < \infty$ by [**7**, Theorem IV.2.3]. We now deduce from Proposition 2.1 and Lemma 2.2 that T is partially continuous. ∎

2.6 Corollary. *Let $\overline{D(T')}^*$ denote the $\sigma(Y', \tilde{Y})$-closure of $D(T')$. Then $\operatorname{cod}_{\overline{D(T')}^*} D(T') < \infty$ if and only if $\overline{D(T')}^* = F(T')$.*

Proof. Suppose $\operatorname{cod}_{\overline{D(T')}^*} D(T') < \infty$. Denote ${}^{\tilde{Y}\perp}D(T')$ by ${}^\perp D$. Considering $S = Q_{{}^\perp D}(J_Y T)$ we note that $S' = (J_Y T)' J_{({}^\perp D)^\perp} = T' J_{\overline{D(T')}^*}$ and hence S is partially continuous by Theorem 2.5. By Lemma 2.2 there is some $F \in \mathscr{F}(\tilde{Y}/{}^\perp D)$ such that $Q_F S$ is continuous. Now let $K = (Q^{\tilde{Y}}_{{}^\perp D})^{-1} F$, where $(Q^{\tilde{Y}}_{{}^\perp D})^{-1}$ is taken in the set theoretic sense. Then K is a closed subspace of $\tilde{Y}$ such that $Q_K(J_Y T) = Q_F S$ is continuous with $K \supset {}^\perp D$ and $\dim(K/{}^\perp D) < \infty$. Hence $(Q_K(J_Y T))' = T' J_{K^\perp}$ is bounded [**7**, II.2.8] with $\dim(\overline{D(T')}^*/K^\perp) < \infty$; that is $K^\perp$ is contained in $D(T')$ and is finite codimensional in $D(T')$. Thus as $K^\perp$ is $\sigma(Y', \tilde{Y})$-closed, so is $D(T')$. ∎

As in [**15**] we define a normed space Z to be *subprojective* if for each closed infinite dimensional subspace M of Z, there exists a closed infinite dimensional subspace N of M which is topologically complemented in Z. Considering such spaces we obtain the following characterisation of partial continuity (we note that this result as well as Corollary 2.18 are in a sense dual to [**3**, Theorem 4]).

2.7 Lemma. *Let M be a closed subspace of Y and let F be a finite dimensional subspace of $\tilde{Y}$ such that $\tilde{M} \oplus F = \tilde{Y}$. Then $M + F$ is closed in $Y + F$.*

Proof. Let P be a bounded projection defined on $\tilde{Y}$ with range $\tilde{M}$ and null space F. Suppose $M + F$ is not closed in $Y + F$. Then there exists a sequence $y_k = x_k + z_k$ where $x_k \in M$, $z_k \in F$ with z_k unbounded such that $y_k \to y + z\,(y \in Y,\ z \in F)$. Then $P(x_k + z_k) = x_k \to Py$. Consequently z_k converges, contradicting the unboundedness of z_k. ∎

2.8 Corollary. $M \in \mathscr{I}_c(Y) \Rightarrow \tilde{M} \in \mathscr{I}_c(\tilde{Y})$.

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Proof. Let $M \in \mathcal{I}_c(Y)$ and suppose that $\tilde{M} \oplus F = \tilde{Y}$, where $\dim F < \infty$. We have $\tilde{M} + F \subset (M+F)^{\sim} \subset \tilde{Y}$, so that $M+F$ is dense in $(Y+F)^{\sim} = \tilde{Y}$. Therefore $M+F$ is a dense subspace of $Y+F$. Hence by the Lemma $M \oplus F = Y + F$. Therefore

$$Y = (M+F) \cap Y = (M \oplus F) \cap Y = M \oplus (F \cap Y),$$

contradicting $M \in \mathcal{I}_c(Y)$. ■

2.9 THEOREM. *Let Y be subprojective. Then T is partially continuous if and only if for each $M \in \mathcal{I}_c(Y)$ there exists $N \in \mathcal{I}_c(\tilde{Y})$, $N \supset M$, such that $Q_N J_Y T$ is continuous.*

