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ANALYTICITY AND QUASI-BANACH VALUED FUNCTIONS

ANTONIO BERNAL AND JOAN CERDÀ

We compare the definitions of analyticity of vector-valued functions and their connections with the topological tensor products of non-locally convex spaces.

1. INTRODUCTION

In this paper we consider quasi-Banach valued functions that are defined on an open subset of the complex plane.

For such a function, there are several definitions for the concept of analyticity. Complex differentiability allows as analytic some functions that are too pathological (see [**1**, **4**, **12**]). Other definitions have been considered by several authors in [**1**, **4**, **5**, **7**, **8**, **11**, **12**], and the proof of the equivalence of these definitions is more or less implicit in these papers.

Our purpose is to give a unified presentation of the basic theory of analytic functions with values in quasi-Banach spaces and to prove the above mentioned equivalences in full detail.

Section 2 contains the preliminary definitions, notation and results on analyticity. In Section 3 we prove that the uniform limits of finite rank analytic functions have power series expansions. This fact has been shown by Kalton in [**8**] but here we give another proof that gives a more elementary appearance to the theorem.

In Sections 4 and 5 we complete the chain of equivalences for the different definitions of analyticity and we observe that the concept of ultra-uniform convergence on compact sets introduced by Etter in [**4**], which is known to be strictly stronger than the uniform convergence on compact sets, is equivalent to it when we consider analytic functions.

The proof of the result of Section 5 uses some facts about the theory of non locally convex tensor products as developed in the papers [**2**, **5**, **6**, **7**, **13**, **14**, **15**, **16**].

Throughout the paper G represents a domain of the complex plane and $(X, \|\cdot\|)$ a complex quasi-Banach space. We refer to [**2**] and [**10**] for precise definitions and basic properties of quasi-Banach spaces. Throughout the paper we shall assume all the quasi-norms to be p-norms for a certain p, $0 < p \leqslant 1$. This can be done since a theorem of

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Aoki and Rolewicz asserts that every quasi-norm on a linear space is equivalent to a p-norm with suitable p, $0 < p \leqslant 1$.

We also make free use of the notations on the non locally convex tensor products $X\widehat{\otimes}_p Y$ and $X\widehat{\otimes}_\varepsilon Y$, Y being a locally convex space with generating system of seminorms $\{\|\cdot\|_\alpha\}_\alpha$ and basis of zero-neighbourhoods $\{U_\beta\}_\beta$.

Recall that $X\widehat{\otimes}_p Y$ stands for the completion of the tensor product $X \otimes Y$ with respect to the quasi-seminorms

$$\|\cdot\| \otimes_p \|\cdot\|_\alpha (Z) = \inf \left(\sum \|x_n\|^p \, \|y_n\|_\alpha^p\right)^{1/p},$$

where the infimum is taken over all finite representations

$$Z = \sum x_n \otimes y_n.$$

Also, $X\widehat{\otimes}_\varepsilon Y$ is the completion of $X \otimes Y$ with respect to the quasi-seminorms

$$q_\beta(Z) = \sup \|(I_X \otimes \omega)(Z)\|,$$

the supremum being taken over all the continuous linear functionals $\omega \in Y'$ which are in the polar of U_β.

2. Definitions of analyticity

In this section we give several definitions for the concept of analyticity and we prove some relations between them.

Let f_n be a sequence of X-valued functions defined on G. In [4], f_n is said to be *ultra-uniformly convergent* to $f: G \to X$ over a set $A \subset G$ if to each zero-neighbourhood W on X there corresponds an index n_0 so that

$$co[(f_n - f)(A)] \subset W$$

whenever $n \geqslant n_0$. Here "co" denotes the convex envelope of the set.

In [4], a function $f: G \to X$ is said to be of class A if there is a sequence of finite rank analytic functions which is ultra-uniformly convergent to f over each compact subset of G.

From this definition an integration theory is developed in [4] such that the Cauchy integral formula holds for functions of class A. However, this integration theory gives no mean value inequality and, if γ is a rectifiable curve in G, no estimate for the quasi-norm of $\int_\gamma f(z)dz$ can be obtained from the supremum of $\|f(z)\|$ over γ.

The fundamental property of functions of class A is the following *factorisation theorem* of [4].

