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COEFFICIENT MULTIPLIERS OF MIXED NORM SPACES

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ABSTRACT. We give a simple characterization of coefficient multipliers from the mixed norm space $H^{p,q,\alpha}$, $2 \le p \le \infty$, into $H^{u,v,\beta}$, $0 < u \le 2$, which includes the main results of Wojtaszczyk in [5]. We also calculate multipliers from the Hardy space H^p, $2 \le p \le \infty$, into H^q, $0 < q \le 2$.

1. **Introduction.** If $0 < p \le \infty$, $0 < q \le \infty$, $0 < \alpha < \infty$, a function f, holomorphic in the unit disc, is said to *belong to the mixed norm space* $H^{p,q,\alpha}$ if

$$\|f\|^q_{p,q,\alpha} = \int_0^1 (1-\varrho)^{q\alpha-1} M_p(\varrho,f)^q \, d\varrho < \infty, \quad (0 < q < \infty),$$
$$\|f\|_{p,\infty,\alpha} = \sup_{0<\varrho<1} (1-\varrho)^\alpha M_p(\varrho,f) < \infty, \quad (q = \infty).$$

As usual,

$$M_p(\varrho,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(\varrho e^{it})|^p \, dt\right)^{1/p}, \quad (0 < p < \infty),$$
$$M_\infty(\varrho,f) = \max_{0\le t\le 2\pi} |f(\varrho e^{it})|.$$

A complex sequence $\{a_n\}$ is of class $\ell(p,q)$, $0 < p, q \le \infty$, if

$$\|\{a_n\}\|^q_{p,q} = \sum_{n=0}^{\infty}\Big(\sum_{k\in I_n} |a_k|^p\Big)^{q/p} < \infty,$$

where $I_0 = \{0\}$, $I_n = \{k \in N : 2^{n-1} \le k < 2^n\}$, $n = 1, 2, \ldots$. In the case where p or q is infinite, replace the corresponding sum by a supremum. Note that $\ell^p = \ell(p,p)$.

The class $\ell(p,q,\alpha)$, $\alpha \in R$, consists of all sequences $\{a_n\}$ for which $\|\{a_n\}\|_{p,q,\alpha} = \|\{(n+1)^\alpha a_n\}\|_{p,q} < \infty$.

For two given vector spaces A, B of sequences, we denote by (A,B) the space of "multipliers" from A to B. More precisely, $(A,B) = \big\{\{\lambda_n\} : \{\lambda_n a_n\} \in B$ for every $\{a_n\} \in A\big\}$. We regard spaces of analytic functions, such as $H^{p,q,\alpha}$, as being sequence spaces (Taylor coefficients).

In [5] P. Wojtaszczyk described the multipliers from $H^{\infty,\infty,\alpha}$ and $H^{p,p,1/p}$, $2 \le p < \infty$, to $H^{q,q,1/q}$, $0 < q \le 2$, by using the general factorization theorems of Grothendieck, Nikishin and Maurey. In this note we calculate multipliers $(H^{p,q,\alpha}, H^{u,v,\beta})$ in the case $2 \le p \le \infty$, $0 < u \le 2$. Our characterization includes the one obtained by Wojtaszczyk but our approach is different and much simpler.

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THEOREM 1. *If* $2 \leq p \leq \infty$, $0 < u \leq 2$, *then* $(H^{p,q,\alpha}, H^{u,v,\beta}) = \ell(\infty, q \circ v, \alpha - \beta)$, *where* $\frac{1}{q \circ v} = \frac{1}{v} - \frac{1}{q}$, *if* $0 < v < q \leq \infty$, *and* $q \circ v = \infty$, *if* $q \leq v$.

2. **Preliminaries.** A sequence space A is said to be *solid*, if whenever it contains $\{a_n\}$ it also contains all sequences $\{b_n\}$ with $|b_n| \leq |a_n|$. For any sequence space A there is a largest solid subspace $s(A)$, contained within it, and a smallest solid superspace $S(A)$, containing it ([2]). In [2] it is also proved that if X is any solid space and A any vector space of sequences then

$$(A, X) = \big(S(A), X\big), \tag{2.1}$$

$$(X, A) = \big(X, s(A)\big). \tag{2.2}$$

To make use of (2.1) and (2.2) we need to determine $s(H^{\infty,q,\alpha})$, $s(H^{p,q,\alpha})$, $0 < p \leq 2$, and $S(H^{p,q,\alpha})$, $2 \leq p \leq \infty$. We will use the following lemma:

LEMMA 2.1 ([4]). *Let* $0 < \alpha < \infty$ *and* $a_k \geq 0$, $k = 0, 1, 2, \ldots$

i) If $0 < q < \infty$ *there is a positive constant* $A_{q,\alpha}$ *such that*

$$A_{q,\alpha}^{-1} \|\{a_k\}\|_{1,q,-\alpha} \leq \left(\int_0^1 (1-\varrho)^{q\alpha - 1} \Big(\sum_{k=0}^{\infty} a_k \varrho^k \Big)^q \, d\varrho \right)^{1/q} \leq A_{q,\alpha} \|\{a_k\}\|_{1,q,-\alpha}.$$

ii) There is a positive constant B_α *such that*

$$B_\alpha^{-1} \sup_{k \geq 0} 2^{-k\alpha} a_k \leq \sup_{0 < \varrho < 1} (1-\varrho)^\alpha \sum_{k=0}^{\infty} a_k \varrho^{2^k} \leq B_\alpha \sup_{k \geq 0} 2^{-k\alpha} a_k.$$

LEMMA 2.2. $s(H^{\infty,q,\alpha}) = \ell(1, q, -\alpha)$.