Proof. Suppose T is partially continuous. By Lemma 2.2 there is some $F \in \mathcal{F}(\tilde{Y})$ such that $Q_F J_Y T$ is continuous. Selecting $M \in \mathcal{I}_c(Y)$ arbitrarily we note that $\tilde{M} \in \mathcal{I}_c(\tilde{Y})$ by Corollary 2.8 and hence $N = \tilde{M} + F \in \mathcal{I}_c(\tilde{Y})$, by the finite dimensionality of F. Observing that $N \supset M$ and that $\|Q_N J_Y T\| \le \|Q_F J_Y T\|$ since $N \supset F$, the first part of the result follows.

Conversely suppose that T is not partially continuous. Hence $H_T \notin F_+$ by Theorem 2.5 and so by [**2**, 2.2] there exists $M \in \mathcal{I}(Y)$ such that $H_T J_M$ is precompact. Since H_T is continuous we may assume M to be a closed subspace of Y. Note that $(H_T J_M)' = Q_{M^\perp} . G_{T'}$ is compact [**7**, III.1.11]. As Y is subprojective there is some subspace $W \in \mathcal{I}(M)$ such that W is topologically complemented in Y by say K. Observe that since $W \subset M$, $H_T J_W$ and hence $Q_{W^\perp} . G_{T'}$ is still precompact. Furthermore letting P be the bounded projection from Y onto W with $N(P) = K$ we see that for any $y' \in Y'$, $y' = y'P + y'(I-P)$ with $y'P \in K^\perp$ and $y'(I-P) \in W^\perp$. Hence $Y' = W^\perp \oplus K^\perp$. Considering [**7**, II.1.14] and [**14**, V.7.29] it follows that $Y'/W^\perp$ is isomorphic to $K^\perp$ under the isomorphism $Q_{W^\perp} . J_{K^\perp}$. Consequently $(J_{K^\perp})^{-1} G_{T'} = (Q_{W^\perp} . J_{K^\perp})^{-1} Q_{W^\perp}|_{K^\perp} . G_{T'}$ is compact by the ideal property of compact operators. Observe that $\operatorname{cod} K = \dim W = \infty$. Finally suppose there is some $N \in \mathcal{I}_c(\tilde{Y})$, $N \supset K$, such that $Q_N J_Y T$ is continuous. Then $(Q_N J_Y T)' = T' J_{N^\perp}$ is bounded [**7**, II.2.8], and therefore $(G_{T'})^{-1} J_{N^\perp}$ is an isomorphism with $N^\perp \in \mathcal{I}(D(T'))$. This now leads to a contradiction as $N^\perp \subset K^\perp$ with $(J_{K^\perp})^{-1} G_{T'}$ a compact operator. Hence the result follows. ■

As an application of the above we obtain the following result.

2.10 COROLLARY. *Let Y be subprojective. Then $SC(X, Y) \subset PB(X, Y)$.*

Proof. Combine 1.2 with 2.9. ■

The following Corollary further illustrates the close link between the properties of partial continuity and strict cosingularity.

2.11 COROLLARY. *Let X and Y be any two non-identical spaces belonging to the class $\{c_0\} \cup \{l_p : 1 < p < \infty\}$ and let $T \in L(X, Y)$ be densely defined. Then T is strictly cosingular if and only if it is partially continuous.*