THEOREM 2.1. *A function $f\colon G \to X$ is of class A if and only if for each $z \in G$ there is a disk $D(z) \subset G$ centred at z, a Banach space L_z, a linear operator $T_z\colon L_z \to X$ and a Banach-valued analytic function*

$$f_z\colon D(z) \to L_z$$

such that

$$f\,|_{D(z)} = T_z \circ f_z.$$

In [4] it is also proved that, if $z_0 \in G$, any function of class A in G has a power series expansion which is ultra-uniformly convergent on any closed disk $\overline{D}(z_0) \subset G$. From this fact it is natural to make the following definition:

A function $f\colon G \to X$ will be said to be *analytic* when for each z_0 there is a disk $D(z_0) \subset G$ centred at z_0 and a power series expansion

$$f(z) = \sum_{n=0}^{\infty} (z - z_0)^n a_n, \quad a_n \in X, \tag{1}$$

which is pointwise convergent on $D(z_0)$.

Thus, any function of class A is analytic.

Since X is a quasi-Banach space, the usual arguments work to prove that the above series is uniformly convergent on compact subsets of $D(z_0)$. We will see that the series (1) is ultra-uniformly convergent on compact sets and that the class A and the class of all analytic functions are the same.

Another definition has been given by Gramsch in [5]. For the moment we will call *locally holomorphic* a function $f\colon G \to X$ if *locally belongs to* $X\widehat{\otimes}_p H^\infty$. More explicitly: if for each $z_0 \in G$ there is an open set $U(z_0) \subset G$ containing z_0, vectors $x_k \in X$ and bounded scalar-valued analytic functions $f_k\colon U(z_0) \to \mathbb{C}$ such that

$$\sum_{k=1}^{\infty} \|x_k\|^p \, \|f_k\|_\infty^p < +\infty \tag{2}$$

and

$$f(z) = \sum_{k=1}^{\infty} f_k(z) x_k \tag{3}$$

in $U(z_0)$. We remark that (2) implies the convergence of (3). At this moment we are not saying that the completed tensor products $X\widehat{\otimes}_p H^\infty(U(z_0))$ are function spaces.

We observe that analytic functions, locally holomorphic functions and functions of class A are the same:

Any function of class A is analytic as we have seen. But analytic functions with values in very general spaces have been studied by Turpin in [**12**] and in the quasi-normed case the factorisation theorem is proved for such functions (the disk $D(z)$ of Theorem 2.1 can be replaced by any open set U so that $\overline{U}$ is a compact subset of G) [**12**, Theorem 9.3.2]. Thus, by Theorem 2.1, any analytic function is of class A. Moreover, taking the power series expansions, it follows that analytic functions are locally holomorphic and in [**5**] the reverse implication has been proved.

PROPOSITION 2.1. *For a function $f : G \to X$ the following properties are equivalent:*

(i) *f is of class A.*
(ii) *f is analytic.*
(iii) *f is locally holomorphic.*

The factorisation property proved by Turpin for analytic functions allows us to prove local properties of analytic functions from the Banach-valued case. For example, it follows that analytic functions are C^∞ and the relations

$$a_k = \frac{1}{k!} f^{(k)}(z_0)$$

hold for the coefficients of the power series (1), and the power series of f about z_0 converges on any disk $D(z_0) \subset G$ centred at z_0.

We remark that in the definition of local holomorphic functions we can use any $p (0 < p \leqslant 1)$ so that X is p-Banach, because it follows from the above proposition that the class of all locally holomorphic functions is independent of p.

We introduce two further definitions:

We shall say that f is *globally holomorphic* if there is a sequence $f_k : G \to \mathbf{C}$ of scalar-valued analytic functions and a sequence of vectors $x_k \in X$ such that, if $K \subset G$ is compact, then

$$\sum_{k=1}^{\infty} \|x_k\|^p \|f_k\|_K^p < +\infty$$

and

$$f(z) = \sum_{k=1}^{\infty} f_k(z) x_k$$

over G. Here $\|f_k\|_K$ stands for the maximum of $|f_k(z)|$ over K. This definition has been considered in [**7**].

