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DIMENSION INVARIANCE OF SUBDIVISIONS

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The dimension of an ordered set is invariant with respect to any subdivision of its completion. This may be applied to support the conjecture (still open) that the problem to determine the (order) dimension of an N-free ordered set is NP-complete.

1. INTRODUCTION

N-free ordered sets are uncommonly useful in the combinatorial theory of ordered sets. Recently, Habib [1] conjectured that the computational time-complexity of the problem to determine the dimension of an N-free ordered set is polynomial. An approach to settling Habib's conjecture in the affirmative uses the popular idea of subdividing the edges of the diagram. Spinrad [3] has shown, by a clever example, that this obvious approach cannot succeed. In fact, our paper is inspired by an effort to settle Habib's conjecture in the negative, by using the same subdivision idea, albeit in a more sophisticated context than before. To this end we show that the dimension of an ordered set is unchanged for any subdivision of its completion, a result that seems to be of independent interest.

THEOREM 1. *For any finite ordered set P, the dimension of P is the same as the dimension of any subdivision of the completion of P.*

Let P be a finite ordered set. The *dimension* of P is defined to be the minimum number of chains into whose product P can be order embedded and the *completion* of P turns out to be the smallest lattice into which P can be order embedded. Loosely speaking, the completion of P is the smallest lattice containing P. A *subdivision* of P is an ordered set obtained from P by adjoining vertices $s(a,b)$ to P corresponding to pairs (a,b) of elements of P such that $a \succ b$, that is, a covers b [$a > b$ and $a > x \geqslant b$ imply $x = b$] ordered by the relations induced from P and $a \succ s(a,b) \succ b$.

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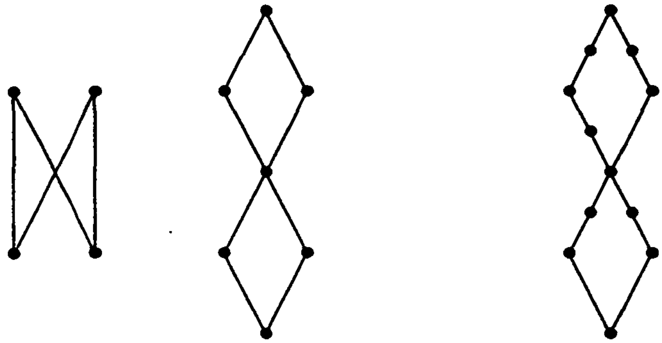
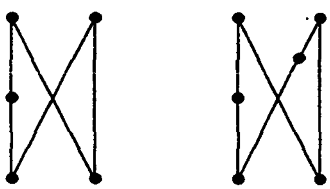


P completion P a subdivision of completion P

Figure 1.

In contrast to Theorem 1, the dimension of a subdivision of P may be strictly larger than the dimension of P (see Figure 2). And, according to Spinrad [3], there is not even a constant c such that dimension (subdivision P) $\leqslant c$ dimension P.



dimension=2 dimension=3

Figure 2.

One important and notable exception is the ordered set S_n of the singleton subsets and $(n-1)$-element subsets of an n-element set ordered by set inclusion.

THEOREM 2. *For any positive integer n, the dimension of any subdivision of S_n equals n, the dimension of S_n.*

Our approach to settling Habib's conjecture (in the negative) uses the idea of a "spanning set" in a lattice. Given a lattice L let P stand for its set of join-irreducible or meet-irreducible elements. A *spanning set* in L is a subset S of L containing P such that, for each a, b in S, if $a \succ b$ in S then $a \succ b$ in L, that is, S is a subdiagram of L containing P. Of course, L itself is a spanning set in L. Thus, the hypercube $L = \mathbf{2}^n$ of all subsets of an n-element set ordered by set inclusion (which is the completion of S_n) is a spanning set of itself. What we seek, however, is a "small" spanning set in L.

CONJECTURE. *There is a constant k such that every finite lattice with n join-irreducible or meet-irreducible elements contains a spanning set with at most n^k elements.*

Theorem 3. *For any positive integer n, the hypercube $\mathbf{2}^n$ contains a spanning set with at most n^2 elements.*

Our approach to Habib's conjecture now runs as follows. Starting with an arbitrary n-element ordered set P, let L be its completion. It is well known that the join-irreducible or meet-irreducible elements of L are contained in P (see [2]). Supposing our conjecture is true, then L contains a spanning set S of size at most n^k elements, k a constant. The rest is easy. Subdivide every edge of the diagram of S to obtain an ordered set T with $|T| \leqslant n^{2k}$. (There cannot be more than $|S|^2$ edges.) It is easy to verify that such a subdivision T of S is N-free, that is, its diagram contains no subdiagram isomorphic to N (see Figure 3).