Proof. The fact that $T \in SC(X, Y)$ implies $T \in PB(X, Y)$ is immediate from Corollary 2.10 and the subprojectivity of Y [**15**, Theorem 3.2]. Hence assume T to be partially continuous. Selecting $M \in \mathcal{I}_c(Y)$ arbitrarily it follows that $D(T' J_{M^\perp}) = D(T') \cap M^\perp$ is infinite dimensional as $\dim M^\perp = \operatorname{cod} M = \infty$ [**7**, I.6.4] and $\operatorname{cod} D(T') < \infty$, by Theorem 2.5. As T' is continuous, by Lemma 2.4, it follows from [**11**] (the remark preceding Theorem 2.a.3) that $T' J_{M^\perp} = (Q_M T)'$ does not have a continuous inverse and hence that $T \in SC(X, Y)$. ■

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2.12 Corollary. *Let Y be subprojective. Then the following are equivalent.*

(I) *There exists $F \in \mathscr{F}(\tilde{Y})$ such that $(Q_F J_Y T)'$ is a bounded isomorphism.*

(II) *For each $M \in \mathscr{I}_c(\tilde{Y})$ there exists $N \in \mathscr{I}_c(\tilde{Y})$, $N \supset M$, such that $(Q_N J_Y T)'$ is a bounded isomorphism.*

(III) *T is partially continuous and there is no $M \in \mathscr{I}_c(\tilde{Y})$ such that $Q_M J_Y T$ is strictly cosingular (nuclear, compact).*

Proof. (I)$\Rightarrow$(II) Suppose there exists $F \in \mathscr{F}(\tilde{Y})$ such that $(Q_F J_Y T)'$ is a bounded isomorphism. Let $M \in \mathscr{I}_c(\tilde{Y})$ be arbitrary and set $N = M + F \in \mathscr{I}_c(\tilde{Y})$. Since $(Q_N J_Y T)'$ is then just a restriction of $(Q_F J_Y T)'$ we conclude that (II) follows from (I).

(II)$\Rightarrow$(III) Suppose (II) holds. For any $K \in \mathscr{I}_c(Y)$, $\tilde{K} \in \mathscr{I}_c(\tilde{Y})$ by Corollary 2.8 and so there exists $N \in \mathscr{I}_c(\tilde{Y})$, $N \supset \tilde{K} \supset K$, such that $(Q_K J_Y T)'$ is bounded. Considering [**7**, II.2.8.] and Theorem 2.9, we conclude that T is partially continuous. It follows from Proposition 1.3 that $T \in F_-$ and hence by Theorem 1.1 there is no $M \in \mathscr{I}_c(\tilde{Y})$ such that $Q_M J_Y T$ is nuclear (compact). The statement about strict cosingularity is an immediate consequence of the definition.

(III)$\Rightarrow$(I) Suppose (III) holds. By Proposition 1.3 and Lemma 2.2 there exists $F_1 \in \mathscr{F}(Y)$ and $F_2 \in \mathscr{F}(\tilde{Y})$ such that $(Q_{F_1} T)'$ has a continuous inverse and $Q_{F_2} J_Y T$ is continuous; that is $(Q_{F_2} J_Y T)'$ is bounded. Let $F = F_1 + F_2$. Then $(Q_F J_Y T)'$ is a restriction of both $(Q_{F_1} T)'$ and $(Q_{F_2} J_Y T)'$ and hence the result follows. ∎

In conclusion we investigate operators with continuous adjoint. Such operators feature prominently in, for example, the study of unbounded Tauberian operators where the continuity of the adjoint is a prerequisite for much of the theory (cf. [**1**]).

2.13 Proposition. *T' is continuous whenever $D(T)$ is finite codimensional in its completion.*

Proof. Let $D(T)$ be finite codimensional in its completion. Denote ${}^{\tilde{Y}\perp}D(T')$ by K. As before we note from [**9**] that $Q_K J_Y T$ is closable in $L(D(T)^{\sim}, \tilde{Y}/K)$ and hence continuous by Lemma 2.4. We conclude that $(Q_K J_Y T)' = T' J_{\overline{D(T')}^{*}}$ is bounded and hence that T' is continuous. ∎