We shall call f *holomorphic* if there is a sequence of analytic functions with finite rank $f_n : G \to X$ uniformly convergent on compact sets of G to f. This definition is due to Peetre in [**11**].

Obviously, any globally holomorphic function is analytic and any analytic function is holomorphic. We shall prove the converse of these two properties.

3. Analyticity of holomorphic functions

In [8, Theorem 6.3] Kalton considers the space $A_0(X)$ of all continuous functions $f\colon \overline{\Delta} \to X$, Δ being the unit disk, which are analytic on Δ, and shows that $A_0(X)$ is complete with respect to the quasi-norm

$$\|f\|_\infty = \max_{|z|\leqslant 1} \|f(z)\|.$$

Thus the uniform limit of a sequence of analytic functions is analytic and any holomorphic function is analytic.

The proof of the completeness of $A_0(X)$ given in [8] is connected with the atomic theory of H^p. It uses a result by Coifman and Rochberg [3] on Bergman spaces.

We shall give another proof which uses a less involved concept of complex analysis: the maximum principle. Since the maximum principle is not valid in general quasi-Banach spaces (see [1] and [9]) we shall use the generalisation of the maximum principle given in [9].

The proof of this generalisation also uses the above mentioned techniques and so our proof is only apparently more elementary than that of [8]. Nevertheless, the class of all quasi-Banach spaces for which the maximum principle is valid contains important examples such as L^p and H^p. These spaces have been called "locally analytically pseudoconvex" by Peetre in [11], "locally holomorphic" by Aleksandrov in [1] and "A-convex" by Kalton in [9]. For an A-convex space our proof is actually more elementary than that of [9].

Recall that X is *A-convex* if it has an equivalent pluri-subharmonic quasi-norm. This property is characterised by the existence of a constant C such that, if f is an X-valued analytic function on Δ which is continuous up to the boundary, then

$$\|f(0)\| \leqslant C \sup_{|z|=1} \|f(z)\|. \tag{4}$$

If X is A-convex, we take an equivalent p-norm $|\cdot|$ and $(X, |\cdot|)$ is a p-Banach space satisfying (4). According to [9, Theorem 3.7], we can suppose that the p-norm is pluri-subharmonic and in (4) we have $C = 1$.

Lemma 3.1. *Let $g\colon \overline{\Delta} \to X$ be continuous on $\overline{\Delta}$ and holomorphic on Δ. Let r satisfy $0 < r < 1$. Then there is a constant $C = C(r, X)$ with*

$$\|f(0)\| \leqslant C \sup_{r\leqslant |z|\leqslant 1} \|g(z)\|.$$

PROOF: If g is analytic, this is [9, Theorem 5.2]. If not, we just make a limiting process approximating f on $\overline{\Delta}$ by finite rank functions which are analytic on a neighbourhood of $\overline{\Delta}$. □

We remark that the last case of the proof will never happen because we shall prove that any holomorphic function is analytic.

Now take a holomorphic function $f\colon G \to X$ and a sequence $f_n\colon G \to X$ of analytic functions with finite rank uniformly convergent to f over the compact sets of G. Fix $z_0 \in G$ and $r_0 > 0$ so that $\overline{D}(z_0, r_0) \subset G$. We have the power series for the functions f_n

$$f_n(z) = \sum_{k=0}^{\infty} (z - z_0)^k a_k^{(n)}.$$

LEMMA 3.2. *For each integer $k \geqslant 0$ the sequence $\{a_k^{(n)}\}_n$ is convergent.*

PROOF: The result is obvious for $k = 0$. Suppose that

$$a_k^{(n)} \to a_k$$

as $n \to \infty$ if $0 \leqslant k < m$. If $z \in \overline{D}(z_0, r_0) \setminus \{z_0\}$ we write

$$\frac{f_n(z)}{(z - z_0)^m} = \frac{a_0^{(n)}}{(z - z_0)^m} + \frac{a_1^{(n)}}{(z - z_0)^{m-1}} + \cdots + \frac{a_{m-1}^{(n)}}{z - z_0} + a_m^{(n)} + (z - z_0)a_{m+1}^{(n)} + \cdots .$$