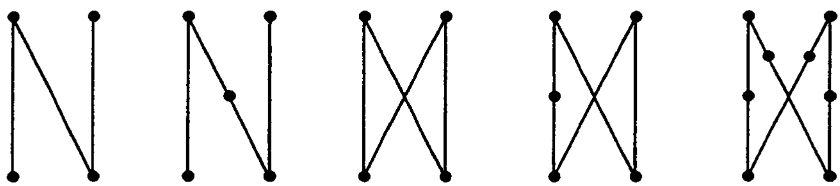


Figure 3.

Now supposing that there is a polynomial (in the number of elements of an ordered set) which bounds the number of steps needed to compute the dimension of an N-free ordered set, then the dimension of this subdivision T can be computed within n^m steps, m a constant. Finally, as a dimension is an isotone operator, dimension $P \leqslant$ dimension $S \leqslant$ dimension $T \leqslant$ dimension(subdivision(completion P)), which, by Theorem 1, implies that the dimension of P is the same as the dimension of T.

Thus, as long as the construction itself is a polynomial in n, it follows that there is a polynomial (in n) to compute the dimension of P, an arbitrary ordered set. This, however, lies in contradiction to the result of Yannakakis [4] who showed that the problem to determine the dimension of an arbitrary ordered set is itself NP-complete.

The fly in the ointment is, of course, the construction of a spanning set which is the substance of our own conjecture. We do, however, have this evidence about Habib's conjecture which, in turn, substantiates our own.

Theorem 4. *Let P be an n-element ordered set with no subdiagram isomorphic to F (Figure 4) and also no subset isomorphic to G (Figure 4) in which either $w_3 \succ w_1$ or $w_3 \succ w_2$ or $w_4 \succ w_1$. Then there is an ordered set Q satisfying these conditions such that*

(i) *Q contains P;*

(ii) Q *is* N*-free;*
(iii) *dimension* $Q =$ *dimension* P;
(iv) $|Q| \leqslant n^4$.

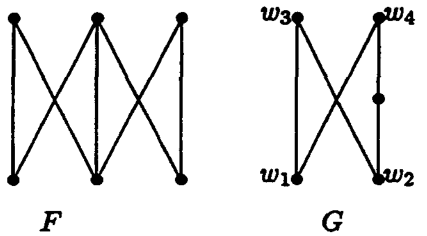


Figure 4.

2. Proof of Theorem 1

It is a well-known, by now, classical fact (see [2]) that, for any ordered set P, the dimension of P is the same as the dimension of the completion of P. What we must, therefore, establish is that for any finite lattice L, the dimension of L is the same as the dimension of any subdivision of L.

To this end let $a \succ b$ in L. We show that the lattice $L' = L \cup \{c\}$ with order induced from L and $a \succ c \succ b$ has the same dimension as L. Repeating this for each edge of L, will complete the proof, for, after each such subdivision, the ordered set is still a lattice.

Let L have dimension k and consider an order embedding f of L into the direct product $C_1 \times C_2 \times \cdots \times C_k$ of k chains $C_1, C_2, \ldots, C_k$. Suppose there is x in $C_1 \times C_2 \times \cdots \times C_k$ such that $f(a) > x > f(b)$ and, for any y in L, $x > f(y)$ implies $f(b) \geqslant f(y)$ and $x < f(y)$ implies $f(a) \leqslant f(y)$. In this case, $f(L) \cup \{x\} \cong L'$, that is, L', too, is order embedded in $C_1 \times C_2 \times \cdots \times C_k$ whence dimension $L' =$ dimension L.