2.14. Lemma. *$R(T')$ is closed if and only if it is $\sigma((D(T))', D(T)^{\sim})$-closed.*

Proof. As $(J_Y T)' = T'$ we may assume without loss of generality that Y is complete and $D(T) = X$. Let S be the operator $Q_{{}^{\perp}D(T')} T$ considered as an element of $L(\tilde{X}, Y/{}^{\perp}D(T'))$. By [**9**], S is closable and so, denoting the minimal closed extension of S by $\tilde{S}$, we note from [**7**, II.2.11] that $\tilde{S}' = S'$ and hence $R(\tilde{S}') = R(T')$, by [**9**, 6.2]. It now follows from [**7**, IV.1.2] that $R(T') = {}^{\perp}N(\tilde{S})$ whenever $R(T')$ is closed. Consequently $R(T')$ is $\sigma(X', \tilde{X})$-closed whenever it is closed. The converse is clear. ∎

2.15 Theorem. *The following are equivalent.*

(I) *T' is continuous.*

(II) *T' is partially continuous.*

(III) *$D(T')$ is closed.*

(IV) *$D(T')$ is $\sigma(Y', \tilde{Y})$-closed.*

(V) *$Q_K J_Y T$ is continuous, where $K = {}^{\tilde{Y}\perp}D(T')$.*

(VI) *For any $M \in \mathscr{I}_c(D(T)^{\sim})$ there is no injective restriction T_1' of T' such that $(T_1')^{-1} J_{M^{\perp}}$ is a bounded nuclear (compact) operator.*

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(VII) *There is some* $\varepsilon > 0$ *such that for any* $M \in \mathcal{I}_c(D(T)^{\sim})$ *there is no injective restriction* T_1' *of* T' *for which* $(T_1')^{-1}J_{M^\perp}$ *is a bounded operator with norm not exceeding* ε.

(VIII) *For every* $M \in \mathcal{I}_c(D(T)^{\sim})$ *there exists* $N \in \mathcal{I}_c(D(T)^{\sim})$, $N \supset M$, *such that* $(J_{N^\perp})^{-1}T'$ *is continuous.*

Proof. (I)$\Leftrightarrow$(II)$\Leftrightarrow$(III) This follows from [**7**, II.2.15] and [**3**, Corollary 11].

(III)$\Leftrightarrow$(IV) Observe that $D(T') = R(G_{T'}) = R((H_T)')$. The equivalence of (III) and (IV) now follows from Lemma 2.14.

(IV)$\Leftrightarrow$(V) Letting $K = {}^{Y\perp}D(T')$ we observe that $(Q_K J_Y T)' = (J_Y T)' J_{\overline{D(T')}^{\bullet}} = T' J_{\overline{D(T')}^{\bullet}}$. The equivalence of (IV) and (V) now follows from [**7**, II.2.8].

(II)$\Leftrightarrow$(VI) First suppose that T' is not partially continuous. Inductively define a sequence of integers $\{a_n\}$ as follows:

$$a_1 = 2, \qquad a_n = 2\left(1 + \sum_{k=1}^{n-1} a_k\right)(n = 2, 3, \ldots).$$

Let $\varepsilon > 0$ be arbitrary. Since T' is not partially continuous, there is some $y_1' \in D(T')$ with $\|y_1'\| \le \varepsilon/4$ and $\|T'y_1'\| = 1$. Select $x_1 \in D(T)$ such that $T'y_1'x_1 = 1$ and $\|x_1\| \le 2$. Suppose that $x_1, x_2, \ldots, x_{n-1}$ and $y_1', y_2', \ldots, y_{n-1}'$ have been found in $D(T)$ and $D(T')$ respectively, such that

$$\begin{gathered} \|y_k'\| \le \varepsilon/(2^k a_k), \qquad \|T'y_k'\| = 1, \qquad \|x_k\| \le a_k, \\ T'y_k'x_j = \delta_{kj} \quad \text{for} \quad 1 \le k, j \le n-1. \end{gathered} \tag{1}$$