We know that the power series $\sum_{\nu=0}^{\infty} (z - z_0)^\nu a_{m+\nu}^{(n)}$ are pointwise convergent on a neighbourhood of $\overline{D}(z_0, r_0)$ and with the usual arguments it follows that they are uniformly convergent on $\overline{D}(z_0, r_0)$. Let us call h_{nm} the functions defined on $\overline{D}(z_0, r_0)$ by the above power series. They are holomorphic on $D(z_0, r_0)$. For $z \in \overline{D}(z_0, r_0) \setminus \{z_0\}$ the following identity holds:

$$h_{nm}(z) = \frac{f_n(z)}{(z - z_0)^m} - \frac{a_0^{(n)}}{(z - z_0)^m} - \cdots - \frac{a_{m-1}^{(n)}}{z - z_0}.$$

By the inductive hypothesis $h_{nm}(z) \to g_m(z)$ pointwise on $\overline{D}(z_0, r_0) \setminus \{z_0\}$, where g_m is the function defined on $\overline{D}(z_0, r_0) \setminus \{z_0\}$ by

$$g_m(z) = \frac{f(z)}{(z - z_0)^m} - \frac{a_0}{(z - z_0)^m} - \cdots - \frac{a_{m-1}}{z - z_0}.$$

Moreover, if $0 < r < r_0$, the convergence is uniform on the ring of centre z_0 and radii r and r_0:

$$\|h_{nm} - g_m(z)\|^p \leqslant \frac{1}{r^{mp}} \|f_n(z) - f(z)\|^p + \cdots + \frac{1}{r^p} \left\|a_{m-1}^{(n)} - a_{m-1}\right\|^p .$$

In particular, the sequence $\{h_{nm}\}_n$ is uniformly Cauchy on the ring of radii r and r_0. From this and from Lemma 3.1 it follows that $a_m^{(n)} = h_{nm}(z_0)$ is convergent in X as $n \to \infty$. This proves the lemma. □

We now have a candidate for power series of f about z_0, formally

$$f(z) = \sum_{\nu=0}^{\infty} (z - z_0)^{\nu} a_{\nu} \tag{5}$$

where, according to the above notation, $a_\nu = \lim_{n\to\infty} a_\nu^{(n)}$. Before proving that the series (5) converges to $f(z)$ on a neighbourhood of z_0, we state a technical lemma.

LEMMA 3.3. *The functions h_m defined on $\overline{D}(z_0, r_0)$ by*

$$h_m(z) = \begin{cases} g_m(z) & \text{if } z \neq z_0 \\ a_m & \text{otherwise} \end{cases}$$

are holomorphic.

PROOF: Since h_{nm} are analytic on $D(z_0, r_0)$, it is enough to prove that $h_{nm} \to h_m$ uniformly on $\overline{D}(z_0, r_0)$.

As we have seen in the proof of Lemma 3.2, $h_{nm} \to h_m$ uniformly on the ring of centre z_0 and radii r_0 and $r_0/2$.

From Lemma 3.1 plus a conformal mapping argument it follows that, if $h\colon \overline{D}(z_0, r_0) \to X$ is continuous and holomorphic on $D(z_0, r_0)$, there is a constant C and a radius r, $0 < r < r_0$, so that

$$\sup_{|z-z_0|\leqslant r_0/2} \|h(z)\| \leqslant C \cdot \sup_{r\leqslant|z-z_0|\leqslant r_0} \|h(z)\|.$$

From this observation it follows that the sequence $\{h_{nm}\}_n$ is uniformly Cauchy on the disk $\overline{D}(z_0, r_0/2)$, but $h_{nm} \to h_m$ pointwise on $D(z_0, r_0)$. □

We can now state the main result.