PROOF. Let $\{a_k\} \in \ell(1, q, -\alpha)$, $\{b_k\} \in \ell^\infty$ and $f(z) = \sum_k a_k b_k z^k$. Since $M_\infty(\varrho, f) \leq C \sum_k |a_k| \varrho^k$ we have $\|f\|_{\infty,q,\alpha} \leq C \|\{a_k\}\|_{1,q,-\alpha}$, by Lemma 2.1. (We use C to denote various constants which may vary from line to line). Thus, $\{a_k\} \in (\ell^\infty, H^{\infty,q,\alpha}) = s(H^{\infty,q,\alpha})$, by Lemma 2 ([2]).

Conversely, let $\{a_k\} \in s(H^{\infty,q,\alpha})$. Then $f(z) = \sum_k |a_k| z^k$ belongs to $H^{\infty,q,\alpha}$. Since $M_\infty(\varrho, f) = \sum_k |a_k| \varrho^k$, we have $\infty > \|f\|_{\infty,q,\alpha} \geq C \|\{a_k\}\|_{1,q,-\alpha}$, by Lemma 2.1.

Using Khintchine's inequality ([6], p. 213) as in [1] (Lemma 2, p. 58), it may be easily proved that

$$s(H^{p,q,\alpha}) = H^{2,q,\alpha} \text{ for } 0 < p \leq 2. \tag{2.3}$$

Observe that $H^{2,q,\alpha} = \ell(2, q, -\alpha)$, by Lemma 2.1.

LEMMA 2.3 ([4]). $S(H^{p,q,\alpha}) = \ell(2, q, -\alpha)$ *for* $p \geq 2$.

As a final preliminary result we need

LEMMA 2.4. *Let* $0 < p, q, u, v \leq \infty$, $0 < \alpha, \beta < \infty$. *Then* $\big(\ell(p, q, \alpha), \ell(u, v, \beta)\big) = \ell(p \circ u, q \circ v, -\alpha + \beta)$.

The lemma follows easily from its special case $\big(\ell(p, q), \ell(u, v)\big) = \ell(p \circ u, q \circ v)$ (see [3]).

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3. **Proof of Theorem 1.** Let $\lambda \in (H^{p,q,\alpha}, H^{u,v,\beta})$. Since $s(H^{\infty,q,\alpha}) \subset H^{p,q,\alpha}$, we have $\lambda \in \big(s(H^{\infty,q,\alpha}), H^{u,v,\beta}\big) = \big(\ell(1,q,-\alpha), H^{u,v,\beta}\big)$, by Lemma 2.2. Obviously, $\ell(1,q,-\alpha)$ is a solid space. Hence, using (2.2), (2.3) and Lemma 2.4 we find that $\lambda \in \big(\ell(1,q,-\alpha), \ell(2,v,-\beta)\big) = \ell(\infty, q\circ v, \alpha-\beta)$.

Conversely, let $\lambda \in \ell(\infty, q\circ v, \alpha-\beta)$. By Lemma 2.4 and Lemma 2.3 $\lambda \in \big(\ell(2,q,-\alpha), \ell(2,v,-\beta)\big) = \big(S(H^{p,q,\alpha}), \ell(2,v,-\beta)\big) = \big(H^{p,q,\alpha}, \ell(2,v,-\beta)\big)$, by (2.1) since $\ell(2,v,-\beta)$ is a solid space. By (2.3) we have $\ell(2,v,-\beta) = s(H^{u,v,\beta})$. Thus, $\lambda \in \big(H^{p,q,\alpha}, s(H^{u,v,\beta})\big) \subset (H^{p,q,\alpha}, H^{u,v,\beta})$.

4. **Multipliers of H^p space.** For $0 < p \leq \infty$, by H^p we denote the Hardy space. It is easy to see that $s(H^\infty) = \ell^1$. An application of Khintchine's inequality shows that $s(H^p) = \ell^2, 0 < p \leq 2$.

As a consequence of Theorem K ([4]), due to Kisliakov, we have $S(H^p) = \ell^2$, $2 \leq p \leq \infty$. Now, using the same method as in §3. We can find multipliers from Hardy space H^p, $2 \leq p \leq \infty$, into H^q, $0 < q \leq 2$.

THEOREM 2. *If* $2 \leq p \leq \infty$ *and* $0 < q \leq 2$, *then* $(H^p, H^q) = \ell^\infty$.

We omit details.

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