Suppose then that, for any x satisfying $f(a) > x > f(b)$ there is y in L such that either $x > f(y)$ but $f(b) \not\geqslant f(y)$ or $x < f(y)$ but $f(a) \not\leqslant f(y)$. Let

$$A = \{x \in C_1 \times \cdots \times C_k \mid f(a) > x > f(b) \text{ and, for some } y \text{ in } L,\ x > f(y) \text{ but } f(b) \not\geqslant f(y)\}$$

and

$$B = \{x \in C_1 \times \cdots \times C_k \mid f(a) > x > f(b) \text{ and, for some } y \text{ in } L,\ x < f(y) \text{ but } f(a) \not\leqslant f(y)\}.$$

If there is x in $A \cap B$ then choose y_A in L maximal satisfying $x > f(y_A)$ but $f(b) \not\geqslant f(y_A)$ and y_B in L minimal satisfying $x < f(y_B)$ but $f(a) \not\leqslant f(y_B)$. Thus, $a > y_A$, $a > b$, $y_B > y_A$, $y_B > b$. As L is a lattice there is z in L such that, in

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particular, $a > z > b$ which, however, is in contradiction to the fact that $a \succ b$ in L. Thus, $A \cap B = \emptyset$. Of course, it may be that $A = \emptyset$ or $B = \emptyset$, too.

Let x be a minimal element in $A \cup \{f(a)\}$. Let z in $C_1 \times C_2 \times \cdots \times C_k$ satisfy that $x \succ z \geqslant f(b)$. Then there is a coordinate i such that the ith coordinates x_i, z_i of x, z satisfy that $x_i \succ z_i$ in C_i, yet, for $j \neq i$, $x_j = z_j$. Now, replace C_i by another chain $C_i' = C_i \cup \{c_i\}$ in which $x_i \succ c_i \succ z_i$.

The lattice L may still be order embedded in $C_1 \times C_2 \times \cdots \times C_{i-1} \times C_i' \times C_{i+1} \times \cdots \times C_k$ by composing f with the identity embedding of $C_1 \times C_2 \times \cdots \times C_k$ into $C_1 \times C_2 \times \cdots \times C_{i-1} \times C_i' \times C_{i+1} \times \cdots \times C_k$. Then we may adjoin the k-tuple $(x_1, x_2, \ldots, x_{i-1}, c_i, x_{i+1}, \ldots, x_k)$ to $f(L)$ to form an ordered subset of $C_1 \times C_2 \times \cdots \times C_{i-1} \times C_i' \times C_{i+1} \times \cdots \times C_k$ isomorphic to L', for this new element is larger, in $f(L)$, only than those elements dominated by $f(b)$ and is smaller in, $f(L)$, only than those dominating $f(a)$. Thus, L' has the same dimension as L. Repeating this construction for any edge of L completes the proof. $\square$

3. Proof of Theorem 2

Let $a_1, a_2, \ldots, a_n$ be the maximal elements of S_n and $b_1, b_2, \ldots, b_n$ its minimal such that $a_i > b_j$ just if $i \neq j$. Let T be the subdivision of S_n obtained by adjoining a new vertex c_{ij} for each covering pair $a_i \succ b_j$, $i \neq j$, such that, in T, $a_i \succ c_{ij} \succ b_j$. We construct n linear extensions $L_1, L_2, \ldots, L_n$ of T such that $x < y$ in T just if $x < y$ in L_i for every $i = 1, 2, \ldots, n$. To this end, let

$$A_i = \{a_{i+1} \succ a_{i+2} \succ \cdots \succ a_{i+(n-1)} \succ a_{i+(n-1)}\},$$

$$B_i = \{b_{i+1} \succ b_{i+2} \succ \cdots \succ b_{i+(n-1)} \succ b_{i+(n-1)}\},$$

where the subscript $i+k$ is replaced by $i+k-n$ if $i+k > n$. Let $X_i = \{x \mid a_i \succ x(T)\}$ and $Y_i = \{y \mid y \succ b_i(T)\}$. Then $X_i \cap Y_i = \emptyset$. Put $S = X_1 \cup X_2 \cup \cdots \cup X_n = Y_1 \cup Y_2 \cup \cdots \cup Y_n$ and $Z_i = S - (X_i \cup Y_i)$. Finally, let $L_i = B_i \oplus \overline{X}_i \oplus \{a_i\} \oplus \overline{Z}_i \oplus \{b_i\} \oplus \overline{Y}_i \oplus A_i$, where $\overline{X}_i, \overline{Y}_i$, and $\overline{Z}_i$ are linear extensions of X_i, Y_i, and Z_i, respectively, for $i = 1, 2, \ldots, n$, and $Y \oplus X$ stands for the linear sum of X and then Y.