THEOREM 3.1. *The series (5) is convergent to $f(z)$ for all z in a certain neighbourhood of z_0. Thus (5) is also convergent on any disk contained in G centred at z_0 and any holomorphic function is analytic.*

PROOF: If $r_0/2 \leqslant |z - z_0| \leqslant r_0$, it follows that

$$h_{nm}(z) - h_m(z) = \frac{f_n(z) - f(z)}{(z-z_0)^m} - \frac{a_0^{(n)} - a_0}{(z-z_0)^m} - \frac{a_1^{(n)} - a_1}{(z-z_0)^{m-1}} - \cdots - \frac{a_{m-1}^{(n)} - a_{m-1}}{z - z_0}.$$

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From this we have

$$\left\|\left(\frac{r_0}{2}\right)^m [h_{nm}(z) - h_m(z)]\right\|^p \leqslant \|f_n(z) - f(z)\|^p + \left\|a_0^{(n)} - a_0\right\|^p + \left\|\frac{r_0}{2}\left(a_1^{(n)} - a_1\right)\right\|^p + \cdots + \left\|\left(\frac{r_0}{2}\right)^{m-1}\left(a_{m-1}^{(n)} - a_{m-1}\right)\right\|^p$$

and, applying Lemma 3.1,

$$\left\|\left(\frac{r_0}{2}\right)^m \left(a_m^{(n)} - a^m\right)\right\|^p \leqslant C\left(\|f_n - f\|^p + \left\|a_0^{(n)} - a_0\right\|^p + \cdots + \left\|\left(\frac{r_0}{2}\right)^{m-1}\left(a_{m-1}^{(n)} - a_{m-1}\right)\right\|^p\right)$$

where $\|f_n - f\|$ stands for the supremum of $\|f_n(z) - f(z)\|$ over the disk $\overline{D}(z_0, r_0)$.

From the last inequality it follows by an inductive argument:

$$\|f_n - f\|^p + \left\|a_0^{(n)} - a_0\right\|^p + \cdots + \left\|\left(\frac{r_0}{2}\right)^{m-1}\left(a_{m-1}^{(n)} - a_{m-1}\right)\right\|^p \leq (1 + C)^{m-1}\left(\|f_n - f\|^p + \left\|a_0^{(n)} - a_0\right\|^p\right)$$

and we arrive at

$$\left\|a_m^{(n)} - a_m\right\| \leqslant C^m \|f_n - f\|,$$

where the constant C is independent of m and n.

The argument up to this point could have been used to obtain estimates for the Taylor coefficients of analytic functions, but the estimates in [8, Theorem 6.1] are more precise.

Now pick z_1 such that $|z_1 - z_0| < (2C)^{-1}$. It follows that

$$\sum_{m=0}^{\infty} C^{mp} \|f_n - f\|^p |z_1 - z_0|^{pm} < +\infty,$$

and so

$$\sum_{m=0}^{\infty} \left\|(z_1 - z_0)^m\left(a_m^{(n)} - a_m\right)\right\|^p \to 0$$

as $n \to \infty$. We have proved that the series $\sum_{m=0}^{\infty} (z_1 - z_0)^m a_m$ has non null radius of convergence. Moreover, we know that

$$\|f_n(z_1) - f(z_1)\| \to 0$$

and

$$\left\|f_n(z_1) - \sum_{m=0}^{\infty} (z_1 - z_0)^m a_m\right\| \to 0,$$

as $n \to \infty$. This completes the proof of the theorem. $\square$

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4. Convergence and ultra-convergence

In [5] an example is given of an l^p-valued function, with $p = 1/3$, that can be uniformly approximated by continuous functions with finite rank but fails to be integrable. Since ultra-uniform convergence preserves integrability of quasi-Banach valued functions as was proved in [4], it follows that the ultra-uniform convergence is a concept strictly stronger than uniform convergence. We prove next that, if we consider only analytic functions, both concepts are essentially equivalent.

If $\{K_n\}_n$ is an exhaustive sequence of compact sets of G and $f\colon G \to X$ is analytic, Etter [4] proves that the supremum

$$u_n(f) = \sup\{|\xi| : \xi \in \Gamma(f(K_n))\}$$

is finite, where $\Gamma(A)$ stands for the set of all finite combinations $\sum \lambda_i a_i$, with $a_i \in A$, $\lambda_i \in \mathbb{C}$ and $\sum |\lambda_i| \leqslant 1$. For a sequence of functions $\{f_\nu\}_\nu$ in $\mathcal{H}(G, X)$ the statements "$u_n(f_\nu) \to 0$ as $\nu \to \infty$ for each n", and "$\{f_\nu\}_\nu$ is ultra-uniformly convergent to zero over each compact subset of G" are equivalent. The functionals u_n are p-norms that generate on $\mathcal{H}(G, X)$ a locally p-convex metrisable topology whose convergence is the ultra-uniform convergence on compact subsets of G.

