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NILPOTENT INJECTORS AND CONJUGACY CLASSES IN SOLVABLE GROUPS

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To Laci Kovács on his 65th birthday

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Abstract

We provide an upper bound for the order of a nilpotent injector of a finite solvable group with Fitting subgroup of order n. We also show that the same bound is an upper bound for the number of conjugacy classes, provided that the $k(GV)$-conjecture holds for solvable G all primes dividing n.

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1. Introduction

There has been much progress lately towards the so-called $k(GV)$-problem, which asserts that if G is a finite p'-group and V is a faithful $GF(p)G$-module, then $k(GV)$, the number of conjugacy classes of the semi-direct product GV, is at most $|V|$. In particular, by the results of [1] and [3] (which themselves built on a good deal of earlier work by several authors), if G is solvable, then the $k(GV)$-problem is answered in the affirmative for all primes p other than 3, 5, 7 or 13 (the other cases still being open to date).

If the $k(GV)$-problem has an affirmative answer for the prime p and solvable G, it follows more generally that we have $k(H) \leq |H|_p$ whenever p is a prime and H is a finite solvable group such that $F(H)$ is a p-group (as usual, for an integer n and a prime p, we let n_p denote the highest power of p which divides n).

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This suggests the problem of finding other bounds for $k(G)$ in terms of distinguished subgroups of G, especially when G is solvable. In a private communication to the author [5], Thompson asked whether it is the case that $k(G) \leq |I|$ whenever G is solvable and I is a nilpotent injector of G. There are several definitions of nilpotent injector in the literature, all of which coincide for solvable groups. For our purposes, we remind the reader that the nilpotent subgroup I of the solvable group G is a nilpotent injector if $I \cap N$ is a maximal nilpotent subgroup of N whenever $N \lhd \lhd G$. Nilpotent injectors were first shown to exist for solvable groups by Fischer. They are unique up to conjugacy, and whenever I is a nilpotent injector of a finite solvable group G, we have $O_p(I) \in \mathrm{Syl}_p(C_G(O_{p'}(G)))$ for each prime p.

In [4], Kovács and the author proved that if G is solvable and $|F(G)| = p^r$ for some integer r, then $k(G) \leq 3^{r-1}|F(G)|$. It is not difficult to extend this result to prove that if G is solvable and $|F(G)|$ has r prime factors (counting multiplicities), then $k(G) \leq 3^{r-1}|F(G)|$. We aim here to improve this bound somewhat.

We define the function $f : \mathbb{N} \to \mathbb{N}$ as follows:
$f(1) = 1$; $f(ab) = f(a)f(b)$ whenever a and b are relatively prime.
When p is an odd prime which is not Fermat, $f(p^s) = p^s(s!)_p$.
Whenever p is a Fermat prime, and s is an integer of the form $(p-1)t + u$ where t, u are integers with $0 \leq u \leq p-2$, we set $f(p^s) = p^{s+t}(t!)_p$.
When $p = 2$ and s is a non-negative integer of the form $3t + u$, where t, u are integers with $0 \leq u \leq 2$, we set $f(2^s) = 2^{s+3t}(t!)_2 u!$.

In this note, we prove the following:

THEOREM. *Let f be the function defined as above on the natural numbers. Then*

(i) *For each finite solvable group H, the order of a nilpotent injector I of H is a divisor of $f(|F(H)|)$. Furthermore, for each positive integer n there exists a finite solvable group H_n with Fitting subgroup of order n and a nilpotent injector of order $f(n)$.*

(ii) *If H is a solvable group such that the $k(GV)$-problem has an affirmative answer for solvable G for all prime divisors of $|F(H)|$, then we have $k(H) \leq f(|F(H)|)$.*

PROOF. It follows from a theorem of Winter [6] for odd primes, and the exposition and expansion of Winter's result in Isaacs' book [2], that whenever p^s is a prime-power, $f(p^s)/p^s$ is an upper bound for the order of a Sylow p-subgroup of a completely reducible p-solvable subgroup of $GL(s, p)$ (actually, for $p = 2$, the result in Isaacs' book gives the slightly weaker bound $2^{4s-3/3}$, which agrees with $f(2^s)/2^s$ when s has the form $3 \cdot 2^m$ for some non-negative integer m. However, careful examination of the arguments gives the result in the above sharper form).