We claim that the L_i's are the required linear extensions of T. First of all we verify that each L_i is a linear extension of T. It is enough to show that

(a) $a_j > b_k(L_i)$ whenever $j \neq k$,
(b) if x in X_j, then $x < a_j(L_i)$,
(c) if y in Y_k, then $y > b_k(L_i)$.

(a) Suppose $j \neq k$. If $j \neq i$, then a_j is in A_i and so $a_j > b_k(L_i)$; if $j = i$, then $k \neq i$ so b_k belongs to B_i and $a_j > b_k(L_i)$. Hence we always have $a_j > b_k(L_i)$ if $j \neq k$.

(b) Suppose $x \in X_j$. If $j \neq i$, then, a_j is in A_i so $x < a_j(L_i)$; if $j = i$, then x is in X_i, so $x < a_i(L_i)$. Hence it is always true that $x < a_j(L_i)$ if x is in X_j.

(c) This is the dual of (b).

Let K be the order on the same set as T given by $x < y(K)$ if and only if $x < y(L_i)$ for each $i = 1, 2, \dots, n$. To complete the proof we must now verify that $K = T$, that is, for each i, j,

(d) a_i is noncomparable to b_i in K,
(e) a_i is noncomparable to x in K, for each x in X_j, $i \neq j$,
(f) b_i is noncomparable to y in K, for each y in Y_j, $i \neq j$,
(g) x_1 is noncomparable to x_2 in K, for any distinct x_1, x_2 in S.

(d) According to the construction of L_i we know that $a_i < b_i(L_i)$. On the other hand, choosing j such that $i \neq j$, gives a_i in A_j and b_i in B_j so $a_i > b_i(L_j)$. Therefore, a_i is noncomparable to b_i in K

(e) Again, according to the construction of L_i if $i \neq j$, and x is in X_j, then x is in $Z_i \cup Y_i$ so $x > a_i(L_i)$ and, on the other hand if $i \neq j$ then a_i is in A_j so $a_i > x(L_j)$. Hence a_i is noncomparable to x in K.

(f) This is the dual of (e).

(g) Let x_1 be in X_i and x_2 be in X_j. If $i = j$ then x_1 is in Y_h and x_2 is in Y_k for some h, k satisfying $h \neq k$. Then $x_1 > x_2(L_h)$ and $x_1 < x_2(L_k)$, so x_1 is noncomparable to x_2 in K. If $i \neq j$ then $x_1 < x_2(L_i)$ and $x_1 > x_2(L_j)$ so x_1 is noncomparable to x_2 in K. This completes the proof. ∎

4. Proof of Theorem 3

Let T_n be the subset of $\mathbf{2}^n$ consisting of all members of

$$\bigcup_{k=0}^{n-2} \{\{i, i+1, \dots, i+k\} \mid 1 \leqslant i \leqslant n\}$$

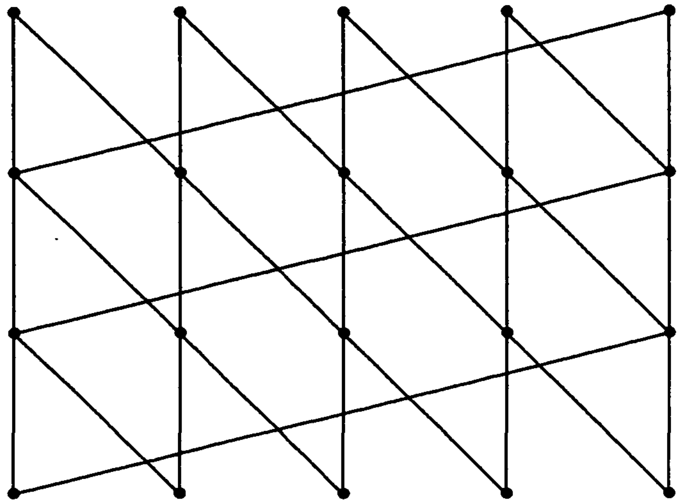
where $i + k$ is replaced by $i + k - n$ if $i + k > n$. Then

$$\min S_n = \{\{1\}, \{2\}, \dots \{n\}\} \subset T_n,$$
$$\max S_n = \{\{1, 2, \dots, n-1\}, \{2, 3, \dots, n\}, \dots \{n, 1, 2, \dots, n-2\}\} \subset T_n,$$

and

$$|T_n| = n(n-1).$$

We know that S_n is the set of join-irreducible or meet-irreducible elements of $\mathbf{2}^n$. Clearly, T_n is a subdiagram of $\mathbf{2}^n$, whence a spanning set in $\mathbf{2}^n$.