Proposition 4.1. *The space $\mathcal{H}(G, X)$ is complete with respect to the quasi-norms u_n.*

Proof: Let $\{f_n\}_n$ be a Cauchy sequence. Consider a subsequence f_{n_m} so that

$$\Gamma[(f_{n_{m+1}} - f_{n_m})(K_m)] \subset B_X\left(0,\, 2^{-m/p}\right).$$

If $m \geqslant 2$, it follows that

$$f_{n_m} = f_{n_1} + \sum_{k=1}^{m-1} (f_{n_{k+1}} - f_{n_k}).$$

The series $\sum_{k=1}^{\infty} (f_{n_{k+1}} - f_{n_k})$ is uniformly convergent over compact subsets of G to a function $\psi\colon G \to X$. If h is a positive integer and $\varepsilon > 0$, pick an integer N so that

$$\sum_{k=N}^{\infty} 2^{-k} < \varepsilon^p \text{ and } K_h \subset K_N.$$

If $m \geqslant N$ and $z \in K_h$, it follows that, denoting $f = f_{n_1} + \psi$,

$$f(z) - f_{n_m}(z) = \sum_{k=m}^{\infty} [f_{n_{k+1}}(z) - f_{n_k}(z)].$$

Then, for any $y \in \Gamma[(f - f_{n_m})(K_h)]$, it will follow that

$$y = \sum_{k=m}^{\infty} \sum_{i=1}^{r} \lambda_i [f_{n_{k+1}}(z_i) - f_{n_k}(z_i)],$$

with $z_i \in K_h$, $\lambda_i \in \mathbb{C}$ and $\sum |\lambda_i| \leqslant 1$.

From this it follows that $\Gamma(f - f_{n_m})(K_h) \subset B_X(0, \varepsilon)$. We have proved that $f_{n_m} \to f$ and $m \to \infty$, ultra-uniformly over compact subsets of G. ∎

THEOREM 4.1. *Let $f_n \colon G \to X$ be analytic. If $\{f_n\}_n$ is uniformly convergent on compact sets of G, then $\{f_n\}_n$ is also ultra-uniformly convergent on compact sets to the same limit.*

PROOF: According to the last proposition, the space $\mathcal{H}(G, X)$ endowed with the p-norms $\{u_k\}_k$ is a complete metrisable linear space. So it is with respect to the p-norms

$$p_n(f) = \sup_{z \in K_n} \|f(z)\|$$

according to Proposition 4.1. Since $p_n \leqslant u_n$, we can use the open mapping theorem. ∎

5. Global holomorphy of analytic functions

Let G, $\{p_n\}_n$, $\{u_n\}_n$ and $\{K_n\}_n$ be as in the preceding section. We can identify $X \otimes \mathcal{H}(G)$ with the space of all finite rank X-valued analytic functions on X. By Theorem 3.1, $\mathcal{H}(G, X)$ is the completion of $X \otimes \mathcal{H}(G)$ with respect to the p-norms $\{p_n\}_n$.

Let U_n be the set of all scalar valued $\phi \in \mathcal{H}(G)$ such that $\sup_{z \in K_n} |\phi(z)| \leqslant 1$, a basic zero-neighbourhood in $\mathcal{H}(G)$. Recall that the functionals q_n defined for $g \in X \otimes \mathcal{H}(G)$ by

$$q_n(g) = \sup_{\omega \in U_n^0} \|(I_X \otimes \omega)(g)\|$$

define the inductive topology on $X \otimes \mathcal{H}(G)$. Also $\delta_z \in U_n^0$ if $z \in K_n$. From this it follows that $p_n(g) \leqslant q_n(g)$ for any $g \in X \otimes \mathcal{H}(G)$.

In the locally convex case the equality of p_n and q_n can be proved. Suppose that X is a Banach space and denote by Δ_n the set of all δ_z with $z \in K_n$. Then $U_n = \Delta_n^0$ and

$$\overline{\Gamma \Delta_n} = U_n^0,$$

and the closure can be taken in the weak topology of the topological dual $\mathcal{H}'(G)$.