Furthermore, it is clear that each of these bounds may be realised in solvable

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subgroups of $GL(s, p)$. For example, if p is odd, but not Fermat, then $C_2 \wr P$ embeds as a completely reducible subgroup of $GL(s, p)$, where P is a Sylow p-subgroup of the symmetric group of degree s. If $p = 2$, then $X \wr P$ embeds as a completely reducible subgroup of $GL(s, 2)$, when s has the form $3t$ for some integer t, where X is the semi-direct product of an extra-special group of order 27 with $SL(2, 3)$ in its natural action, and P is a Sylow 2-subgroup of the symmetric group of degree t (and $(X \wr P) \times S_3$ embeds in $GL(s + 2, 2)$. If p is a Fermat prime, then the semi-direct product of an extra-special 2-group of order $2(p - 1)^2$ with a cyclic group of order p (with non-trivial action) embeds as an absolutely irreducible subgroup of $GL(p - 1, p)$. Its wreath product with a Sylow p-subgroup of the symmetric group of degree r embeds completely reducibly in $GL(r(p - 1), p)$.

Taking an appropriate semi-direct product produces, for each prime p and each non-negative integer s, a finite solvable group H with $F(H)$ of order p^s having a Sylow p-subgroup (which in this situation is a nilpotent injector) of order $f(p^s)$. Taking suitable direct products produces, for each positive integer n, a finite solvable group H_n with $F(H_n)$ of order n such that H_n has a nilpotent injector of order $f(n)$.

On the other hand, we prove that a finite solvable group H with $F(H)$ of order n has a nilpotent injector of order dividing $f(n)$. It is useful for what follows to note the obvious fact that $f(p^s)/p^s$ increases with s. Consequently, as $F(H/\Phi(H)) = F(H)/\Phi(H)$, it suffices to assume that $\Phi(H) = 1$, which we do. In that case, $F(H)$ is Abelian of squarefree exponent, and $H/F(H)$ acts completely reducibly on it.

Let I be a nilpotent injector of H. Then $O_p(I)/O_p(H)$ acts faithfully on $O_p(H)$ as $O_p(I)$ acts trivially on $O_{p'}(H)$. If $|O_p(H)| = p^s$, then $H/C_H(O_p(H))$ acts completely reducibly on $O_p(H)$, so that $|O_p(I)| \leq f(p^s)$. Hence I has order dividing $f(n)$, as p was arbitrary.

It remains to prove that $k(H) \leq f(n)$ when H is solvable with $|F(H)| = n$. Let I be a nilpotent injector of H. We recall the well-known inequality $k(H) \leq k(N)k(H/N)$ whenever $N \lhd H$. This again allows us to assume that $\Phi(H) = 1$, which we do. Set $F = F(H)$ and $N = O_p(F)$ for a prime p dividing $|F|$. Then F is completely reducible as H/F-module. Hence there is a subgroup L of H which complements N. It is routine to check that $F = N \times F(C_L(N))$. Set $C = C_L(N) \lhd H$.

We have $O_p(C) \leq O_p(H) = N$, so that $F(C) = O_{p'}(F(C)) = O_{p'}(F(H))$. Hence $O_{p'}(I) = I \cap C$ is a nilpotent injector of C. We may assume by induction that $k(C) \leq f(|F(C)|)$. Now H/C is the semi-direct product of N with $L/C_L(N)$, with the latter group acting completely reducibly on N. Hence a Sylow p-subgroup of H/C has order at most $f(|N|)$. Using the hypothesis that the $k(GV)$-problem has an affirmative answer for p, we see that $k(H/C) \leq f(|N|)$. Hence we see that $k(H) \leq k(C)k(H/C) \leq f(|F(C)|)f(|N|) = f(|F(H)|)$, as required to complete the proof. □

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