Let $F = \operatorname{span}\{Tx_1, Tx_2, \ldots, Tx_{n-1}\}$. Then $\operatorname{cod} F^\perp = \dim F < \infty$ and hence $T'J_{F^\perp}$ is not continuous. Consequently there exists $y_n' \in \{Tx_1, Tx_2, \ldots, Tx_{n-1}\}^\perp \cap D(T')$ such that $\|y_n'\| \le \varepsilon/(2^n a_n)$ and $\|T'y_n'\| = 1$. Select $x \in D(T)$ so that $T'y_n'x = 1$ and $\|x\| \le 2$. Let

$$x_n = x - \sum_{k=1}^{n-1} (T'y_k'x)x_k.$$

Then $T'y_k'x_j = \delta_{kj}$ for $1 \le k, j \le n$ and $\|x_n\| \le \|x\|\left(1 + \sum_{k=1}^{n-1} \|x_k\|\right) \le a_n$. Hence by induction we may construct $\{x_n\} \subset D(T)$ and $\{y_n'\} \subset D(T')$ such that (1) is satisfied for all $n \in \mathbb{N}$. Now define a nuclear (compact) operator $B \in L[Y, D(T)^{\sim}]$ as follows:

$$By = \sum_{k=1}^{\infty} y_k'(y)x_k \quad \text{for each} \quad y \in Y.$$

For each $k \in \mathbb{N}$ we now have that

$$T'y_k'(BTx) = y_k'Tx = T'y_k'x$$

and hence each $T'y_k'$ annihilates $\overline{R(BT - J_D)} = M$, where J_D denotes the injection of $D(T)$ into its completion. From (1) we conclude that M is infinite codimensional in $D(T)^{\sim}$. Since $M^\perp = N((BT - J_D)')$, by [**7**], we have that $(BT - J_D)' = (BT)' - I = T'B' - I = 0$ everywhere on $M^\perp$ (I denotes the identity on $D(T)'$). Therefore $T'B'J_{M^\perp} = J_{M^\perp}$ and consequently there is some injective restriction T_1' of T' such that $(T_1')^{-1}J_{M^\perp}$ agrees with the nuclear (compact) operator $B'J_{M^\perp}$. The converse is a consequence of the fact that one cannot have a nuclear (compact) isomorphism on an infinite dimensional space.

(VII)$\Rightarrow$(II) This follows from the fact that the operator B constructed above has norm not exceeding ε.

(I)$\Rightarrow$(VII) Suppose T' is continuous and that (VII) is false. Hence we may select $M \in \mathcal{I}_c(D(T)^{\sim})$ such that, for some injective restriction T_1' of T', $(T_1')^{-1}J_{M^\perp}$ is bounded with norm less than $(2\,\|T'\|)^{-1}$. For any $y' \in R((T_1')^{-1}J_{M^\perp})$ we then have that

$$2\,\|T'\|\,\|y'\| = 2\,\|T'\| \,.\, (\|(T_1')^{-1}J_{M^\perp} \,.\, T'y'\|) < \|T'y'\| \leq \|T'\|\,\|y'\|;$$

an obvious contradiction. The result follows.

(I)$\Leftrightarrow$(VIII) The converse being trivial, suppose that T' is not continuous. Hence there exists $M \in \mathcal{I}_c(D(T)^{\sim})$ such that $(T_1')^{-1}J_{M^\perp}$ is a bounded compact operator where T_1' is some injective restriction of T'. Clearly there can be no $N \in \mathcal{I}_c(D(T)^{\sim})$, $N \supset M$, such that $(J_{N^\perp})^{-1}T'$ and hence $(J_{N^\perp})^{-1}T_1'$ is continuous (note that $N^\perp \in \mathcal{I}(M^\perp)$). ■

The following Corollary generalises [**7**, II.4.8].