If λ_i, $1 \leqslant i \leqslant r$, are scalars so that $\sum |\lambda_i| \leqslant 1$, $z_i \in K_n$, $g = \sum x_i \otimes \phi_i \in X \otimes \mathcal{H}(G)$ and $\omega = \lambda_1 \delta_{z_1} + \cdots + \lambda_r \delta_{z_r}$, then

$$\left\| \sum \langle \omega, \phi_i \rangle x_i \right\| \leqslant \sum |\lambda_j| \left\| \sum \phi_i(z_j) x_i \right\| \leqslant p_n(g). \tag{6}$$

If $\omega \in U_n^0$, we approximate by $\omega_\alpha \in \Gamma(\Delta_n)$ and the same result follows. Thus $q_n(g) \leqslant p_n(g)$.

But in our case we do not have inequality (6). It is true however that the topologies generated in $X \otimes \mathcal{H}(G)$ by $\{p_n\}_n$ and $\{q_n\}_n$ are the same:

Proposition 5.1. *The continuous extension*

$$\Phi \colon X \widehat{\otimes}_\varepsilon \mathcal{H}(G) \to \mathcal{H}(G, X)$$

of the identity $\qquad X \otimes_\varepsilon \mathcal{H}(G) \to X \otimes \mathcal{H}(G),\ \{p_n\}_n$

is a topological isomorphism.

Proof: This result has been proved in [7] using some considerations on the tensor product $X \widehat{\otimes}_\varepsilon \mathcal{C}(G)$. We give a direct proof.

The inequality $p_n \leqslant q_n$ extends to the completion

$$p_n(\Phi(u)) \leqslant \check{q}_n(u),$$

for each $u \in X \widehat{\otimes}_\varepsilon \mathcal{H}(G)$.

We prove that Φ is one to one. Let $u \in X \widehat{\otimes}_\varepsilon \mathcal{H}(G)$ be so that $\Phi(u) = 0$. We show that $\check{q}_n(u) = 0$ for each n. The quasi-norms $\check{q}_n$ have the representation

$$\check{q}_n(u) = \sup_{\omega \in U_n^0} \|I_X \check{\otimes} \omega(u)\|,$$

where $I_X \check{\otimes} \omega$ denotes the extension to $X \widehat{\otimes}_\varepsilon \mathcal{H}(G)$ of the continuous linear mapping

$$I_X \otimes \omega \colon X \otimes_\varepsilon \mathcal{H}(G) \to X.$$

See [7] or [2, Proposition 3.1].

Thus we have to prove that

$$I_X \check{\otimes} \omega(u) = 0$$

if $\omega \in U_n^0 = \overline{\Gamma \Delta_n}$. Consider first $\omega \in \Gamma \Delta_n$, $\varepsilon > 0$, $\omega = \sum_{i=1}^{r} \lambda_i \delta_{z_i}$, with $\lambda_i \in \mathrm{C} \setminus \{0\}$ so that $\sum |\lambda_i| \leqslant 1$ and $z_i \in K_n$. Pick $u' \in X \otimes \mathcal{H}(G)$ with $\check{q}_n(u - u') < \varepsilon(1 + \sum |\lambda_i|^p)^{-1/p}$. Then

$$p_n(u') = p_n(\Phi(u - u')) \leqslant \check{q}_n(u - u') < \varepsilon\left(1 + \sum |\lambda_i|^p\right)^{-1/p}$$

and

$$\|I_X \check{\otimes} \omega(u)\|^p \leqslant \check{q}_n(u - u')^p + \|I_X \otimes \omega(u')\|^p.$$

Now suppose that $u' = \sum x_j \otimes \phi_j \in X \otimes \mathcal{H}(G)$. Then

$$\begin{aligned}\|(I_X \otimes \omega)(u')\|^p &= \left\|\sum_i \lambda_i \sum_j \phi_j(z_i) x_j\right\|^p \\ &\leqslant \sum_i |\lambda_i|^p \left\|\sum_j \phi_j(z_i) x_j\right\|^p \\ &\leqslant p_n(u')^p \sum_i |\lambda_i|^p < \varepsilon^p.\end{aligned}$$

Thus $\|I_X \check{\otimes} \omega(u)\| = 0$ if $\omega \in \Gamma\Delta_n$.