T_5

Figure 5.

5. Proof of Theorem 4

For subsets B and C of an ordered set P, we write $B < C$ $(B \prec C)$ if $b < c$ $(b \prec c)$ for any b in B and any c in C. For the purposes of this proof, an N is regarded as a quadruple (a, B, C, d) of elements a, d and subsets B, C of P such that $\{a\} \prec B \succ C \prec \{d\}$ and $a \| d$, which means that a is noncomparable to d. The pair (B, C) is called an *N-edge*. An N (a, B, C, d) or an N-edge (B, C), is said to be *maximal* if there is no other N-edge (B', C') with $B \subset B'$ and $C \subset C'$.

Lemma 1. *Let P be an ordered set with no subset isomorphic to G (Figure 4) in which either $w_3 \succ w_1$ or $w_3 \succ w_2$ or $w_4 \succ w_1$. If (a, B, C, d) is a maximal N in P and $\overline{P} = P \cup \{p\}$ is the ordered set with order induced from P and $B \succ \{p\} \succ C$, then P and $\overline{P}$ have the same dimension.*

Proof: We first show that $u = \inf B \succ \sup C = v$ in the completion of P. Suppose that there is z in the completion of P with $u > z > v$. Of course, z does not belong to P and, therefore z is doubly reducible. We choose a minimal element s in P such that $s \geqslant z$ but $s \ngeqslant u$ and a maximal element t in P such that $t \leqslant z$ but $t \nleqslant v$. Then $\{s\} \succ C$, $\{t\} \prec B$ and $s \succ t$, for otherwise P contains a forbidden copy of G. Now $\big(a, B, C \cup \{t\}, s\big)$ is an N in P as long as $a \| s$, and $\big(a, B \cup \{s\}, C, d\big)$ is an N in P as long as $a \prec s$, both contradicting the maximality of the N-edge (B, C). On the other hand, P will contain G if $a < s$ but $a \not\prec s$. By Theorem 1, subdividing the edge (u, v) by p gives us a lattice $L' =$ completion $P \cup \{p\}$ with dimension $L' =$

dimension P. Now, let P' be the ordered set on $P \cup \{p\}$ induced from L'. Clearly, dimension P' is equal to dimension P.

Finally, we need to show that $P' = \overline{P}$. Suppose not. By duality we assume that there is an upper cover y of p in P' with y not in B. Then $a < y$. If $a \not\prec y$ then P will contain G; otherwise, $(a, B \cup \{y\}, C, d)$ is an N in P, contradicting the maximality of (B, C). $\square$

For an ordered set P, let

$$N(P) = \{(x_1, x_2, x_3, x_4) \in P^4 \mid x_1 \prec x_2 \succ x_3 \prec x_4,\ x_1 \| x_4\}$$

Lemma 2. *Let P be an ordered set containing neither a subdiagram isomorphic to F (Figure 4) nor a subset isomorphic to G (Figure 4) in which either $w_3 \succ w_1$ or $w_3 \succ w_2$ or $w_4 \succ w_1$. If (a, B, C, d) is maximal in P and $\overline{P} = P \cup \{p\}$ is the ordered set with order induced from P and $B \succ \{p\} \succ C$, then $\overline{P}$ too contains no subdiagram isomorphic to F, no subset isomorphic to G with either $w_3 \succ w_1$ or $w_3 \succ w_2$ or $w_4 \succ w_1$, and $|N(P)| > |N(\overline{P})|$.*

Proof: We divide the proof into three parts.

I. $\overline{P}$ contains no subdiagram isomorphic to F.

By duality and symmetry we have only two cases to check (see Figure 6).

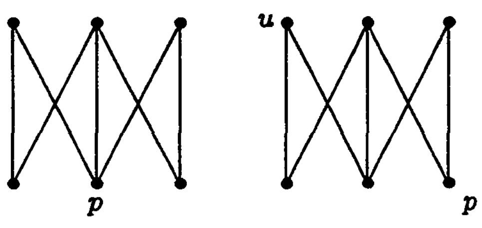


Figure 6.

In case (i), by replacing p by c from C, we get F in P. As to case (ii), if $u \| c$ for some c in C, then choose c in place of p to get F in P. Hence $\{u\} \succ C$, for otherwise P contains a forbidden G. If $a \| u$ then P contains a subdiagram isomorphic to F by replacing p by a. Thus $a < u$. If $a \prec u$ then u belongs to B, and otherwise P contains a forbidden G. Now the conclusion follows.