2.16 Corollary. *Let Y be complete and T closable. Then the following are equivalent.*

(I) *T' is continuous ($D(T')$ is closed).*

(II) *T' is bounded ($D(T') = Y'$).*

(III) *T is continuous.*

Proof. The equivalence of (II) and (III) is a consequence of [**7**, II.2.8]. As T is closable, $D(T')$ is total [**7**, II.2.11]; that is ${}^\perp D(T') = \{0\}$. The equivalence of (I) an (III) now follows from Theorem 2.15. ■

Defining quasi-complementation as in [**12**], we obtain the following result.

2.17 Proposition. *Let Y be either separable (more generally let ${}^\perp D(T')$ be quasi-complemented) or reflexive and let T be such that for each $M \in \mathcal{I}_c(Y)$ there exists $N \in \mathcal{I}_c(Y)$, $N \supset M$, with $Q_N T$ closable. Then the $\sigma(Y', Y)$-closure of $D(T')$ is finite codimensional in Y' (or equivalently* $\dim {}^\perp D(T') < \infty$ [**7**, I.6.4]).

Proof. Suppose that $\dim {}^\perp D(T') = \infty$. In the case where Y is separable we select a quasi-complement of ${}^\perp D(T')$, say M [**12**]. Then $M \in \mathcal{I}_c(Y)$ with $M^\perp \cap ({}^\perp D(T'))^\perp = (M \oplus {}^\perp D(T'))^\perp = \{0\}$; that is $M^\perp \cap D(T') = \{0\}$. Alternatively if Y is reflexive, we select a linearly independent sequence $\{y_n'\} \subset Y'$ such that for any $n \in \mathbb{N}$, $y_n' \notin ({}^\perp D(T'))^\perp$ (observe that $\dim {}^\perp D(T') = \mathrm{cod}({}^\perp D(T'))^\perp = \infty$). Note that $\overline{\mathrm{span}}\{y_1', y_2', \ldots\}$ is separable and hence let E be a quasi-complement of $\overline{\mathrm{span}}\{y_1', y_2', \ldots\} \cap ({}^\perp D(T'))^\perp$ in $\overline{\mathrm{span}}\{y_1', y_2', \ldots\}$ [**12**]. Then $E \in \mathcal{I}(Y')$ with E $\sigma(Y', Y)$-closed, by the reflexivity of Y'. Hence letting $M = {}^\perp E$, we note that $\mathrm{cod}\, M = \dim({}^\perp E)^\perp = \dim E = \infty$ with $M^\perp \cap ({}^\perp D(T'))^\perp = \{0\}$; that is $M^\perp \cap D(T') = \{0\}$. Observe that in each case we obtain $M \in \mathcal{I}_c(Y)$ such that $M^\perp \cap D(T') = \{0\}$. Suppose there exists $N \in \mathcal{I}_c(Y)$, $N \supset M$, such that $Q_N T$ is closable. Then $D((Q_N T)') = D(T'J_{N^\perp})$ is $\sigma(Y', Y)$-dense in $N^\perp \equiv (Y/N)'$ by [**7**, II.2.11]. This gives a contradiction since $D(T'J_{N^\perp}) \subset D(T') \cap M^\perp = \{0\}$. ■

By considering [**7**, II.2.11], we see that the property of having the $\sigma(Y', Y)$-closure of $D(T')$ finite codimensional in Y' is closely related to T being closable. Thus Proposition 2.17 provides a sufficient condition for T to be "almost closable".