If $\omega \in U_n^0$, we take again $u'' \in X \otimes \mathcal{H}(G)$ with

$$\check{q}_n(u - u'')^p < \varepsilon.$$

With u'' fixed, we take $\omega' \in \Gamma\Delta_n$ with

$$\|I_X \otimes (\omega - \omega')(u'')\|^p < \varepsilon.$$

Now we have

$$\begin{aligned}\|I_X \check{\otimes} \omega(u)\|^p \leqslant{}& \|I_X \check{\otimes} \omega(u - u'')\|^p + \|I_X \check{\otimes}(\omega - \omega')(u'')\|^p \\ &+ \|I_X \check{\otimes} \omega'(u - u'')\|^p + \|I_X \check{\otimes} \omega'(u)\|^p.\end{aligned}$$

As we have seen before, the last term of the sum vanishes and we have $(\omega, \omega' \in U_n^0)$:

$$\|I_X \check{\otimes} \omega(u)\|^p < 3\varepsilon,$$

and the one-to-one character of Φ is proved.

By the open mapping theorem, we only have to prove that the map

$$\Phi \colon X \widehat{\otimes}_\varepsilon \mathcal{H}(G) \to \mathcal{H}(G, X)$$

is onto.

Let $f \in \mathcal{H}(G, X)$. We take a sequence $f_n \in X \otimes \mathcal{H}(G)$ so that

$$\Gamma[(f - f_n)(K_n)] \subset B_X\left(0, 2^{-n/p}\right).$$

From this, it follows that, if $n < m$,

$$\Gamma[(f_n - f_m)(K_n)] \subset B_X\left(0, 2^{-(n+1)/p}\right).$$

Now, if $\omega \in \Gamma\Delta_n$, we have that

$$(I_X \otimes \omega)(f_n - f_m) \in \Gamma(f_n - f_m)(K_n) \subset B_X\Big(0, 2^{-(n+1)/p}\Big).$$

If $\omega \in U_n^0$, we approximate it by $\omega_\alpha \in \Gamma\Delta_n$ in the weak topology and we obtain

$$\|(I_X \otimes \omega)(f_n - f_m)\| \leqslant 2^{-(n+1)/p}.$$

So, if $n < m$, then

$$q_n(f_n - f_m) \leqslant 2^{-(n+1)/p}$$

and $\{f_n\}_n$ is a Cauchy sequence in $X \otimes_\varepsilon \mathcal{H}(G)$. Let $f_n \to f^*$ in $X \widehat{\otimes}_\varepsilon \mathcal{H}(G)$ and $f_n = \Phi(f_n) \to \Phi(f^*)$ and $\mathcal{H}(G, X)$. Since $f_n \to f$ in $\mathcal{H}(G, X)$, the result is proved. □

Finally, we recall that $\mathcal{H}(G)$ is a nuclear space and, by a result of Waelbroeck [16], the p-projective and the inductive topologies on $X \otimes \mathcal{H}(G)$ coincide.

Thus, we have the following result:

Theorem 5.1. *The spaces $X \widehat{\otimes}_p \mathcal{H}(G)$, $X \widehat{\otimes}_\varepsilon \mathcal{H}(G)$ and $\mathcal{H}(G, X)$ are topologically isomorphic. Any analytic function is globally holomorphic, and the uniform convergence on compact sets, the ultra-uniform convergence on compact sets, and the convergence with respect to the functionals $\check{q}_n$ and $|\cdot| \widehat{\otimes}_p \|\cdot\|_n$, coincide, where*

$$\|\phi\|_n = \sup_{z \in K_n} |\phi(z)|, \quad \phi \in \mathcal{H}(G).$$

We remark that in [7] there is a proof of Theorem 5.1 that does not use nuclearity but only the approximation property of $\mathcal{H}(G)$. The inclusion

$$X \widehat{\otimes}_p \mathcal{H}(G) \hookrightarrow \mathcal{H}(G, X)$$

is a consequence of [7, Lemma 2.1] and Proposition 5.1. To obtain the equality, an argument involving projective limits is used.

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Department de Matemàtica Aplicada i Anàlisi
Universitat de Barcelona
08071 Barcelona
Spain