II. $\overline{P}$ contains no forbidden G

By duality we have three cases to consider (see Figure 7).

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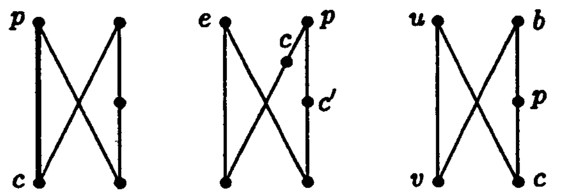

Figure 7.

(i) p is adjacent to a covering edge of G, that is, $p \succ c$. By taking b in B in place of p, we get G in P.

(ii) p is not adjacent to a covering edge of G. We have c and c' from C which are not below e. If $e \| b$ for some b in B then pick b instead of p to get G in P. Hence, $\{e\} < B$. On the other hand, if $e \| d$ then replace p by d to get G in P. If $e < d$, then $e \prec d$, and $\{e\} \prec B$, for otherwise P contains G. Now, $(a, B, C \cup \{e\}, d)$ is an N in P, a contradiction.

(iii) p is one of the middle elements, Observe that $\{b, c, u, v\}$ is a cover-preserving order subset of P. Now this case is divided into four further cases.

Case 1. $\{v\} < B$ and $\{u\} > C$.

In fact $\{u\} \succ C$ and $\{v\} \prec B$. If $a < u$, then again $a \prec u$ and so $(a, B \cup \{u\}, C, d)$ is an N in P, which is a contraction. Hence $a \| u$, which implies that $(a, B, C \cup \{v\}, u)$ is again an N in P.

Case 2. $\{v\} < B$ and $u \| c'$ for some c' in C.

If $v < d$ and so $(a, B, C \cup \{v\}, d)$ is an N in P. Thus $v \| d$. Then (u, v, b, c, d, c') in P is a subdiagram isomorphic to F.

Case 3. $\{u\} < C$ and $v \| b'$ for some b' in B.

This is the dual of Case 2.

Case 4. $u \| c'$ and $v \| b'$ for some b' in B and c' in C.

In fact, $\{u, v, b, c, b', c'\}$ in P is isomorphic to F.

III. $|N(P)| > |N(\overline{P})|$.

Pick b from B and c from C. Define a map f on $N(\overline{P})$ to P^4 by

$$f(x_1, x_2, x_3, x_4) = \begin{cases} (c, x_2, x_3, x_4) & x_1 = p, \\ (x_1, b, x_3, x_4) & x_2 = p, \\ (x_1, x_2, c, x_4) & x_3 = p, \\ (x_1, x_2, x_3, b) & x_4 = p, \\ (x_1, x_2, x_3, x_4) & \text{otherwise.} \end{cases}$$

We shall show that f is in fact a map from $N(\overline{P})$ to $N(P)$. Since the cases except $x_1 = p$ and $x_4 = p$ are trivial, we consider only the case $x_1 = p$ by duality. Suppose $c < x_4$. Since P contains no forbidden G, we have $c \prec x_4$, which also contradicts II. Hence (c, x_2, x_3, x_4) belongs to $N(P)$. It is now routine to verify that f is one-to-one and (a, b, c, d) does not belong to $f(N(\overline{P}))$. For instance, if $(x_1, x_2, x_3, x_4) \neq (y_1, y_2, y_3, y_4)$ in $N(\overline{P})$ with $x_1 = p$ and $y_2 = p$, then x_3 is not in C and y_3 is in C so $f(x_1, x_2, x_3, x_4) = (c, x_2, x_3, x_4) \neq (y_1, b, y_3, y_4) = f(y_1, y_2, y_3, y_4)$. ∎

Finally we are ready to complete the proof of Theorem 4. If there is no N in P then $Q = P$ and we are done. Suppose there is an N in P. Then we can find a maximal N-edge (B, C) in P. By Lemmas 1 and 2, we construct an ordered set $\overline{P} = P \cup \{p\}$ with order induced from P and $B \succ \{p\} \succ C$ such that dimension $\overline{P} =$ dimension P and $|N(\overline{P})| < |N(P)|$. Since $N(P) \subset P^4$, by adding elements to P one by one, as above, we finally can reach a desired ordered set Q in at most n^4 steps.

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