2.18 Corollary. *Let Y be either a separable or a reflexive Banach space and let T' be continuous. Then T is partially continuous if and only if for every $M \in \mathcal{I}_c(Y)$ there exists $N \in \mathcal{I}_c(Y)$, $N \supset M$, such that $Q_N T$ is continuous.*

Proof. Suppose that for each $M \in \mathcal{I}_c(Y)$ there exists $N \in \mathcal{I}_c(Y)$, $N \supset M$, such that $Q_N T$ is continuous. By Proposition 2.17 ${}^{\perp}D(T') \in \mathcal{F}(Y)$. Since T' is continuous, we note from Theorem 2.15 that $Q_{{}^{\perp}D(T')}T$ is continuous. Hence T is partially continuous by Lemma 2.2. The converse follows as in Theorem 2.9. ∎

2.19 Corollary. *Let X and Y be Banach spaces with Y either separable or reflexive. Then $SC[X, Y] \subset PB[X, Y]$.*

Proof. This is an easy consequence of Corollary 2.18 and Propositions 2.17 and 1.2. ∎

2.20 Example. *There exists an unbounded partially continuous strictly cosingular operator.* Consider, for example, any unbounded finite rank operator. Less trivially we may construct the required operator as follows. Let $A \in SC[X, Y]$ be an arbitrary bounded strictly cosingular operator and let E be a dense subspace of X of codimension 1 (the kernel of a discontinuous linear functional). Select $x_0 \in X$ such that $x_0 \notin E$ and define $T \in L[X, Y]$ as follows: $TJ_E = AJ_E$ with either $Tx_0 = 0$ if $Ax_0 \neq 0$, or $Tx_0 = y_0$ if $Ax_0 = 0$, where y_0 is some arbitrarily chosen non-zero element of Y. It is clear that T is partially continuous with x_0 being a point of discontinuity. It remains to verify that T is in fact strictly cosingular. Supposing the contrary it follows that there exists $M \in \mathcal{I}_c(Y)$ such that $T'J_{M^\perp}$ has a continuous inverse. Letting $F = \operatorname{span}\{Tx_0, Ax_0\}$ we conclude that $M + F \in \mathcal{I}_c(Y)$ with $(Q_{M+F}T)' = T'J_{M^\perp \cap F^\perp}$ having a continuous inverse. However this is a contradiction as $Q_{M+F}T$ agrees with $Q_{M+F}A$. Consequently $T \in SC[X, Y]$. ∎

The following example gives more insight into unbounded strictly cosingular operators.

2.21 Example. *We construct an unbounded strictly cosingular operator in the class $L(c_0, l_\infty)$.* Let $D(T)$ be the span of the unit vectors

$$e_k = (0, 0, \ldots, 1, 0, 0, \ldots)$$

in c_0. Now define $T: D(T) \subset c_0 \to l_\infty$ as follows:

$$T(x_1, x_2, \ldots, x_n, 0, 0, \ldots) = \left(\sum_{k=1}^{n} kx_k, x_2, x_3, \ldots, x_n, 0, 0, \ldots \right).$$

Clearly T is unbounded. Next suppose there exists $M \in \mathcal{I}_c(Y)$ such that $(Q_M T)' = T'J_{M^\perp}$ has a continuous inverse. Denoting $\operatorname{span}\{e_1\}$ by F it follows that $M + F \in \mathcal{I}_c(Y)$ and that $(Q_{M+F}T)' = T'J_{M^\perp \cap F^\perp}$ has a continuous inverse. Next observe that $Q_{M+F}T$ is continuous as $M + F \supset F$ and that $(\overline{Q_{M+F}T})' = (Q_{M+F}T)'$. However this leads to a contradiction since $\overline{Q_{M+F}T}$ agrees with $Q_{M+F}J$, where J denotes the canonical injection of c_0 into l_∞ and it is known that J is strictly cosingular [**13**, Example 2]. Hence T is strictly cosingular. (Note that, by Lemma 2.2, T is in fact partially continuous as $Q_F T$ is continuous.) ∎

It is an open problem whether the characterisation of partial continuity given in Theorem 2.9 holds for arbitrary normed linear spaces Y. An affirmative answer to this would settle another open problem; that is whether there exists a non-partially continuous strictly cosingular operator.

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